

## ON THE REGULARITY OF CROSSED PRODUCTS

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ABSTRACT. We study some generalizations of the notion of regular crossed products  $K * G$ . For the case when  $K$  is an algebraically closed field, we give necessary and sufficient conditions for the twisted group ring  $K * G$  to be an  $n$ -weakly regular ring, a  $\xi^*N$ -ring or a ring without nilpotent elements.

## 1. INTRODUCTION

Let  $G$  be a group,  $U(K)$  the group of units of the associative ring  $K$  with identity and let  $\sigma : G \rightarrow \text{Aut}(K)$  be a map of  $G$  into the group  $\text{Aut}(K)$  of automorphisms of  $K$ . Let  $K * G = K_\rho^\sigma G = \{\sum_{g \in G} u_g \alpha_g \mid \alpha_g \in K\}$  be the crossed product (in the sense of [1]), of the group  $G$  over the ring  $K$  with respect to the factor system

$$\rho = \{\rho(g, h) \in U(K) \mid g, h \in G\}$$

and the map  $\sigma : G \rightarrow \text{Aut}(K)$ . Moreover we assume that the factor system  $\rho$  is normalized, i.e.  $\rho(g, 1) = \rho(1, g) = \rho(1, 1) = 1$  for any  $g \in G$ .

In particular, if  $\sigma = 1$ , then the crossed product  $K * G$  is called a *twisted group ring*, which we denote by  $K_\rho G$ . If the factor system  $\rho$  is unitary, i.e.  $\rho(g, h) = 1$  for all  $g, h \in G$ , then  $K * G$  is called a *skew group ring* and is denoted by  $K^\sigma G$ . In the case, when  $\rho = 1$  and  $\sigma = 1$ , then  $K * G$  is the ordinary group ring  $KG$ .

In the present paper we study properties of crossed products  $K * G$  which are generalizations of the notion of a regular ring. For the case when  $K * G$  is a twisted group ring over the algebraically closed field  $K$ , we give necessary and sufficient conditions for  $K * G$  to be an  $n$ -weakly regular ring ( $n \geq 2$ ), a  $\xi^*N$ -ring or a ring without nilpotent elements. Our investigation can be considered as a generalization of certain results of [2, 3, 4, 7, 11, 12] earlier obtained for group rings. Note that we exclude the case when  $K * G$  is a skew group ring, so we do not cite any reference from that topic.

## 2. TWISTED GROUP ALGEBRAS WITHOUT NILPOTENT ELEMENTS

Denote the  $K$ -basis of  $K * G$  by  $U_G = \{u_g \mid g \in G\}$ . The multiplication of  $u_g, u_h \in U_G$  is defined by  $u_g u_h = \rho(g, h) u_{gh}$ , where  $\rho(g, h) \in \rho$  and  $g, h \in G$ . The factor system  $\rho$  of the crossed product  $K * G$  is called *symmetric*, if for all elements  $g, h \in G$  the condition  $gh = hg$  yields  $\rho(g, h) = \rho(h, g)$ . The finite subset  $\text{Supp}(a) = \{g \in G \mid \alpha_g \neq 0\}$  of  $G$  is called the *support* of the element  $a \in K * G$ .

We shall freely use the following.

**Lemma 1.** *Let  $K * G$  be a crossed product and suppose that  $axb = c$  for some  $x, a, b, c \in K * G$ . If  $H$  is the subgroup of  $G$  generated by  $\text{Supp}(a)$ ,  $\text{Supp}(b)$  and  $\text{Supp}(c)$ , then there exists an element  $y \in K * H$ , such that  $ayb = c$ .*

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*Proof.* Indeed, if  $x = y + z$ , then  $ayb + azb = c$ , where  $y = \sum_{h \in H} u_h \alpha_h$  and  $z = \sum_{g \notin H} u_g \beta_g$ . This shows that  $\text{Supp}(azb) \subseteq H$ . Since  $fg h \notin H$  for  $f \in \text{Supp}(a)$ ,  $g \in \text{Supp}(z)$  and  $h \in \text{Supp}(b)$ , we conclude that  $azb = 0$  and  $ayb = c$ , as it was requested.  $\square$

**Corollary 1.** *If  $g \in G$  has infinite order, then  $u_g - 1$  is neither a one-sided zero divisor, nor a one-sided invertible element of the crossed product  $K * G$ .*

*Proof.* In fact, if  $u_g - 1$  is either a one-sided zero divisor, or a one-sided invertible element of  $K * G$ , then by Lemma 1, we may assume that  $u_g - u_1$  is also such an element of  $K * H$ , where  $H = \langle g \rangle$  is an infinite cyclic group. But  $H$  is an ordered group, a contradiction.  $\square$

For twisted group algebras we give a refinement of Corollary 2 and Lemma 2 of [7] (see p.68) which were earlier proved for group rings.

**Theorem 1.** *Let  $K_\rho G$  be a twisted group algebra of a torsion group  $G$  over the algebraically closed field  $K$ . The ring  $K_\rho G$  does not contain nilpotent elements if and only if the following conditions hold:*

- (i)  $G$  is an abelian group;
- (ii) the order of every elements in  $G$  is invertible in  $K$ ;
- (iii) the factor system  $\rho$  is symmetric.

*Proof.* Assume that the conditions (i), (ii) and (iii) hold. Then the twisted group ring  $K_\rho G$  is commutative. If  $x \in K_\rho G$  is a nonzero nilpotent element and  $H = \langle \text{Supp}(x) \rangle$ , we conclude that  $K_\rho H$  is a commutative artinian ring with a nonzero nilpotent element  $x$ . So, by Theorem 2.2 of ([8], p.415), we get a contradiction.

Conversely, let  $K_\rho G$  be a twisted group ring without nilpotent elements. If  $g \in G$  is of order  $n$  and  $u_g^n = u_1 \alpha_g$ , where  $\alpha_g \in U(K)$ , then there exists an element  $\mu_g \in U(K)$  such that  $\mu_g^n = \alpha_g^{-1}$ , because  $K$  is algebraically closed. So for the element  $v_g = u_g \mu_g$  we have  $v_g^n = 1$ . Obviously,

$$x = (v_g - 1)u_h(1 + v_g + v_g^2 + \cdots + v_g^{n-1})$$

is a nilpotent element of  $K_\rho G$  for all  $h \in G$  as far as  $x^2 = 0$ . Thus  $x = 0$ , so we conclude that

$$(1) \quad u_h = v_g u_h v_g^i \quad (0 \leq i \leq n-1).$$

Examining the supports we can deduce that  $h^{-1}gh = g^{-i}$  ( $h \in G$ ). Therefore all cyclic subgroups of  $G$  are normal. This implies that  $G$  is either abelian or hamiltonian. If  $gh = hg$ , then  $i = n-1$  and by (1) it follows that  $u_h v_g = v_g u_h$ , since  $v_g^n = 1$  is the identity element of  $K_\rho G$ . So we conclude that  $\rho(g, h) = \rho(h, g)$ , i.e. the factor system  $\rho$  is symmetric and condition (iii) holds.

If  $\text{char}(K) = p > 0$  and  $G$  contains an element  $g$  of order  $p$ , then

$$(1 + v_g + v_g^2 + \cdots + v_g^{p-1})^p = 0$$

and we get a contradiction. This implies that condition (ii) also follows.

Assume that  $G$  is hamiltonian and  $\langle g, h \mid g^4 = h^4 = 1, g^2 = h^2, g^h = g^{-1} \rangle \cong Q_8$  is the quaternion group of order 8. Then  $h^{-1}gh = g^{-1}$  and  $i = 1$ . Therefore in this case by (1) we have  $u_h = v_g u_h v_g$ , i.e.

$$(2) \quad v_h = v_g v_h v_g,$$

where  $v_h = u_h \mu_h$  and  $v_g^4 = v_h^4 = 1$ . Since  $G$  contains 2-elements, it follows from (ii) that  $\text{char}(K) \neq 2$ .

$K$  being an algebraically closed field, it is clear that there exist nonzero elements  $\alpha, \beta \in K$  for which  $\alpha^2 + \beta^2 = 0$ . Then by (2) it is easy to verify that

$$w = \alpha(v_g^2 v_h - v_h) + \beta(v_g^3 v_h - v_g v_h)$$

is a nonzero nilpotent element of  $K_\rho G$ .

Indeed,  $h \in \text{Supp}(\alpha(v_g^2 v_h - v_h))$ , but  $h \notin \text{Supp}(\beta(v_g^3 v_h - v_g v_h))$ . Thus we have  $w \neq 0$ . Moreover, by (2) we obtain that  $u_h^2 v_g = v_g u_h^2$  and  $u_h v_g^2 = v_g^2 u_h$ . Then  $w^2 = (v_g^2 - 1)^2 (\alpha v_h + \beta v_g v_h)^2$ . Since  $(v_g^2 - 1)^2 = 2(1 - v_g^2)$  and

$$\begin{aligned} (\alpha v_h + \beta v_g v_h)^2 &= (\alpha^2 + \beta^2) v_h^2 + \alpha \beta v_h^2 (v_g^2 + 1) v_g \\ &= \alpha \beta v_h^2 (v_g^2 + 1) v_g, \end{aligned}$$

we obtain  $w^2 = 2(1 - v_g^2) \alpha \beta v_h^2 (1 + v_g^2) v_g = 0$ , which is impossible. Hence condition (i) follows, as requested.  $\square$

### 3. REGULAR CROSSED PRODUCTS

An associative ring  $R$  with unity is called *regular (strongly regular)* if for every  $a \in R$  there is an element  $b \in R$ , such that  $aba = a$  ( $ba^2 = a$ , respectively). A ring  $R$  is called  $\xi^*$ -ring ( $\xi^*N$ -ring) if for every  $a \in R$  there exists  $b \in R$  such that  $aba - a$  is a central (central nilpotent, respectively) element of  $R$ . It is clear that every regular ring is a  $\xi^*N$ -ring and every  $\xi^*N$ -ring is a  $\xi^*$ -ring (see [7, 12]).

By the theorem of Auslander, Connell and Willamayor (see [3], Theorem 3, p.660), it is well known that a group ring is regular if and only if  $K$  is regular,  $G$  is a locally finite group and the order of every element  $g \in G$  is invertible in  $K$ .

Our first result for this section is the following.

**Theorem 2.** *Let  $K * G$  be a crossed product of the group  $G$  over the ring  $K$  such that one of the following conditions is satisfied:*

- (i)  $K * G$  is a  $\xi^*N$ -ring;
- (ii)  $K * G$  is  $n$ -weakly regular.

*Then  $G$  is a torsion group.*

*Proof.* (i) Suppose that  $g \in G$  is an element of infinite order. Then there exists a  $b \in K * G$  and a natural number  $n \geq 1$  such that

$$x = (u_g - 1)b(u_g - 1) - (u_g - 1)$$

is a central element of  $K * G$  and  $x^n = 0$ . If  $n = 1$ , then  $x = 0$  and

$$(u_g - 1)[b(u_g - 1) - 1] = 0.$$

Since, by Corollary 1, the element  $u_g - 1$  is not a left zero divisor in  $K * G$ , we obtain that  $b(u_g - 1) = 1$ , i.e.  $u_g - 1$  is a left invertible element in  $K * G$ , which is also impossible. Therefore  $n > 1$  and

$$x^n = (u_g - 1)[b(u_g - 1) - 1]x^{n-1} = 0.$$

In the same way we obtain that  $z_1 = [b(u_g - 1) - 1]x^{n-1} = 0$ . Suppose that for some  $k \geq 1$  we have  $z_k = [b(u_g - 1) - 1]^k x^{n-k} = 0$ . If  $1 < k < n$ , as far as  $x$  is

central,

$$\begin{aligned} z_k &= x[b(u_g - 1) - 1]^k x^{n-k-1} \\ &= (u_g - 1)[b(u_g - 1) - 1]^{k+1} x^{n-k-1} = 0. \end{aligned}$$

Now applying Corollary 1 we obtain that

$$z_{k+1} = [b(u_g - 1) - 1]^{k+1} x^{n-k-1} = 0.$$

Thus, by induction we conclude that  $z_n = [b(u_g - 1) - 1]^n = 0$ .

The last equality shows that there exists  $z \in K * G$  such that  $z(u_g - 1) = 1$ , which, by Corollary 1, is impossible.

(ii) Suppose that  $g \in G$  is an element of infinite order. Then for some  $b, c \in K * G$  we have  $u_g - 1 = (u_g - 1)b(u_g - 1)^n c$ . By Corollary 1 we have

$$(u_g - 1)[1 - b(u_g - 1)^n c] = 0,$$

we conclude that  $b(u_g - 1)^n c = 1$ . Hence it follows that  $b(u_g - 1)x = 1$ , where  $x = (u_g - 1)^{n-1} c$ . If  $e = xb(u_g - 1)$ , then

$$e^2 = x[b(u_g - 1)x]b(u_g - 1) = xb(u_g - 1) = e,$$

i.e.  $e$  is a central idempotent of  $K * G$ . Thus we have

$$\begin{aligned} 1 &= b(u_g - 1)x = b(u_g - 1)[xb(u_g - 1)]x \\ &= xb(u_g - 1)[b(u_g - 1)x] = xb(u_g - 1), \end{aligned}$$

i.e.  $u_g - 1$  has a left invertible element  $xb \in K * G$ . Now again by Corollary 1 we obtain a contradiction, so the proof is complete.  $\square$

**Corollary 2.** *If the crossed product  $K * G$  is a regular ring, then  $K$  is also a regular ring and  $G$  is a torsion group.*

*Proof.* The claim follows from Theorem 2 and Lemma 1.  $\square$

Observe that the theorem of Auslander, Connell and Willamayor (see [3], Theorem 3, p.660) does not apply for crossed products. Indeed, if  $K$  is a non-perfect field of characteristic  $p > 0$  and  $G$  is the  $p^\infty$ -group, then there exists a twisted group ring  $K_\rho G$ , which must be a field (see [9], Proposition 4.2).

If  $G$  satisfies the maximum condition for finite normal subgroups and the group ring  $KG$  is a  $\xi^*N$ -ring, then  $G$  is locally finite (see [11], Theorem 3, p.16).

We shall prove the local finiteness of  $G$  without the assumption of the maximum condition when  $K$  is a field. First we recall that (see [10], p.308)

$$\Delta(G) = \{g \in G \mid [G : C_G(g)] < \infty\}$$

is a subgroup of  $G$ , where  $C_G(g)$  is the centralizer of  $g$  in  $G$ . Furthermore, we put

$$\Delta^p(G) = \langle g \in \Delta(G) \mid g \text{ is a } p\text{-element} \rangle,$$

that is the subgroup of  $\Delta(G)$  which is generated by all  $p$ -elements of  $\Delta(G)$ .

Now we are ready to prove the following.

**Theorem 3.** *Let  $KG$  be the group algebra of a group  $G$  over a field  $K$ . If  $KG$  is a  $\xi^*N$ -ring, then  $G$  is a locally finite group. Moreover, if  $\text{char}(K) = p > 0$  then  $\Delta^p(G)$  contains all  $p$ -elements of  $G$ .*

*Proof.* Let  $\mathfrak{N}(KG)$  be the union of all nilpotent ideals of  $KG$ . In particular, the central nilpotent elements of  $KG$  are in  $\mathfrak{N}(KG)$  and, consequently,  $KG/\mathfrak{N}(KG)$  is a regular ring.

Assume  $\text{char}(K) = p > 0$ . By Theorem 8.19 ([10], p.309),

$$\mathfrak{N}(KG) = \mathfrak{Rad}(K[\Delta^p(G)])KG,$$

where  $\mathfrak{Rad}(K[\Delta^p(G)])$  is the Jacobson radical of the group ring  $K[\Delta^p(G)]$ . Obviously, the augmentation ideal  $\omega(K[\Delta^p(G)])$  is a maximal ideal of  $K[\Delta^p(G)]$ , so

$$\mathfrak{N}(KG) = \mathfrak{Rad}(K[\Delta^p(G)])KG \subseteq \omega(K[\Delta^p(G)])KG.$$

It is well-known (see [3], Theorem 3, p.660) that

$$K[G/(\Delta^p(G))] \cong KG/\omega(K[\Delta^p(G)])KG$$

and therefore the group algebra  $K[G/\Delta^p(G)]$  is regular, as a homomorphic image of  $KG/\mathfrak{N}(KG)$ . This implies, by the theorem of Auslander, Connell and Willamayor (see [3], Theorem 3, p.660), that  $G/\Delta^p(G)$  is locally finite and has no  $p$ -element. Thus we obtain that  $\Delta^p(G)$  contains all the  $p$ -elements of  $G$  and the group  $G$  is locally finite (see [5], Theorem 23.1.1, p.215).

If  $\text{char}(K) = 0$ , then  $\mathfrak{N}(KG) = 0$  and  $KG$  is regular. According to Auslander-Connell-Villamayor's theorem the proof is complete.  $\square$

#### 4. $n$ -WEAKLY REGULAR TWISTED GROUP ALGEBRAS

Let  $n \geq 2$  be a fixed natural number. A ring  $R$  is called  *$n$ -weakly regular* [4] if for every  $a \in R$  there exist elements  $b, c \in R$  such that  $a = aba^n c$ .

Obviously, an  $n$ -weakly regular ring  $R$  has no nonzero nilpotent element. Indeed, if  $R$  contains a nonzero nilpotent element, then there exists a nonzero nilpotent element  $a \in R$  with  $a^2 = 0$ . Hence  $a = aba^n c = 0$ , which is impossible. From this fact we can conclude that all idempotents of an  $n$ -weakly regular ring are central.

In [2] (Theorem 2, p.119) it was proved that the group algebra  $KG$  over a field  $K$  is  $n$ -weakly regular ( $n \geq 2$ ) if and only if  $K$  and  $G$  satisfy at least one of the following two conditions:

- (i)  $\text{char}(K) = p > 0$  and  $G$  is an abelian torsion group without  $p$ -elements;
- (ii)  $\text{char}(K) = 0$  and  $G$  is either an abelian torsion group or a hamiltonian  $\text{gr}G = Q \times E \times A$ , where  $A$  is an abelian torsion group without 2-elements and the equation  $x^2 + y^2 + z^2 = 0$  in  $KA$  has only the trivial solution.

In the case when  $K$  is an algebraically close field, this result can be extended to.

**Theorem 4.** *A twisted group algebra  $K_\rho G$  of a group  $G$  over the algebraically closed field  $K$  is  $n$ -weakly regular ( $n \geq 2$ ) if and only if the following conditions hold:*

- (i)  $G$  is an abelian torsion group;
- (ii) the order of every element of  $G$  is invertible in  $K$ ;
- (iii) the factor system  $\rho$  is symmetric.

*Proof.* Suppose that  $K_\rho G$  is  $n$ -weakly regular. Then conditions (i), (ii) and (iii) hold by Theorems 1 and 2.

Conversely, if  $K$  and  $G$  satisfy the conditions (i), (ii) and (iii), then  $K_\rho G$  is a commutative ring. Let  $a \in K_\rho G$  be an arbitrary element. Then  $a \in K_\rho H$ , where  $H = \langle \text{Supp}(a) \rangle$  is a finite abelian group. Since  $K_\rho H$  is a commutative semisimple artinian ring ([8], Theorem 2.2), we conclude that  $K_\rho H$  is a direct product of

fields, so  $K_\rho H$  is  $n$ -weakly regular. This implies that  $K_\rho G$  is  $n$ -weakly regular, as requested.  $\square$

Analyzing the result of [2] (see Theorem 2, p.119) on  $n$ -weakly regular group rings and [7] (see Corollary 2, p.70) about strongly regular group rings we deduce that when  $K$  is a field, then these two classes coincide.

In the case of twisted group algebras over an algebraically closed basic field we have the following.

**Corollary 3.** *Let  $K_\rho G$  be a twisted group algebra of a group  $G$  over an algebraically closed field  $K$ . The following statements are equivalent:*

- (i)  $K_\rho G$  is strongly regular;
- (ii)  $K_\rho G$  is  $n$ -weakly regular for every natural number  $n \geq 2$ ;
- (iii)  $K_\rho G$  is  $n$ -weakly regular for some natural number  $n \geq 2$ ;
- (iv)  $G$  is an abelian torsion group, the order of every element of  $G$  is invertible in  $K$  and the factor system  $\rho$  is symmetric.

*Proof.* Suppose that  $K_\rho G$  is a strongly regular ring. If  $a \in K_\rho G$  and  $a = a^2b$ , then  $a = aba$ , because  $K_\rho G$  does not contain nilpotent elements. Now by induction it follows that  $a = ab^n c$  for some  $c \in K_\rho G$  and for every natural number  $n \geq 1$ . So (i) implies (ii) and, obviously, (ii) implies (iii). By the preceding theorem, (iii) implies (iv). Finally, by the Auslander-Connell-Villamayor theorem and by (iv) it follows that  $K_\rho G$  is a commutative von Neumann ring and so (iv) implies (i).  $\square$

## 5. $\xi N$ -TWISTED GROUP ALGEBRAS

A ring  $R$  is called a  $\xi N$ -ring if for any  $a \in R$  there exists  $b \in R$  such that  $a^2b - a$  is a central nilpotent element of  $R$  (see [11]).

Obviously, every  $\xi N$ -ring is a  $\xi$ -ring and, therefore, (see [6], Theorem 1, p.714) we deduce that every  $\xi N$ -ring is a  $\xi^* N$ -ring. Moreover, (see [6], Lemma 2, p.715) it follows that in  $\xi N$ -rings all nilpotent elements are central.

$\xi N$ -group rings over commutative rings are described in [11] (Theorem 2, p.15). From this description, it follows that a group ring  $KG$  over a field  $K$  of characteristic  $p > 0$  is a  $\xi N$ -ring if and only if  $G$  is an abelian torsion group.

Finally we prove the following.

**Theorem 5.** *A twisted group algebra  $K_\rho G$  of a group  $G$  over the algebraically closed field  $K$  is a  $\xi N$ -ring if and only if the following conditions hold:*

- (i)  $G$  is an abelian torsion group;
- (ii) the factor system  $\rho$  is symmetric.

*Proof.* Let  $K_\rho G$  be a  $\xi N$ -ring. Then ([6], Theorem 1, p.714) the ring  $K_\rho G$  is a  $\xi^* N$ -ring and, in view of Theorem 2, we conclude that  $G$  is a torsion group. As far as  $K$  is an algebraically closed field, for every element  $g \in G$  of order  $n$  there exists an  $\mu_g \in U(K)$ , such that  $v_g = u_g \mu_g$  ( $u_g \in U_G$ ) and  $v_g^n = 1$ . Then we put

$$z = (v_g - 1)v_h(1 + v_g + v_g^2 + \cdots + v_g^{n-1}), \quad (h \in H).$$

Clearly,  $z^2 = 0$  and therefore  $z$  is a central element of  $K_\rho G$ . Thus  $zv_h = v_h z$  and, so we obtain the equality

$$(3) \quad \begin{aligned} 2v_h v_g v_h + \sum_{i=1}^{n-1} v_g^i v_h v_g v_h + \sum_{i=2}^{n-1} v_h v_g^i v_h \\ = \sum_{i=1}^{n-1} v_g^i v_h^2 + \sum_{i=0}^{n-1} v_h v_g^i v_h v_g. \end{aligned}$$

If  $\text{char}(K) = 2$ , then  $2v_h v_g v_h = 0$ . Consequently for the product  $v_h v_g^2 v_h$  and for the corresponding supports we obtain the following three cases:

- (A1)  $v_h v_g^2 v_h = v_g^i v_h v_g v_h$ ,  $hg^2 h = g^i h g h$  and  $h g h^{-1} = g^i$  ( $1 \leq i \leq n-1$ );
- (A2)  $v_h v_g^2 v_h = v_h v_g^i v_h v_g$ ,  $hg^2 h = h g^i h g$  and  $h g h^{-1} = g^{2-i}$  ( $1 \leq i \leq n-1$ );
- (A3)  $v_h v_g^2 v_h = v_g^i v_h$ ,  $hg^2 h = g^i h^2$  and  $h g^2 h^{-1} = g^i$  ( $1 \leq i \leq n-1$ ).

This shows that  $\langle g^2 \rangle$  is a normal cyclic subgroup of  $G$ .

If  $g$  is a 2-element of  $G$ , then  $1 + v_g$  is nilpotent and by Lemma 2 of [6] we deduce that  $1 + v_g$  is a central element of  $K_\rho G$ . Therefore  $v_g v_h = v_h v_g$  for every  $h \in G$ .

If  $g$  is an element of odd order, then  $\langle g^2 \rangle = \langle g \rangle$  and from (A1), (A2) and (A3) we obtain that every cyclic subgroup of  $G$  is normal, i.e.  $G$  is either abelian, or hamiltonian. Since the 2-elements of  $G$  are central, we conclude that  $G$  is an abelian torsion group, i.e. condition (i) holds. Now by (A1) and (A2) it follows that  $i = 1$  and  $v_g v_h = v_h v_g$ . In case (A3) we have  $i = 2$  and  $v_h v_g^2 = v_g^2 v_h$ . But  $\langle v_g^2 \rangle = \langle v_h \rangle$ , so  $v_h$  commutes with  $v_g^i$  for all  $i = 1, \dots, n-1$ . Therefore condition (ii) also holds.

Now, suppose that  $\text{char}(K) \neq 2$ . Then by (3), we conclude that for the product  $v_h v_g v_h$  we have the following four cases:

- (B1)  $v_h v_g v_h = v_g^i v_h^2$ ,  $h g h = g^i h^2$  and  $h g h^{-1} = g^i$  ( $1 \leq i \leq n-1$ );
- (B2)  $v_h v_g v_h = v_h v_g^i v_h v_g$ ,  $h g h = h g^i h g$  and  $h g h^{-1} = g^{1-i}$  ( $0 \leq i \leq n-1$ );
- (B3)  $v_h v_g v_h = -v_h v_g^i v_h$ ,  $h g h = h g^i h$  and  $g^{i-1} = 1$ , which is impossible, because  $2 \leq i \leq n-1$  and  $g$  is of order  $n$ ;
- (B4)  $v_h v_g v_h = -v_g^i v_h v_g v_h$ ,  $h g h = g^i h g h$  and  $g^i = 1$ , which is impossible, because  $1 \leq i \leq n-1$ .

Therefore  $\langle g \rangle$  is a normal cyclic subgroup of  $G$  for every  $g \in G$ . Hence  $G$  is either abelian or a hamiltonian group.

Assume that  $G$  is hamiltonian and  $\langle g, h \mid g^4 = 1, h^2 = g^2, h g h^{-1} = g^{-1} \rangle \cong Q_8$ . Then by (B1) and (B2), it follows that either  $i = 3$  or  $i = 2$ , respectively. Hence we obtain that  $v_h v_g = v_g^3 v_h$ , where  $v_g^4 = v_h^4 = 1$ .

Let  $(\alpha, \beta)$  be a nontrivial solution of the equation  $x^2 + y^2 = 0$  in  $K$ . Then as in the proof of Theorem 1 we establish that

$$w = \alpha(v_g^2 v_h - v_h) + \beta(v_g^3 v_h - v_g v_h)$$

is a nonzero nilpotent element of  $K_\rho G$  with  $z^2 = 0$ . Therefore  $w$  is a central element of  $K_\rho G$ . But  $w v_h \neq v_h w$ , so we obtain a contradiction. Thus  $G$  is abelian and condition (i) holds. If  $g h = h g$ , then by (B1) and (B2) it follows that either  $i = 1$  or  $i = 2$ , respectively. Hence we obtain that  $v_h v_g = v_g v_h$  for all  $g, h \in G$  and so condition (ii) also follows.

Conversely, if the conditions (i) and (ii) hold, then  $K_\rho G$  is a commutative ring. For every element  $a \in K_\rho G$  with  $H = \langle \text{Supp}(a) \rangle$ , the ring  $K_\rho H$  is artinian and

$R \cong K_\rho H / \mathfrak{Nil}(K_\rho H)$  is a finite sum of fields. Therefore  $R$  is strongly regular and hence  $K_\rho H$  is a  $\xi N$ -ring. Since  $a \in K_\rho H$ , we deduce that  $K_\rho G$  is a  $\xi N$ -ring.

Note that if  $K_\rho G$  is a  $\xi N$ -ring, then the periodicity of  $G$  can be proved directly. Indeed, if  $g \in G$  is an element of infinite order and  $z = (u_g - 1)^2 x - (u_g - 1)$  is a central nilpotent element of  $K_\rho G$ , then  $z^n = 0$  for some  $n \geq 1$ . By Corollary 1 we deduce that  $[(u_g - 1)x - 1]z^{n-1} = 0$ .

Using the fact that  $z$  is central, we can prove by induction that

$$[(u_g - 1)x - 1]^k z^{n-k} = 0$$

for every  $k \geq 1$ . Therefore  $[(u_g - 1)x - 1]^n = 0$ . This equality shows that  $u_g - 1$  is right invertible in  $K_\rho G$ , which again is impossible by Corollary 1.  $\square$

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