Annales Mathematicae et Informaticae
59 (2023) pp. 117–130
DOI: 10.33039/ami.2023.12.002
URL: https://ami.uni-eszterhazy.hu

Units of the semisimple group algebras $\mathcal{F}_q SL(2,8)$ and $\mathcal{F}_q SL(2,9)$

Sivaranjani N U[a], **E. Nandakumar**[a], **Gaurav Mittal**[b], **R. K. Sharma**[c]

[a]Department of Mathematics, College of Engineering and Technology, SRM Institute of Science & Technology, Kattankulathur - 603 203, Tamil Nadu, INDIA
sn0926@srmist.edu.in,nanda1611@gmail.com

[b]Defence Research and Development Organization, Near Metcalfe House, Delhi - 110054, INDIA
gaurav.mittaltwins@yahoo.com

[c]Department of Mathematics, Indian Institute of Technology, Delhi - 110016, INDIA
rksharmaiitd@gmail.com
Corresponding Author: E. Nandakumar

Abstract. In this paper, we characterize the structure of the unit group of the semisimple group algebras $\mathcal{F}_q SL(2,8)$ and $\mathcal{F}_q SL(2,9)$ of the special linear groups of 2×2 matrices with determinant 1 over the finite fields of order 8 and 9, respectively.

Keywords: Unit group, Group algebra, Special linear group, Wedderburn Decomposition

AMS Subject Classification: 16U60, 20C05

1. Introduction

The group algebra of the finite group G over the finite field $\mathcal{F}_q$ is denoted by $\mathcal{F}_q G$. Let q and p be the order and characteristics of the finite field $\mathcal{F}_q$, respectively and $q = p^k$. Let $\mathcal{U}(\mathcal{F}_q G)$ be the group of units of the group algebra $\mathcal{F}_q G$. Group theory frequently runs into the issues with unit group of the group algebra. The characterization of the unit groups is crucial for a number of applications, including the study of the isomorphism problem [14], one of the most significant research problems in the theory of group algebras, the development of convolutional codes in group algebra (see [5, 9]) and other applications. The structure of the unit

Submitted: September 20, 2023
Accepted: December 6, 2023
Published online: December 13, 2023

group of the semisimple group algebra $\mathcal{F}_qG$ has been extensively studied (see [3, 4, 10, 12–18, 20, 22]). The study by Bakshi et al. [4], in which the unit groups of the semisimple group algebras of all metabelian groups are studied, is one of the most significant ones in this field. As a result, the majority of research in this field focuses on understanding the unit group of non-metabelian group algebras. The unit groups of the group algebras of non-metabelian groups up to order 72 were described by Mittal et al. in [16]. Further, Sharma et al. determined the unit group of the semisimple group algebra (SGA) of the respective groups $SL(2,3)$ (special linear group over the finite field of 3 elements) and $SL(2,5)$ (See [11] and [19]). In continuation, Sivaranjani et al. [21] determined the unit group of the SGA of the group $SL(2,7)$. In addition, Arvind et al. [1] investigated the unit group of the SGA of the group $SL(3,\mathbb{Z}_2)$. The main objective of this paper is to derive the unit group of the SGA of the groups $SL(2,8)$ and $SL(2,9)$, respectively. We notice that the difficulty of exactly identifying the unit group of the SGA increases as the size of the group increases. One may refer [15, 17] for some of the recent works in this area. Our first goal in determining the unit group is to infer the Wedderburn decomposition (WD) of $\mathcal{F}_qSL(2,8)$ and $\mathcal{F}_qSL(2,9)$, respectively. Further, it is easy to derive the unit group from the WD. The rest of this paper is structured as follows. The prerequisites for the article are covered in Section 2. In sections 3 and 4, we deduce the unit group of the group algebras $\mathcal{F}_qSL(2,8)$ and $\mathcal{F}_qSL(2,9)$ in the form of theorems 3.1 and 4.1, respectively. Section 5 concludes the paper.

2. Preliminaries

Throughout this paper, $SL(n,r)$ denotes the special linear group of $n\times n$ matrices with determinant 1 over the finite field of order r. The order of $SL(n,r)$ is given by

$$(r^n-1)(r^n-r)\cdots(r^n-r^{n-1})(r-1)^{-1}.$$

Next, we discuss some notations and results from [7]. Let $J(\mathcal{F}_qG)$ denote the Jacobson radical of $\mathcal{F}_qG$. Let s be the least common multiple of the orders of p-regular elements of group G and let η be the primitive s^{th} root of unity over a finite field $\mathcal{F}$. Let $T_{G,\mathcal{F}}=\{t:\eta\to\eta^t \text{ is an automorphism of } \mathcal{F}(\eta) \text{ over } \mathcal{F}\}$. Since the Galois group $\mathrm{Gal}(\mathcal{F}(\eta):\mathcal{F})$ is cyclic, for any $\sigma\in\mathrm{Gal}(\mathcal{F}(\eta):\mathcal{F})$, there exists a positive integer s such that $\sigma(\eta)=\eta^s$. For any p-regular element $g\in G$ (i.e., p does not divide order of g), we define $\gamma_g=\sum h$, where h runs over all the elements in the conjugacy class C_g of g. The cyclotomic $\mathcal{F}$-class of γ_g is defined as $S\mathcal{F}(\gamma_g)=\{\gamma_{g^t}\mid t\in T_{G,\mathcal{F}}\}$. The following theorem characterizes the set $T_{G,\mathcal{F}}$.

Theorem 2.1 ([16, Theorem 2.3])**.** *Let $\mathcal{F}$ be a finite field with prime power order d such that $\gcd(d,s)=1$ and $e=\mathrm{order}_s(d)$ is the multiplicative order of d modulo s, then $T_{G,\mathcal{F}}=\{1,d,\dots,d^{e-1}\} \mod s$.*

To uniquely identify the Wedderburn decomposition (WD) of the group algebra, the following six results will play an important role.

Proposition 2.2 ([7, Proposition 1.2])**.** *The number of non isomorphic simple components of $\mathcal{F}G/J(\mathcal{F}G)$ is equal to the number of cyclotomic $\mathcal{F}$-classes in G.*

Theorem 2.3 ([7, Theorem 1.3])**.** *Assume that G has t cyclotomic $\mathcal{F}$-classes and $\mathrm{Gal}(\mathcal{F}(\eta) : \mathcal{F}))$ is a cyclic group, then $|S_i| = [\mathcal{F}_i : \mathcal{F}]$ with appropriate index ordering if $S_1, S_2, \cdots, S_t$ are the cyclotomic $\mathcal{F}$-classes of G and $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t$ are the simple components of $Z(\mathcal{F}G/J(\mathcal{F}G))$.*

Proposition 2.4 ([14, Proposition 3.6.11])**.** *Let G' be the commutator subgroup of G and let $\mathcal{F}G$ be a semisimple group algebra, then*

$$\mathcal{F}G \simeq \mathcal{F}(G/G') \oplus \triangle(G, G'),$$

where $\mathcal{F}(G/G')$ is the sum of all commutative simple components of $\mathcal{F}G$ and $\triangle(G, G')$ is the sum of all others.

Proposition 2.5 ([14, Proposition 3.6.7])**.** *Let N be a normal subgroup of G and let $\mathcal{F}G$ be a semisimple group algebra (SGA), then*

$$\mathcal{F}G \simeq \mathcal{F}(G/N) \oplus \triangle(G, N),$$

where $\triangle(G, N)$ is an ideal of $\mathcal{F}G$ generated by the set $\{n - 1 : n \in N\}$.

Proposition 2.6 ([6, Proposition 1])**.** *Let $\mathcal{F}G$ be a finite SGA, where characteristics of $\mathcal{F}$ is p. Let $\mathcal{F}G \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i)$, where $\mathcal{F}_i$ are finite extensions of $\mathcal{F}$ and r is a positive integer. Then p does not divide any of the n_i.*

Lemma 2.7 ([24])**.** *Let p_1 and p_2 be two primes. Let $\mathcal{F}_{q_1}$ be a field with $q_1 = p_1^{k_1}$ elements and let $\mathcal{F}_{q_2}$ be a field with $q_2 = p_2^{k_2}$ elements, where $k_1, k_2 \geq 1$. Let both the group algebras $\mathcal{F}_{q_1}G, \mathcal{F}_{q_2}G$ be semisimple. Suppose that*

$$\mathcal{F}_{q_1}G \cong \oplus_{i=1}^{t}\mathbf{M}(n_i, \mathcal{F}_{q_1}),\ n_i \geq 1$$

and $M\big(n, \mathcal{F}_{q_2^r}\big)$ is a Wedderburn component of the group algebra $\mathcal{F}_{q_2}G$ for some $r \geq 1$ and any positive integer n, i.e.,

$$\mathcal{F}_{q_2}G \cong \oplus_{i=1}^{s-1}\mathbf{M}(m_i, \mathcal{F}_{q_{2,i}}) \oplus \mathbf{M}(n, \mathcal{F}_{q_2^r}),\ m_i \geq 1.$$

Here $\mathcal{F}_{q_{2,i}}$ is a field extension of $\mathcal{F}_{q_2}$. Then $M(n, \mathcal{F}_{q_1})$ must be a Wedderburn component of the group algebra $\mathcal{F}_{q_1}G$ and it appears atleast r times in the WD of $\mathcal{F}_{q_1}G$.

Proposition 2.8 ([1, Corollary 3.8])**.** *Let $\mathcal{F}G$ be a finite SGA, where characteristics of $\mathcal{F}$ is p. If there exists an irreducible representations of degree n over $\mathcal{F}$, then one of the Wedderburn component of $\mathcal{F}G$ is $\mathbf{M}(n, \mathcal{F})$.*

3. Unit group of $\mathcal{F}_q SL(2,8)$

Let $G_1 := SL(2,8)$. Clearly, the order of G_1 is 504. The group algebra $\mathcal{F}_q G_1$ is semi simple for $p \neq 2,3,7$ by Maschke's theorem [14]. Also, One can note that the degrees of irreducible representations of G_1 are $1,7,8$ and 9 whenever $|S\mathcal{F}_q(\gamma_g)| = 1, \forall g \in G_1$ (see [23]). The group G_1 has 9 conjugacy classes (let the representative of these classes be denoted by g_i for $i = 1,\dots,9$). The representatives (R) of the conjugacy classes, sizes (S) and the orders (O) of representatives are tabulated below.

R	I_2	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & x^2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix}$	$\begin{bmatrix} x & 0 \\ 0 & \beta \end{bmatrix}$	$\begin{bmatrix} x^2 & 0 \\ 0 & \alpha+1 \end{bmatrix}$	$\begin{bmatrix} x+1 & 0 \\ 0 & \alpha \end{bmatrix}$
S	1	63	56	56	56	56	72	72	72
O	1	2	3	9	9	9	7	7	7

Here I_2 is 2×2 identity matrix, x is the generator of multiplicative group of finite field of order 8, $\alpha = x^2 + x$ and $\beta = x^2 + 1$. Also, G_1 can be generated by two elements a and b, where

$$a = \begin{bmatrix} x & 0 \\ 0 & x^2+1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.1}$$

The exponent of G_1 is 126. In this section, we characterize the unit group of the group algebra $\mathcal{F}_q G_1$ for $p \neq 2,3,7$ such that the group algebra $\mathcal{F}_q G_1$ is semisimple and $q = p^k$. In the following theorems, $\mathcal{F}_i$ denotes the finite extensions of $\mathcal{F}_q$ and n_i, r are positive integers.

Theorem 3.1. *The unit group of $\mathcal{F}_q G_1$, where $q = p^k$ and $p \neq 2,3,7$ is given as follows:*
(1) *for* $p^k \equiv \{1, 55, 71, 125\} \mod 126$, *we have*

$$U(\mathcal{F}_q G_1) \simeq \mathcal{F}_q^* \oplus GL(7,\mathcal{F}_q)^4 \oplus GL(8,\mathcal{F}_q) \oplus GL(9,\mathcal{F}_q)^3.$$

(2) *for* $p^k \equiv \{13, 29, 41, 43, 83, 85, 97, 113\} \mod 126$, *we have*

$$U(\mathcal{F}_q G_1) \simeq \mathcal{F}_q^* \oplus GL(7,\mathcal{F}_q) \oplus GL(8,\mathcal{F}_q) \oplus GL(9,\mathcal{F}_q)^3 \oplus GL(7,\mathcal{F}_{q^3}).$$

(3) $p^k \equiv \{17, 19, 37, 53, 73, 89, 107, 109\} \mod 126$, *we have*

$$U(\mathcal{F}_q G_1) \simeq \mathcal{F}_q^* \oplus GL(7,\mathcal{F}_q)^4 \oplus GL(8,\mathcal{F}_q) \oplus GL(9,\mathcal{F}_{q^3}).$$

(4) $p^k \equiv \{5,11,23,25,31,47,59,61,65,67,79,95,101,103,115,121\} \mod 126$, *we have*

$$U(\mathcal{F}_qG_1) \simeq \mathcal{F}_q^* \oplus GL(7,\mathcal{F}_q) \oplus GL(8,\mathcal{F}_q) \oplus GL(7,\mathcal{F}_{q^3}) \oplus GL(9,\mathcal{F}_{q^3}).$$

Proof. The group algebra $\mathcal{F}_qG_1$ is semi simple, it follows from the Wedderburn decomposition theorem [14] that $\mathcal{F}_qG_1 \simeq \oplus_{i=1}^r \mathbf{M}(n_i,\mathcal{F}_i)$. The derived subgroup G_1' of G_1 is G_1 itself (i.e., G_1 is a perfect one). This accompanying with Proposition 2.2 gives

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \bigoplus_{i=1}^{r-1} \mathbf{M}(n_i,\mathcal{F}_i), \quad n_i \geq 2. \tag{3.2}$$

Using Theorem 2.1, we construct the set $T_{G,\mathcal{F}}$ of group G_1 and divide the proof into the following 4 cases.
Case 1: $p^k \equiv \{1,55,71,125\} \mod 126$. In this case, we note that the cardinality of cyclotomic $\mathcal{F}_q$-class of γ_g is 1, for all g in G_1. By employing this with Proposition 2.2 and Theorem 2.3, we further rewrite (3.2) as

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \bigoplus_{i=1}^{8} \mathbf{M}(n_i,\mathcal{F}_q) \implies 503 = \sum_{i=1}^{8} n_i^2,\ n_i \geq 2. \tag{3.3}$$

We have discussed earlier in this section that the degrees of irreducible representations of G_1 are $1,7,8$ and 9, whenever $\mid S\mathcal{F}_q(\gamma_g) \mid = 1, \forall g \in G_1$. We note that there are 158 choices of n_i's fulfilling (4.3). The only choice that only contains $7,8$ and 9 is $(7^4,8,9^3)$. Hence, the Wedderburn decomposition (WD) is

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \oplus \mathbf{M}(7,\mathcal{F}_q)^4 \oplus \mathbf{M}(8,\mathcal{F}_q) \oplus \mathbf{M}(9,\mathcal{F}_q)^3.$$

Case 2: $p^k \equiv \{13,29,41,43,83,85,97,113\} \mod 126$. In this case, the cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\},\ \text{for}\ i = 1,2,3,7,8,9, \quad S\mathcal{F}_q(\gamma_{g_4}) = \{\gamma_{g_4},\gamma_{g_5},\gamma_{g_6}\}.$$

By incorporating Proposition 2.2, we derive from (3.2) that

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \bigoplus_{i=1}^{5} \mathbf{M}(n_i,\mathcal{F}_q) \oplus \mathbf{M}(n_6,\mathcal{F}_{q^3}) \implies 503 = \sum_{i=1}^{5} n_i^2 + 3n_6^2,\ n_i \geq 2. \tag{3.4}$$

Due to Lemma 2.7, it follows from (3.4) that $n_i \geq 7$. Consequently, the possible choices of n_i's fulfilling (3.4) are $(7^4,8,9)$, $(7^3,8,10,8)$, $(7,8,9^3,7)$ and $(8^4,10,7)$. Again, Lemma 2.7 implies that $\mathbf{M}(10,\mathcal{F}_q)$ can not be a Wedderburn component. Therefore, we are only remaining with two choices of n_i's given by $(7^4,8,9)$ and $(7,8,9^3,7)$. Next, to uniquely identify the correct choice, we show that $\mathbf{M}(9,\mathcal{F}_q)$ will always be a Wedderburn component in this case. In particular, we take $p = 13$

and consider the following mapping from G_1 to $GL(9, \mathcal{F}_{13})$:

$$a \to \begin{bmatrix} 5 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 4 & 8 & 3 & 0 & 0 & 0 & 0 \\ 6 & 12 & 11 & 2 & 12 & 11 & 0 & 0 & 0 \\ 0 & 6 & 2 & 11 & 9 & 4 & 2 & 4 & 0 \\ 10 & 5 & 10 & 7 & 7 & 6 & 12 & 7 & 11 \\ 3 & 2 & 2 & 4 & 9 & 10 & 12 & 9 & 8 \\ 11 & 1 & 3 & 8 & 3 & 5 & 11 & 8 & 9 \\ 4 & 5 & 3 & 12 & 6 & 10 & 11 & 6 & 1 \end{bmatrix},$$

$$b \to \begin{bmatrix} 8 & 9 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 7 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 12 & 2 & 4 & 3 & 1 & 12 & 0 & 0 \\ 3 & 10 & 1 & 4 & 6 & 0 & 8 & 1 & 11 \\ 2 & 5 & 3 & 8 & 1 & 10 & 0 & 10 & 2 \\ 9 & 8 & 5 & 6 & 2 & 8 & 1 & 5 & 11 \\ 6 & 1 & 5 & 0 & 9 & 2 & 8 & 8 & 0 \\ 12 & 4 & 0 & 10 & 8 & 12 & 6 & 7 & 10 \end{bmatrix}.$$

This mapping is a homomorphism from G_1 to $GL(9, \mathcal{F}_{13})$ (as a and b given in (3.1) generates G_1). It should be noted that this map is an irreducible representation of G_1 over $\mathcal{F}_{13}$. Therefore, according to Proposition 2.8, $\mathbf{M}(9, \mathcal{F}_{13})$ will always be a Wedderburn component of $\mathcal{F}_q G_1$. Hence, it follows that $(7, 8, 9^3, 7)$ is the only possible value for n_i. Hence, the WD is

$$\mathcal{F}_q G \simeq \mathcal{F}_q \oplus \mathbf{M}(7, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_q) \oplus \mathbf{M}(9, \mathcal{F}_q)^3 \oplus \mathbf{M}(7, \mathcal{F}_{q^3}).$$

Case 3: $p^k \equiv \{17, 19, 37, 53, 73, 89, 107, 109\} \bmod 124$. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } 1, 2, 3, 4, 5, 6, \quad S\mathcal{F}_q(\gamma_{g_7}) = \{\gamma_{g_7}, \gamma_{g_8}, \gamma_{g_9}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (3.2) that

$$\mathcal{F}_q G_1 \simeq \mathcal{F}_q \bigoplus_{i=1}^{5} \mathbf{M}(n_i, \mathcal{F}_q) \oplus \mathbf{M}(n_6, \mathcal{F}_{q^3}) \implies 503 = \sum_{i=1}^{5} n_i^2 + 3n_6^2, \ n_i \geq 2. \quad (3.5)$$

By proceeding on the similar lines of case 2, one can show that we need to deduce the unique choices among the 2 choices $(7^4, 8, 9)$ and $(7, 8, 9^3, 7)$. For this, we take

$p = 17$ and show that there are two distinct homomorphisms from G_1 to $GL(7, \mathcal{F}_{17})$. We consider the following mappings:

$$a \to \begin{bmatrix} 13 & 10 & 0 & 0 & 0 & 0 & 0 \\ 1 & 11 & 10 & 4 & 0 & 0 & 0 \\ 0 & 8 & 6 & 11 & 7 & 9 & 0 \\ 0 & 1 & 14 & 12 & 14 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 & 15 & 4 \\ 0 & 0 & 1 & 1 & 4 & 3 & 10 \\ 0 & 0 & 0 & 12 & 12 & 3 & 12 \end{bmatrix}, \quad b \to \begin{bmatrix} 8 & 14 & 16 & 0 & 0 & 0 & 0 \\ 9 & 2 & 6 & 13 & 4 & 0 & 0 \\ 0 & 5 & 14 & 1 & 6 & 15 & 4 \\ 0 & 2 & 3 & 11 & 1 & 13 & 15 \\ 0 & 1 & 12 & 14 & 4 & 3 & 12 \\ 0 & 0 & 12 & 8 & 10 & 14 & 10 \\ 0 & 0 & 1 & 9 & 1 & 11 & 16 \end{bmatrix},$$

$$a \to \begin{bmatrix} 14 & 14 & 0 & 0 & 0 & 0 & 0 \\ 13 & 16 & 0 & 4 & 0 & 0 & 0 \\ 16 & 12 & 12 & 15 & 0 & 0 & 0 \\ 5 & 8 & 9 & 0 & 15 & 7 & 2 \\ 10 & 9 & 13 & 13 & 9 & 1 & 12 \\ 9 & 1 & 6 & 3 & 13 & 15 & 0 \\ 1 & 5 & 11 & 8 & 2 & 14 & 2 \end{bmatrix}, \quad b \to \begin{bmatrix} 6 & 8 & 16 & 0 & 0 & 0 & 0 \\ 12 & 1 & 16 & 14 & 0 & 16 & 0 \\ 11 & 12 & 13 & 7 & 4 & 12 & 0 \\ 0 & 15 & 9 & 6 & 0 & 13 & 0 \\ 9 & 7 & 2 & 9 & 13 & 13 & 14 \\ 12 & 14 & 10 & 6 & 9 & 9 & 4 \\ 5 & 9 & 14 & 13 & 4 & 16 & 4 \end{bmatrix}.$$

These mappings are 2 irreducible representations of G_1 over $\mathcal{F}_{17}$. Therefore, Proposition 2.8 derives that $\mathbf{M}(7, \mathcal{F}_{17})^2$ is a summand of the group algebra $\mathcal{F}_{17}G_1$. Thus, the required choices of n_i's fulfilling (3.5) is $(7^4, 8, 9)$ Hence, the WD is

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \oplus \mathbf{M}(7, \mathcal{F}_q)^4 \oplus \mathbf{M}(8, \mathcal{F}_q) \oplus \mathbf{M}(9, \mathcal{F}_{q^3}).$$

Case 4: $p^k \equiv \{5, 11, 23, 25, 31, 47, 59, 61, 67, 79, 101, 103, 115, 121, 65, 95\} \bmod 124$. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } 1,2,3, \; S\mathcal{F}_q(\gamma_{g_7}) = \{\gamma_{g_7}, \gamma_{g_8}, \gamma_{g_9}\}, S\mathcal{F}_q(\gamma_{g_4}) = \{\gamma_{g_4}, \gamma_{g_5}, \gamma_{g_6}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (3.2) that

$$\mathcal{F}_qG_1 \simeq (\mathcal{F}_q) \bigoplus_{i=1}^{2} \mathbf{M}(n_i, \mathcal{F}_q) \oplus \mathbf{M}(n_3, \mathcal{F}_{q^3}) \oplus \mathbf{M}(n_4, \mathcal{F}_{q^3})$$

$$\implies 503 = \sum_{i=1}^{2} n_i^2 + 3(n_3^2 + n_4^2), \; n_i \geq 2. \tag{3.6}$$

By following the procedure as in case 1, we can show that the $n_i \geq 7$ in (3.6). Hence, the only possible choice of n_i's is $(7, 8, 7, 9)$, which means that

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \oplus \mathbf{M}(7, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_q) \oplus \mathbf{M}(7, \mathcal{F}_{q^3}) \oplus \mathbf{M}(9, \mathcal{F}_{q^3}).$$

This completes the proof. □

4. Unit group of $\mathcal{F}_q SL(2,9)$

Let $G_2 := SL(2,9)$. The order of G_2 is 720. Since $p > 5$, it does not divide the order of G_2, the group algebra $\mathcal{F}_q G_2$ is semisimple. Also, from [8] one can note that G_2 has irreducible representations of degrees $1, 4, 5, 8, 9$ and 10 whenever $|S\mathcal{F}_q(\gamma_g)| = 1, \forall g \in G_2$. The group G_2 has 13 conjugacy classes and it is represented as $g_i' s$. The representative of the conjugacy classes, size and the order of representatives are tabulated below:

R	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 2x+2\\x+2 & 1\end{bmatrix}$	$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$	$\begin{bmatrix}0 & 2\\1 & 2\end{bmatrix}$	$\begin{bmatrix}0 & 2x+2\\x+2 & 2\end{bmatrix}$
S	1	40	40	1	40	40
O	1	6	6	2	3	3

$\begin{bmatrix}0 & 2\\1 & 2x+2\end{bmatrix}$	$\begin{bmatrix}0 & 2\\1 & x+2\end{bmatrix}$	$\begin{bmatrix}0 & 2\\1 & x+1\end{bmatrix}$	$\begin{bmatrix}0 & 2\\1 & 2x+1\end{bmatrix}$	$\begin{bmatrix}2x+2 & 0\\0 & 2x+1\end{bmatrix}$
72	72	72	72	90
5	5	10	10	8

$\begin{bmatrix}x & 0\\0 & 2x\end{bmatrix}$	$\begin{bmatrix}x+2 & 0\\0 & x+1\end{bmatrix}$
90	90
4	8

Here x is the generator of multiplicative group of finite field of order 9. Also, G_2 can be generated by two elements a and b, where

$$a = \begin{bmatrix}x+1 & 0\\0 & x+2\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}2 & 1\\2 & 0\end{bmatrix}. \tag{4.1}$$

In this section, we characterize the unit group of the group algebra $\mathcal{F}_q G_2$ for $p > 5$ such that the group algebra $\mathcal{F}_q G_2$ is semisimple and $q = p^k$. It is clear from the above table that the exponent of G_2 is 120.

Theorem 4.1. *The unit group of $\mathcal{F}_q G_2$ is as follows:*
(1) *for* $p^k \equiv \{1, 31, 41, 49, 71, 79, 89, 119\} \mod 120$, *we have*

$$U(\mathcal{F}_q G_2) \simeq \mathcal{F}_q^* \oplus GL(4,\mathcal{F}_q)^2 \oplus GL(5,\mathcal{F}_q)^2 \oplus GL(8,\mathcal{F}_q)^4 \oplus GL(9,\mathcal{F}_q) \oplus GL(10,\mathcal{F}_q)^3.$$

(2) *for* $p^k \equiv \{7, 17, 23, 47, 73, 97, 103, 113\} \mod 120$, *we have*

$$U(\mathcal{F}_q G_2) \simeq \mathcal{F}_q^* \oplus GL(4,\mathcal{F}_q)^2 \oplus GL(5,\mathcal{F}_q)^2 \oplus GL(9,\mathcal{F}_q) \oplus GL(10,\mathcal{F}_q)^3 \oplus GL(8,\mathcal{F}_{q^2})^2.$$

(3) $p^k \equiv \{11, 19, 29, 59, 61, 91, 101, 109\} \mod 120$, *we have*

$$U(\mathcal{F}_q G_2) \simeq \mathcal{F}_q^* \oplus GL(4,\mathcal{F}_q)^2 \oplus GL(5,\mathcal{F}_q)^2 \oplus GL(9,\mathcal{F}_q) \oplus GL(8,\mathcal{F}_q)^4$$

$$\oplus GL(10,\mathcal{F}_q) \oplus GL(10,\mathcal{F}_{q^2}).$$

(4) $p^k \equiv \{13, 37, 43, 53, 67, 77, 83, 107\} \mod 120$, *we have*

$$\begin{aligned} U(\mathcal{F}_qG_2) \simeq \mathcal{F}_q^* &\oplus GL(4,\mathcal{F}_q)^2 \oplus GL(5,\mathcal{F}_q)^2 \oplus GL(9,\mathcal{F}_q) \oplus GL(10,\mathcal{F}_q) \\ &\oplus GL(10,\mathcal{F}_{q^2}) \oplus GL(8,\mathcal{F}_{q^2})^2. \end{aligned}$$

Proof. It follows from the Wedderburn decomposition theorem that $\mathcal{F}_qG_2 \simeq \bigoplus_{i=1}^{r} \mathbf{M}(n_i,\mathcal{F}_i)$. Also, G_2 is a perfect group. This and Proposition 2.2 imply that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{r-1} \mathbf{M}(n_i,\mathcal{F}_i), \quad n_i \geq 2. \tag{4.2}$$

As in the previous theorem, we construct the set $T_{G,\mathcal{F}}$ of group G_2 and divide the proof into the following 4 cases.

Case 1: $p^k \equiv \{1, 31, 41, 49, 71, 79, 89, 119\} \mod 120$. In this case, it can be verified that $|S\mathcal{F}_q(\gamma_g)| = 1, \forall g \in G_2$. By utilizing this along with Proposition 2.2, we further rewrite (4.2) as

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{12} \mathbf{M}(n_i,\mathcal{F}_q) \implies 719 = \sum_{i=1}^{12} n_i^2, \ n_i \geq 2. \tag{4.3}$$

Next, we consider the normal subgroup N of G_2 generated by $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$. One can observe that with $G_2/N \simeq A_6$. We recall from [2, Proposition 4.7] that

$$\mathcal{F}_qA_6 \simeq \mathcal{F}_q \oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q) \oplus \mathbf{M}(8,\mathcal{F}_q)^2. \tag{4.4}$$

Utilizing (4.4) and Proposition 2.5 in (4.3) to derive that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q) \oplus \mathbf{M}(8,\mathcal{F}_q)^2 \bigoplus_{i=1}^{6} \mathbf{M}(n_i,\mathcal{F}_q) \tag{4.5}$$

with $360 = \sum_{i=1}^{6} n_i^2, \ n_i \geq 2$. We note that G_2 has irreducible representations of degrees $1, 4, 5, 8, 9$ and 10. This means n_i's in (4.5) are among the set $\{4, 5, 8, 9, 10\}$. Among all the possible choices of n_i's fulfilling $360 = \sum_{i=1}^{6} n_i^2$, the only choice that contains elements from the set $\{4, 5, 8, 9, 10\}$ is $(4^2, 8^2, 10^2)$. Hence, (4.5) implies that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(4,\mathcal{F}_q)^2 \oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(8,\mathcal{F}_q)^4 \oplus \mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q)^3.$$

Case 2: $p^k \equiv \{7, 17, 23, 47, 73, 97, 103, 113\} \mod 120$. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for 1-6, 11-3}, \ S\mathcal{F}_q(\gamma_{g_7}) = \{\gamma_{g_7}, \gamma_{g_8}\}, \ S\mathcal{F}_q(\gamma_{g_9}) = \{\gamma_{g_9}, \gamma_{g_{10}}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (4.2) that

$$\begin{aligned} \mathcal{F}_q G_2 &\simeq \mathcal{F}_q \bigoplus_{i=1}^{8} \mathbf{M}(n_i, \mathcal{F}_q) \oplus \mathbf{M}(n_9, \mathcal{F}_{q^2}) \oplus \mathbf{M}(n_{10}, \mathcal{F}_{q^2}), \\ 719 &= \sum_{i=1}^{8} n_i^2 + 2(n_9^2 + n_{10}^2), \end{aligned} \tag{4.6}$$

where $n_i \geq 2$. We observe the WD of $\mathcal{F}_q A_6$ in this case is (see [2, Proposition 4.7])

$$\mathcal{F}_q A_6 \simeq \mathcal{F}_q \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_{q^2}). \tag{4.7}$$

Using (4.7) and Proposition 2.5, we further obtain from (4.3) that

$$\begin{aligned} \mathcal{F}_q G_2 \simeq \mathcal{F}_q &\oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q) \\ &\oplus \mathbf{M}(8, \mathcal{F}_{q^2}) \bigoplus_{i=1}^{4} \mathbf{M}(n_i, \mathcal{F}_q) \oplus \mathbf{M}(n_5, \mathcal{F}_{q^2}), \end{aligned} \tag{4.8}$$

with

$$360 = \sum_{i=1}^{4} n_i^2 + 2n_5^2, \;\; n_i \geq 2. \tag{4.9}$$

According to Lemma 2.7 and Case 1, $4 \leq n_i \leq 10$. Moreover, Proposition 2.6 confirms that $n_i \neq 7$ in this case. Thus, we are remaining with the three choices of n_i's fulfilling (4.9) given by $(4^2, 8^2, 10), (4^2, 10^2, 8)$ and $(8^2, 10^2, 4)$. Next, to uniquely identify the correct choice, we show that $\mathbf{M}(4, \mathcal{F}_q)$ will always be a Wedderburn component in this case. In particular, we take $p = 7$ and consider the following mapping from G_2 to $GL(4, \mathcal{F}_7)$:

$$a \to \begin{bmatrix} 3 & 6 & 2 & 0 \\ 5 & 6 & 0 & 0 \\ 0 & 5 & 4 & 2 \\ 5 & 2 & 1 & 1 \end{bmatrix}, \quad b \to \begin{bmatrix} 4 & 2 & 5 & 2 \\ 4 & 5 & 4 & 0 \\ 4 & 6 & 3 & 5 \\ 0 & 3 & 4 & 0 \end{bmatrix}.$$

This mapping is a homomorphism from G_2 to $GL(4, \mathcal{F}_7)$ (as a and b given in (4.1) generates G_2). It should be noted that this map is an irreducible representation of G_2 over $\mathcal{F}_7$. Therefore, according to Proposition 2.8, $\mathbf{M}(4, \mathcal{F}_7)$ will always be a Wedderburn component of $\mathcal{F}_q G_2$. Consequently, we are left with two possible choices of n_i's given by $(4^2, 8^2, 10)$ and $(4^2, 10^2, 8)$. Finally, we show that $\mathbf{M}(10, \mathcal{F}_q)^2$

is a summand of $\mathcal{F}_qG_2$. For this, we define the following two maps:

$$a \to \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0\\ 6&2&3&4&0&0&0&0&0&0\\ 4&6&3&3&0&0&0&0&0&0\\ 2&5&0&2&0&0&0&0&0&0\\ 5&1&0&6&0&1&6&0&1&0\\ 3&3&0&5&1&1&3&3&4&0\\ 3&5&6&0&4&1&2&5&0&3\\ 6&0&4&4&2&3&6&6&1&5\\ 6&6&1&4&4&1&2&3&5&1\\ 3&2&5&1&1&0&4&1&4&4 \end{bmatrix}, \ b \to \begin{bmatrix} 0&4&4&0&0&0&0&0&0&0\\ 2&1&4&3&5&0&0&0&0&0\\ 6&3&4&3&3&4&0&0&0&0\\ 2&5&2&5&2&4&3&0&0&0\\ 6&0&3&3&1&4&6&0&0&0\\ 3&2&6&2&5&6&1&0&0&0\\ 4&3&0&3&2&2&5&0&0&0\\ 4&1&6&5&5&1&2&4&3&4\\ 2&3&6&6&0&0&6&0&1&3\\ 4&1&0&3&5&4&4&3&3&2 \end{bmatrix}$$

$$a \to \begin{bmatrix} 4&5&0&0&0&0&0&0&0&0\\ 0&4&3&5&0&0&0&0&0&0\\ 6&6&2&4&6&0&4&0&0&0\\ 4&1&4&5&1&3&1&6&0&0\\ 4&3&0&3&4&3&4&0&4&5\\ 3&6&2&2&5&0&4&4&6&5\\ 2&0&5&3&3&3&4&1&0&5\\ 3&2&3&1&0&3&0&0&0&6\\ 5&4&0&1&6&2&2&5&3&0\\ 5&1&0&4&1&6&3&1&0&5 \end{bmatrix}, \ b \to \begin{bmatrix} 1&0&4&0&0&0&0&0&0&0\\ 0&6&1&6&3&0&0&0&0&0\\ 0&4&0&5&3&5&6&0&0&0\\ 0&1&0&5&2&2&0&3&6&0\\ 0&3&3&0&6&5&4&2&1&2\\ 0&3&2&2&4&1&1&4&1&0\\ 0&1&3&1&0&0&2&6&3&6\\ 0&1&3&3&6&1&5&1&0&1\\ 0&6&4&2&2&6&5&4&1&6\\ 0&5&1&5&0&6&5&3&0&6 \end{bmatrix}$$

These mappings are 2 irreducible representations of G_2 over $\mathcal{F}_7$. Therefore, Proposition 2.8 derives that $\mathbf{M}(10,\mathcal{F}_7)^2$ is a summand of the group algebra $\mathcal{F}_7G_2$. Consequently, the required choice of n_i's is $(4^2,10^2,8)$. Hence, using (4.8), we get

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(4,\mathcal{F}_q)^2 \oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q)^3 \oplus \mathbf{M}(8,\mathcal{F}_{q^2})^2.$$

Case 3: $p^k \equiv \{11,19,29,59,61,91,101,109\} \bmod 120$. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for 1-10, 12}, \ S\mathcal{F}_q(\gamma_{g_{11}}) = \{\gamma_{g_{11}}, \gamma_{g_{13}}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (4.2) that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{10} \mathbf{M}(n_i,\mathcal{F}_q) \oplus \mathbf{M}(n_{11},\mathcal{F}_{q^2}), \ 719 = \sum_{i=1}^{10} n_i^2 + 2n_{11}^2, \ n_i \geq 2. \tag{4.10}$$

We observe the WD of $\mathcal{F}_qA_6$ in this case same as in Case 1. Using this and Proposition 2.5, we further obtain from (4.10) that

$$\begin{aligned} \mathcal{F}_qG_2 \simeq \mathcal{F}_q &\oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(8,\mathcal{F}_q)^2\mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q) \\ &\bigoplus_{i=1}^{4} \mathbf{M}(n_i,\mathcal{F}_q) \oplus \mathbf{M}(n_5,\mathcal{F}_{q^2}). \end{aligned} \tag{4.11}$$

By proceeding as in the previous case, we are remaining with the three choices of n_i's fulfilling (4.9) given by $(4^2, 8^2, 10), (4^2, 10^2, 8)$ and $(8^2, 10^2, 4)$. Further, on the similar lines of the previous case, we can show that the final choice of n_i's is $(4, 4, 8, 8, 10)$. Hence, it follows from (4.11) that the WD is

$$\begin{aligned}\mathcal{F}_qG_2 \simeq \mathcal{F}_q &\oplus \mathbf{M}(4, \mathcal{F}_q)^2 \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_q)^4 \\ &\oplus \mathbf{M}(10, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_{q^2}).\end{aligned}$$

Case 4: $p^k \equiv \{13, 37, 43, 53, 67, 77, 83, 107\}$ mod 120. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } i = 1\text{-}6, 12, S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}, \gamma_{g_{i+1}}\} \text{ for } i = 7, 9,$$

$$S\mathcal{F}_q(\gamma_{g_{11}}) = \{\gamma_{g_{11}}, \gamma_{g_{13}}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (4.2) that

$$\begin{aligned}\mathcal{F}_qG_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{6} \mathbf{M}(n_i, \mathcal{F}_q) \bigoplus_{i=7}^{9} \mathbf{M}(n_i, \mathcal{F}_{q^2}) \\ \implies 719 = \sum_{i=1}^{6} n_i^2 + 2\sum_{i=7}^{9} n_i^2,\ n_i \geq 2.\end{aligned} \tag{4.12}$$

In this case, the WD of $\mathcal{F}_qA_6$ is given by (4.7). Using this and proposition 2.5, we further obtain from (4.12) that

$$\begin{aligned}\mathcal{F}_qG_2 \simeq \mathcal{F}_q &\oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_{q^2}) \\ &\bigoplus_{i=1}^{2} \mathbf{M}(n_i, \mathcal{F}_q) \bigoplus_{i=3}^{4} \mathbf{M}(n_i, \mathcal{F}_{q^2}),\end{aligned}$$

with $360 = n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2,\ n_i \geq 2$. According to lemma 2.7 and case 1, $4 \leq n_i \leq 10$ for each i. Furthermore, lemma 2.7 and case 1 implies that $n_i \neq 7$ for any i. This leaves us with three possible values of n_i's given by $(4, 4, 8, 10)$, $(8, 8, 4, 10)$ and $(10, 10, 4, 8)$. Next, to uniquely identify the correct choice, we show that $\mathbf{M}(4, \mathcal{F}_q)$ will always be a Wedderburn component in this case. In particular, we take $p = 13$ and consider the following mapping from G_2 to $GL(4, \mathcal{F}_{13})$:

$$a \to \begin{bmatrix} 0 & 12 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 9 \\ 0 & 1 & 2 & 8 \end{bmatrix}, \quad b \to \begin{bmatrix} 0 & 2 & 7 & 0 \\ 9 & 3 & 3 & 3 \\ 1 & 10 & 0 & 1 \\ 0 & 10 & 1 & 8 \end{bmatrix}.$$

This mapping is an irreducible representation of G_2 over $\mathcal{F}_{13}$. Therefore, according to proposition 2.8, $\mathbf{M}(4, \mathcal{F}_q)$ will always be a Wedderburn component of $\mathcal{F}_qG_2$. Hence, we get

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(4, \mathcal{F}_q)^2 \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q)$$

$$\oplus \mathbf{M}(10, \mathcal{F}_{q^2}) \oplus \mathbf{M}(8, \mathcal{F}_{q^2})^2.$$

This completes the proof. □

5. Conclusion

In this paper, we focused on deriving the unit groups of the semisimple group algebras of groups $SL(2,8)$ and $SL(2,9)$. In order to derive these, we computed the Wedderburn decomposition using the findings from the classical theory of group algebras. Having the wide range of possible Wedderburn components, it is evident that it becomes more and more challenging to characterize the Wedderburn decomposition with increasing group size. Finally, this paper further motivates to deduce the unit groups of special linear groups of higher order.

References

[1] N. Arvind, S. Panja: *The unit group of $\mathcal{F}_q SL(3,2)$, $p > 11$*, Arxiv (2021), pp. 1–9, URL: https://arxiv.org/abs/2106.07261.

[2] N. Arvind, S. Panja: *Unit group of some finite semisimple group algebras*, Journal of Egyptian Mathematical Society 3 (2022), p. 17, URL: https://joems.springeropen.com/articles/10.1186/s42787-022-00151-0.

[3] Y. Bai, Y. Li, J. Peng: *Unit groups of finite group algebras of abelian groups of order 17 to 20*, AIMS Mathematics 6 (2021), pp. 7305–7317, URL: https://www.aimspress.com/article/doi/10.3934/math.2021428.

[4] G. K. Bakshi, S. Gupta, I. B. S. Passi: *The algebraic structure of finite metabelian group algebras*, Communications in Algebra 43.1 (2015), pp. 2240–2257, DOI: 10.1080/00927872.2014.888566.

[5] A. Bovdi, J. Kurdics: *Lie properties of the group algebra and the nilpotency class of the group of units*, Journal of Algebra 212.1 (1999), pp. 28–64, DOI: 10.1006/jabr.1998.7617.

[6] C. Dietzel, G. Mittal: *Summands of finite group algebras*, Czechoslovak Mathematical Journal 71.4 (2021), pp. 1011–1014, DOI: 10.21136/CMJ.2020.0171-20.

[7] R. A. Ferraz: *Simple components of the center of $FG/J(FG)$*, Communications in Algebra 36.9 (2008), pp. 3191–3199, DOI: 10.1080/00927870802103503.

[8] J. Humphreys: *Representations of $SL(2,p)$*, American Mathematical Monthly 82.1 (1975), pp. 21–39, URL: https://www.tandfonline.com/doi/abs/10.1080/00029890.1975.11993765.

[9] T. Hurley: *Convolutional codes from units in matrix and group rings*, International Journal of Information and Coding Theory 50.3 (2009), pp. 431–463, URL: https://www.ijpam.eu/contents/2009-50-3/9/index.html.

[10] M. Khan, R. K. Sharma, J. B. Srivastava: *The unit group of FS_4*, Acta Mathematica Hungarica 118.1-2 (2008), pp. 105–113, DOI: 10.1007/s10474-007-6169-4.

[11] S. Maheshwari, R. K. Sharma: *The unit group of the group algebra $\mathcal{F}_q SL(2,\mathbb{Z}_3)$*, Journal of Algebra Combinatorics Discrete Structures and Applications 3 (2016), pp. 1–6, URL: https://jacodesmath.com/index.php/jacodesmath/article/view/24/0.

[12] N. Makhijani, R. K. Sharma: *The unit group of algebra of circulant matrices*, International Journal of Group Theory 3.4 (2014), pp. 13–16, URL: https://web.iitd.ac.in/~rksharma/Research%20Publications/Journal/Neha_Circulant.pdf.

[13] N. Makhijani, R. K. Sharma, J. B. Srivastava: *The unit group of some special semisimple group algebras*, Quaestiones Mathematicae 39.1 (2016), pp. 9–28, URL: https://www.tandfonline.com/doi/abs/10.2989/16073606.2015.1024410.

[14] C. P. Milies, S. K. Sehgal: *An Introduction to Group Rings*, Netherlands: Springer Dordrecht, 2002, URL: https://link.springer.com/book/9781402002380.

[15] G. Mittal, R. K. Sharma: *Computation of the Wedderburn decomposition of semisimple group algebras of groups up to order 120*, Annales Mathematicae et Informaticae Preprint (2023), DOI: 10.33039/ami.2023.07.001.

[16] G. Mittal, R. K. Sharma: *On unit group of finite group algebras of non-metabelian groups upto order 72*, Mathematica Bohemica 146.4 (2021), pp. 429–455, URL: https://articles.math.cas.cz/10.21136/MB.2021.0116-19.

[17] G. Mittal, R. K. Sharma: *Wedderburn decomposition of a semisimple group algebra FqG from a subalgebra of factor group of G*, International Electronic Journal of Algebra 32 (2022), pp. 91–100, DOI: 10.24330/ieja.1077582.

[18] R. K. Sharma, M. Khan, J. B. Srivastava: *The unit group of $\mathcal{F}S_3$*, Acta Mathematica Academiae Paedagogicae Nyıregyhaziensis 23.2 (2007), pp. 129–142, URL: https://www.emis.de/journals/AMAPN/vol23_2/amapn23_14.pdf.

[19] R. K. Sharma, G. Mittal: *Unit group of semisimple group algebra $F_qSL(2,5)$*, Mathematica Bohemica 147.1 (2022), pp. 1–10, URL: https://articles.math.cas.cz/10.21136/MB.2021.0104-20.

[20] R. K. Sharma, P. Yadav: *Unit group of algebra of circulant matrices*, International Journal of Group Theory 2.4 (2013), pp. 1–6, URL: https://web.iitd.ac.in/~rksharma/Research%20Publications/Journal/Pooja_circulant.pdf.

[21] N. U. Sivaranjani, E. Nandakumar, G. Mittal, R. K. Sharma: *The structure of the unit group of the group algebra $\mathcal{K}_qSL(2,\mathbb{Z}_7)$*, Preprint (2023).

[22] F. Tang, Y. Wei, Y. Li: *Unit groups of group algebras of some small groups*, Czechoslovak Mathematical Journal 64 (2014), pp. 149–157, URL: https://link.springer.com/article/10.1007/s10587-014-0090-0.

[23] C. L. Uchtman: *Representations of $SL(2,q)$*, Kent: Dissertation, Kent State University, 2015, URL: https://etd.ohiolink.edu/acprod/odb_etd/etd/r/1501/10?clear=10&p10_accession_num=kent1433952120.

[24] R. Waxman: *On the Wedderburn component of a group algebra* (2019), URL: https://math.stackexchange.com/questions/4677421/on-the-wedderburn-component-of-a-group-algebra.