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SOME OBSERVATIONS CONCERNING THE NON-EXISTENCE OF BOUNDED DIFFERENTIALS ON WEIGHTED l^1 ALGEBRAS

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Abstract. We consider Banach algebras of weighted absolutely summing complex sequences, endowed with Cauchy type product. In this frame it is proved that the only bounded differentials are zero.

1. Introduction

Differential operators have been extensively studied in the frame of spaces of formal series in several variables [2]. If D is the usual derivative or any *degree reducing operator* acting on a space of formal series, the problem of characterization of classes of operators commuting with D has been treated by several authors [1], [8], [10]. Similar questions in the context of Banach and Frechet spaces have been studied mainly by Grabiner and, more recently, by J. Prada (cf. [3], [9]). A very wide generalization of *integro - differentiation processes* is attained by means of *Hadamard products* of suitable analytic functions (cf. [4], pp. 428–431). For researches in this direction see also [6], [5], [7]. Considering Banach algebras of analytic functions, a matter of special interest is the problem of existence and, if it is possible, determination and characterization of bounded differentiations. In particular, if $\mathbb{D}$ denotes the open unit disk centered at zero in the complex plane we are concerned for the algebra $\mathcal{A}(\bar{\mathbb{D}})$ of continuous functions on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$, with the usual $\|\circ\|_\infty$ - norm and the convolution product

$$(f * g)(z) = z \int_0^1 f(tz)\ g((1-t)z)\ dt,\ \ f,\ g\ \in \mathcal{A}(\bar{\mathbb{D}}),\ z \in \bar{\mathbb{D}}.$$

More generally, by identifying each element of $\mathcal{A}(\bar{\mathbb{D}})$ with its uniquely determined sequence of Taylor coefficients this problem can be considered in the context of Banach weighted algebras. So, let $\alpha = (\alpha_n)_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers such that the inequalities $\alpha_{n+m} \leq \alpha_n\ \alpha_m$ hold for $n,\ m \in \mathbb{N}_0$. For instance, some examples of such sequences are:

(1) $\{[(n+1)\ \pi]\}_{n\in\mathbb{N}_0}$, where $[x]$ is the usual greatest integral function of a real number x.
(2) $\{t\ (n+1)\}_{n\in\mathbb{N}_0}$ if $t \geq 2$.
(3) $\{(n+1)^s\}_{n\in\mathbb{N}_0}$ or $\{s^n\}_{n\in\mathbb{N}_0}$ if $s > 1$.
(4) $\{4n\,(1+\varkappa_F(n))\}_{n\in\mathbb{N}_0}$, where F is a non empty subset of $\mathbb{N}_0$.

It will be considered the space $l^1(\alpha)$ of formal complex series $a = \sum_{n=0}^\infty a_n\ \mathbf{x}^n$ such that $\sum_{n=0}^\infty |a_n|\ \ \alpha_n < \infty$. Also, perhaps, it should be useful to think of the

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elements of $\mathrm{l}^1(\alpha)$ as certain complex functions on $\mathbb{N}_0$. Such vector space, endowed with the norm $\|a\|_{\mathrm{l}^1(\alpha)} = \sum_{n=0}^{\infty} |a_n| \ \alpha_n$ defined for $a \in \mathrm{l}^1(\alpha)$, becomes a Banach space. Indeed, the Cauchy product given by

$$a * b = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_{n-m} \ b_m \right) \ \mathbf{x}^n, \ a, \ b \ \in \mathrm{l}^1(\alpha),$$

induces an Abelian Banach algebra structure with unity $\mathbf{1}$. If $\alpha^{-1} = \left(\alpha_n^{-1}\right)_{n\in\mathbb{N}_0}$ it is well known that $\mathrm{l}^1(\alpha)$ is isometrically isomorphic to $c_0\left(\alpha^{-1}\right)^*$, where $c_0\left(\alpha^{-1}\right)$ is the Banach space of complex sequences $\eta = (\eta_n)_{n\in\mathbb{N}_0}$ so that

$$\lim_{n\to\infty} \eta_n/\alpha_n = 0$$

with the norm $\|\eta\|_{c_0(\alpha^{-1})} = \sup_{n\in\mathbb{N}_0} |\eta_n| / \alpha_n$. Of course, a linear operator Δ on $\mathrm{l}^1(\alpha)$ is called a *differentiation* if the Leibnitz rule

$$\Delta(a * b) = \Delta(a) * b + a * \Delta(b)$$

holds for all $a, \ b \in \mathrm{l}^1(\alpha)$. Since 1955 it is known the image of any bounded derivation on a commutative Banach algebra is contained in the radical (cf. [11]). Therefore, the only bounded derivation on a commutative semi - simple Banach algebra is zero. The commutative Banach algebras $\mathrm{l}^1(\alpha)$ are not semi - simple. Indeed, if $\alpha_n^{1/n} \to 0$ then rad $\left[\mathrm{l}^1(\alpha)\right]$ consists of those elements $a \in \mathrm{l}^1(\alpha)$ such that $a_0 = 0$. In this article it is proved that the set $\mathcal{BD}\left(\mathrm{l}^1(\alpha)\right)$ of bounded differentiations is trivial. To this end, in a first place, necessary and sufficient conditions of boundedness will be stated in Theorem 2. Later, in Proposition 5, it will be proved that non triviality of $\mathcal{BD}\left(\mathrm{l}^1(\alpha)\right)$ would impose some growth condition on α. Finally, our main result follows from Corollary 7, where it is shown that the above growth condition cannot hold by the required intrinsic properties of the sequence of weights linked to the algebra structure of $\mathrm{l}^1(\alpha)$.

2. $\mathcal{BD}\left(\mathrm{l}^1(\alpha)\right)$ IS TRIVIAL

Lemma 1. *If* $r \in \mathbb{N}$ *and* $a_0, \ldots, \ a_r, b_0, \ldots, b_r, c_0, \ldots, c_r$ *are complex numbers then*

$$\sum_{h=1}^{r+1} h \ a_{r-h+1} \sum_{k=0}^{h} b_{h-k} \ c_k = \sum_{h=0}^{r} \sum_{k=1}^{h+1} k \ a_{h-k+1} \left(b_{r-h} \ c_k + c_{r-h} \ b_k\right).$$

Proof. For $0 \le k \le r$ we ´ll set $b'_k = k \ b_k$ and $c'_k = k \ c_k$. Then

$$\sum_{h=1}^{r+1} h \ a_{r-h+1} \sum_{k=0}^{h} b_{h-k} \ c_k = \sum_{h=1}^{r+1} a_{r-h+1} \sum_{k=0}^{h} (b'_{h-k} \ c_k + b_{h-k} \ c'_k). \tag{1}$$

In particular,

$$\sum_{h=1}^{r+1} a_{r-h+1} \sum_{k=0}^{h} b'_{h-k} \ c_k = \sum_{h=1}^{r+1} a_{r-h+1} \sum_{k=0}^{h-1} b'_{h-k} \ c_k$$

$$= \sum_{k=0}^{r} c_k \sum_{h=k+1}^{r+1} a_{r-h+1} \ b'_{h-k} = \sum_{k=0}^{r} c_{r-k} \sum_{h=1}^{k+1} a_{k-h+1} \ b'_h.$$

On operating analogously with the other sum in (1) it is obtained

$$\sum_{h=1}^{r+1} h\ a_{r-h+1} \sum_{k=0}^{h} b_{h-k}\ c_k = \sum_{h=0}^{r} \left(b_{r-h} \sum_{k=1}^{h+1} a_{h-k+1}\ c'_k + c_{r-h} \sum_{k=1}^{h+1} a_{h-k+1}\ b'_k \right)$$

$$= \sum_{h=0}^{r} \sum_{k=1}^{h+1} k\ a_{h-k+1} \left(b_{r-h}\ c_k + c_{r-h}\ b_k\right)$$

and the claim follows. □

Theorem 2. *In order that a linear map* Δ *on* $l^1(\alpha)$ *be a bounded differentiation it is necessary and sufficient that the extended number*

$$\mathfrak{h}_\alpha(\Delta) = \sup_{n\in\mathbb{N}} \frac{n}{\alpha_n} \sum_{k=0}^{\infty} |\Delta\mathbf{x}(k)|\ \alpha_{k+n-1}$$

be finite. In this case, $\|\Delta\| = \mathfrak{h}_\alpha(\Delta)$ *and*

$$\Delta a = \sum_{n=1}^{\infty} (n\ a_n) \cdot \left(\mathbf{x}^{n-1} * \Delta\mathbf{x}\right),\ a \in \mathrm{l}^1(\alpha). \tag{2}$$

Proof. By Leibnitz rule $\Delta\mathbf{1} = \boldsymbol{\Delta}(\mathbf{1} * \mathbf{1}) = 2 \cdot \Delta\mathbf{1}$ and so $\Delta\mathbf{1} = 0$. Moreover, for all non negative exponents is $\mathbf{x}^n * \mathbf{x}^m = \mathbf{x}^{n+m}$ and by a recurrence application of Leibnitz rule we get $\Delta\mathbf{x}^n = n \cdot \left(\mathbf{x}^{n-1} * \Delta\mathbf{x}\right)$, $n \in \mathbb{N}$. If $a \in \mathrm{l}^1(\alpha)$ then $\lim_{m\to\infty} \sum_{n=0}^{m} a_n\ \mathbf{x}^n = a$ because

$$\left\| a - \sum_{n=0}^{m} a_n\ \mathbf{x}^n \right\|_{\mathrm{l}^1(\alpha)} = \sum_{n>m} |a_n|\ \alpha_n \to 0 \text{ as } m \to \infty.$$

Therefore, if Δ is bounded (2) holds. Now, if $b \in \mathrm{l}^1(\alpha)$ and $n \in \mathbb{N}$ then

$$\mathbf{x}^{n-1} * b = \sum_{m=n-1}^{\infty} b_{m-n+1}\ \mathbf{x}^m,$$

i.e.

$$\alpha_n\ \|\Delta\| \geq \|\Delta\mathbf{x}^n\|_{\mathrm{l}^1(\alpha)} = n\ \left\|\mathbf{x}^{n-1} * \Delta\mathbf{x}\right\|_{\mathrm{l}^1(\alpha)} = n \sum_{m=n-1}^{\infty} \left|(\Delta\mathbf{x})_{m-n+1}\right|\ \alpha_m. \tag{3}$$

Since n is arbitrary $\mathfrak{h}_\alpha(\Delta) \leq \|\Delta\| < \infty$ and the condition is necessary. On the other hand, if $\mathfrak{h}_\alpha(\Delta)$ is finite, $a \in \mathrm{l}^1(\alpha)$ and $m,\ p \in \mathbb{N}_0$ then

$$\left\| \sum_{n=m}^{m+p} (n\ a_n) \cdot \left(\mathbf{x}^{n-1} * \Delta\mathbf{x}\right) \right\|_{\mathrm{l}^1(\alpha)} \leq \sum_{n=m}^{m+p} n\ |a_n|\ \left\|\mathbf{x}^{n-1} * \Delta\mathbf{x}\right\|_{\mathrm{l}^1(\alpha)}$$

$$\leq \mathfrak{h}_\alpha(\Delta) \sum_{n=m}^{m+p} |a_n|\ \alpha_n.$$

By completeness we deduce that (2) is a well defined element of $\mathrm{l}^1(\alpha)$. Indeed, Δ is obviously linear and since

$$\|\Delta a\|_{\mathrm{l}^1(\alpha)} \leq \sum_{n=1}^{\infty} n\ |a_n|\ \left\|\mathbf{x}^{n-1} * \Delta\mathbf{x}\right\|_{\mathrm{l}^1(\alpha)} \leq \mathfrak{h}_\alpha(\Delta)\ \|a\|_{\mathrm{l}^1(\alpha)} \tag{4}$$

it is also bounded. Now, if $c \in \mathrm{l}^1(\alpha)$ and $c = \sum_{n=0}^{\infty} c_n \ \mathbf{x}^n$ for each $m \in \mathbb{N}_0$ is $|c_m| \leq \|c\|_{\ \mathrm{l}^1(\alpha)} / \alpha_m$, i.e. the natural projections $c \to c_m$ are bounded forms. Hence,

$$(\Delta c)_m = \sum_{n=1}^{\infty} n \ c_n \ \left(\mathbf{x}^{n-1} * \Delta \mathbf{x}\right)_m = \sum_{n=1}^{m+1} n \ c_n \ (\Delta \mathbf{x})_{m-n+1}\,, \ m \in \mathbb{N}_0. \tag{5}$$

In consequence of (5) and Lemma1 Leibnitz rule holds for Δ. Finally, by (4) is $\|\Delta\| \leq \mathfrak{h}_\alpha(\Delta)$, the opposite inequality being true by (3). □

Corollary 3. *If Δ is a bounded differentiation then $(\Delta \mathbf{x})_0 = (\Delta \mathbf{x})_1 = 0$.*

Remark 4. We observe that Corollary 3 follows in any weighted space $\mathrm{l}^1(\alpha)$. As expected, the usual differentiation D such that $D\mathbf{x} = \mathbf{1}$ is always not bounded.

Poposition 5. *If $\mathcal{BD}\left(\mathrm{l}^1(\alpha)\right)$ is not trivial the sequence $\{n/\alpha_n\}_{n \in \mathbb{N}_0}$ is bounded.*

Proof. For $a \in \mathrm{l}^1(\alpha)$ the relation

$$\Psi_a \eta = \left\{\sum_{n=m}^{\infty} \eta_n \ \overline{a}_{n-m}\right\}_{m \in \mathbb{N}_0}, \ \eta \in c_0\left(\alpha^{-1}\right),$$

defines an element $\Psi_a \in \mathcal{B}\left(c_0\left(\alpha^{-1}\right)\right)$ whose adjoint is $\Psi_a^*(b) = a * b, \ b \in \mathrm{l}^1(\alpha)$. Therefore, if Δ is a non zero bounded differentiation and n is a positive integer we have

$$\|\Delta \mathbf{x}^n\|_{\mathrm{l}^1(\alpha)} = n \sup_{\eta : \|\eta\|_{c_0(\alpha^{-1})} = 1} \left|\left\langle \eta, \mathbf{x}^{n-1} * \Delta \mathbf{x}\right\rangle\right| = n \sup_{\eta : \|\eta\|_{c_0(\alpha^{-1})} = 1} \left|\left\langle \Psi_{\mathbf{x}^{n-1}} \eta, \Delta \mathbf{x}\right\rangle\right| \tag{6}$$

$$= n \sup_{\eta : \|\eta\|_{c_0(\alpha^{-1})} = 1} \left|\left\langle \{\eta_{m+n-1}\}_{m \in \mathbb{N}_0}, \Delta \mathbf{x}\right\rangle\right| = n \ \|\Delta \mathbf{x}\|_{\mathrm{l}^1(\alpha)}\,,$$

i.e. $\|\Delta \mathbf{x}^n / \alpha_n\|_{\mathrm{l}^1(\alpha)} = (n/\alpha_n) \ \|\Delta \mathbf{x}\|_{\mathrm{l}^1(\alpha)} \leq \|\Delta\|$ and since $\Delta \mathbf{x} \neq 0$ the claim holds. □

Corollary 6. *For $\Delta \in \mathcal{BD}\left(\mathrm{l}^1(\alpha)\right)$ is*

$$\|\Delta\| = \mathfrak{h}_\alpha(\Delta) = \sup_{n \in \mathbb{N}} \|\Delta(\mathbf{x}^n/\alpha_n)\|_{\mathrm{l}^1(\alpha)} = \|\Delta \mathbf{x}\|_{\mathrm{l}^1(\alpha)} \ \sup_{n \in \mathbb{N}} \ n/\alpha_n.$$

Proof. It suffices to consider Th. 2 and (6). □

Corollary 7. $\mathcal{BD}\left(\mathrm{l}^1(\alpha)\right)$ *is trivial.*

Proof. Let us assume the existence of a non zero bounded differentiation Δ on $\mathrm{l}^1(\alpha)$. By Proposition 5, there is a positive constant κ_1 such that $n/\kappa_1 \leq \alpha_n$ for all $n \in \mathbb{N}_0$. By Theorem 2 and Corollary 3 there is a positive constant κ_2 so that $n \ |\Delta \mathbf{x}(k)| \ \alpha_{k+n-1} \leq \alpha_n \ \kappa_2$ if $n, \ k \in \mathbb{N}$.Let k be an integer greater than 2 such that $\Delta \mathbf{x}(k) \neq 0$. Since $\{\alpha_n\}_{n \in \mathbb{N}_0}$ becomes unbounded, there is a positive integer n_k so that $\kappa_1 \, |\Delta \mathbf{x}(k)| \ \alpha_{k+n-1}/\kappa_2 \geq 1$ if $n \geq n_k$. Then

$$\alpha_n \geq n \ \alpha_{k-1+n} \ \frac{|\Delta \mathbf{x}(k)|}{\kappa_2}$$

and therefore we obtain

$$\alpha_{n_k} \geq n_k \ \alpha_{k-1+n_k} \ \frac{|\Delta \mathbf{x}(k)|}{\kappa_2},$$

$$\alpha_{k+n_k-1} \geq (k+n_k-1) \ \alpha_{2(k-1)+n_k} \ \frac{|\Delta \mathbf{x}(k)|}{\kappa_2},$$

$$\alpha_{2k+n_k-2} \geq (2k+n_k-2) \ \alpha_{3(k-1)+n_k} \ \frac{|\Delta \mathbf{x}(k)|}{\kappa_2},$$

$$\cdots$$

So, if $j \in \mathbb{N}$ we have

$$\alpha_{n_k} \geq \alpha_{n_k+j(k-1)} \left(\frac{|\Delta \mathbf{x}(k)|}{\kappa_2} \right)^j \prod_{l=0}^{j-1} [l\,(k-1)+n_k]$$

$$\geq \frac{(j-1)!}{\kappa_1} \ \left(\frac{|\Delta \mathbf{x}(k)|\,(k-1)}{\kappa_2} \right)^{j-1},$$

which is impossible because j is any natural number. Thus $\Delta \mathbf{x}\,(k)$ must be zero for all k in contradiction with our initial assumption. □

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