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# FEKETE-SZEGÖ FUNCTIONAL FOR NON-BAZILEVIČ FUNCTIONS

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ABSTRACT. Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be an analytic function in the unit disk $\mathcal{U}$ and let the class of non-Bazilevič functions, for $0 < \lambda < 1$, be described with $\operatorname{Re}\left\{f'(z)\,(z/f(z))^{1+\lambda}\right\} > 0$, $z \in \mathcal{U}$. In this paper we obtain sharp upper bound of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for the class of non-Bazilevič functions and for some of its subclasses.

## 1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ normalized such that $f(0) = f'(0) - 1 = 0$, i.e., of type $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$.

A function $f \in \mathcal{A}$ is said to be of *Bazilevič type* if for a starlike function $g$ ($g \in \mathcal{A}$ is starlike if and only if $\operatorname{Re}\{zg'(z)/g(z)\} > 0$, $z \in \mathcal{U}$) we have

$$\operatorname{Re}\left\{f'(z)(f(z)/z)^{\alpha+i\gamma-1}(g(z)/z)^{-\alpha}\right\} > 0,$$

$z \in \mathcal{U}$ (see more in [1]). This class and its subclasses were widely studied in the past decades. Specially, in [4] sharp upper bound of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ is obtained for all real $\mu$ when $\gamma = 0$. That result was partially extended in [2] to a wider subclass satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{f^{1-\alpha}(z)g^{\alpha}(z)}\right\} > \beta, \quad z \in \mathcal{U},$$

where $\alpha > 0$ and $0 \le \beta < 1$.

In [5], Obradović introduced a class of functions $f \in \mathcal{A}$ that for $0 < \lambda < 1$ is defined by

$$\operatorname{Re}\left\{f'(z)(z/f(z))^{1+\lambda}\right\} > 0, \quad z \in \mathcal{U}.$$

Recently, in his talk at the Conference 'Computational Methods and Function Theory 2001', he called this functions to be of *non-Bazilevič type*. By now, this class was studied in a direction of finding necessary conditions over $\lambda$ that embeds this class into the class of univalent function or its subclasses, which is still an open problem. Here we will find sharp upper bound of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for the class of non-Bazilevič functions and for some its subclasses. In that purpose we will need the following lemma.

**Lemma 1.** ([6], p.166, formula (10)) ([3], p.41) *Let $p \in \mathcal{P}$, that is, $p$ be analytic in $\mathcal{U}$, be given by $p(z) = 1 + \sum_{n=1}^{\infty} p_nz^n$ and $\operatorname{Re} p(z) > 0$ for $z \in \mathcal{U}$. Then*

$$\left|p_2 - p_1^2/2\right| \le 2 - |p_1|^2/2$$

*and $|p_n| \le 2$ for all $n \in \mathbb{N}$.*

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## 2. Main results

**Theorem 1.** *Let $f \in \mathcal{A}$, $0 < \lambda < 1$ and $0 \le \alpha < 1$. If*

$$\operatorname{Re}\left\{f'(z)(z/f(z))^{1+\lambda}\right\} > \alpha, \quad z \in \mathcal{U}, \tag{1}$$

*then $|a_2| \le 2(1-\alpha)/(1-\lambda)$ and for all $\mu \in \mathbb{C}$ the following bound is sharp*

$$|a_3 - \mu a_2^2| \le \frac{2(1-\alpha)}{2-\lambda} \max\left\{1, \left|1 + \frac{2(1+\lambda-\mu)(2-\lambda)(1-\alpha)}{(1-\lambda)^2}\right|\right\}.$$

*Proof.* Condition (1) is equivalent to

$$f'(z) = (f(z)/z)^{1+\lambda}[(1-\alpha)p(z) + \alpha], \quad z \in \mathcal{U},$$

for some $p \in \mathcal{P}$. Equating coefficients we obtain $a_2 = p_1(1-\alpha)/(1-\lambda)$,

$$a_3 = \frac{1-\alpha}{2-\lambda} p_2 + \frac{(1-\alpha)^2(1+\lambda)}{2(1-\lambda)^2} p_1^2$$

and further

$$a_3 - \mu a_2^2 = \frac{1-\alpha}{2-\lambda}\left(p_2 - \frac{1}{2}p_1^2\right) + \\ + \frac{(1-\alpha)(1-\lambda)^2 + (1-\alpha)^2(1+\lambda-2\mu)(2-\lambda)}{2(2-\lambda)(1-\lambda)^2} p_1^2.$$

Now, using Lemma 1 we receive $|a_3 - \mu a_2^2| \le H(x) = A + ABx^2/4$ where $x = |p_1| \le 2$, $A = 2(1-\alpha)/(2-\lambda) > 0$, $B = \left(|C| - (1-\lambda)^2\right)/(1-\lambda)^2$ and $C = (1-\lambda)^2 + (1-\alpha)(1+\lambda-2\mu)(2-\lambda)$. So, we have

$$|a_3 - \mu a_2^2| \le \begin{cases} H(0) = A, & |C| \le (1-\lambda)^2 \\ H(2) = A|C|/(1-\lambda)^2, & |C| \ge (1-\lambda)^2 \end{cases}.$$

Equality is attained for functions given by

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\lambda} = \frac{1 + z^2(1-2\alpha)}{1-z^2}$$

and

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\lambda} = \frac{1 + z(1-2\alpha)}{1-z}$$

respectively. □

For $\alpha = 0$ we have the following corollary.

**Corollary 1.** *Let $f \in \mathcal{A}$ and $0 < \lambda < 1$. If*

$$\operatorname{Re}\left\{f'(z)(z/f(z))^{1+\lambda}\right\} > 0, \quad z \in \mathcal{U},$$

*then $|a_2| \le 2/(1-\lambda)$ and for all $\mu \in \mathbb{C}$ the following bound is sharp*

$$|a_3 - \mu a_2^2| \le \frac{2}{2-\lambda} \max\left\{1, \left|1 + \frac{(1+\lambda-2\mu)(2-\lambda)}{(1-\lambda)^2}\right|\right\}.$$

Now we will consider one subclass of the class of non-Bazilevič function.

**Theorem 2.** *Let $f \in \mathcal{A}$, $0 < \lambda < 1$ and $0 < k \le 1$. If*

$$\left|f'(z)(z/f(z))^{1+\lambda} - 1\right| < k, \quad z \in \mathcal{U}, \tag{2}$$

*then $|a_2| \le k/(1-\lambda)$ and for all $\mu \in \mathbb{C}$ the following bound is sharp*

$$|a_3 - \mu a_2^2| \le \frac{k}{(2-\lambda)} \max\left\{1, \frac{k(2-\lambda)}{(1-\lambda)^2}\left|\frac{1+\lambda}{2} - \mu\right|\right\}.$$

*Proof.* Similarly as in the proof of Theorem 1, condition (2) implies that there exists a function $p \in \mathcal{P}$ such that for all $z \in \mathcal{U}$

$$f'(z) = (f(z)/z)^{1+\lambda}(2k/(1+p(z)) + 1 - k).$$

Equating the coefficients we obtain $a_2 = -kp_1/(2(1-\lambda))$,

$$a_3 = \frac{k^2}{8}\frac{1+\lambda}{(1-\lambda)^2}p_1^2 - \frac{k}{2(2-\lambda)}\left(p_2 - \frac{p_1^2}{2}\right)$$

and

$$a_3 - \mu a_2^2 = -\frac{k}{2(2-\lambda)}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{k^2 p_1^2}{4(1-\lambda)^2}\left(\frac{1+\lambda}{2} - \mu\right).$$

So, $|a_3 - \mu a_2^2| \le H(x) = A + Bx^2/4$ where $x = |p_1| \le 2$, $A = k/(2-\lambda) > 0$, $B = k^2|C|/(1-\lambda)^2 - k/(2-\lambda)$ and $C = (1+\lambda)/2 - \mu$. Therefore

$$|a_3 - \mu a_2^2| \le \begin{cases} H(0) = A, & |C| \le (1-\lambda)^2/(k(2-\lambda)) \\ H(2) = Ak(2-\lambda)|C|/(1-\lambda)^2, & |C| \ge (1-\lambda)^2/(k(2-\lambda)) \end{cases}.$$

Here equality is attained for the functions given by $f'(z)(z/f(z))^{1+\lambda} = 1 - kz^2$ and $f'(z)\left(z/f(z)\right)^{1+\lambda} = 1 - kz$, respectively. □

For $k = 1$ we receive the following corollary.

**Corollary 2.** *Let $f \in \mathcal{A}$ and $0 < \lambda < 1$. If*

$$\left|f'(z)(z/f(z))^{1+\lambda} - 1\right| < 1, \quad z \in \mathcal{U},$$

*then $|a_2| \le 1/(1-\lambda)$ and for all $\mu \in \mathbb{C}$ the following bound is sharp*

$$|a_3 - \mu a_2^2| \le \frac{1}{(2-\lambda)^2}\max\left\{1, \frac{2-\lambda}{(1-\lambda)^2}\left|\frac{1+\lambda}{2} - \mu\right|\right\}.$$

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