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$(n-1)$-DIMENSIONAL GENERALIZED NULL SCROLLS IN R_1^n

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Abstract. In this paper, we obtained relationships between the principal curvatures of an $(n-1)$-dimensional generalized null scroll M which is a ruled hypersurface in R_1^n. We calculated the normal curvature of M and a characterized the curvature lines.

1. Preliminaries

Let M be an m-dimensional Lorentzian submanifold of R_1^n. Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections of R_1^n and M, respectively. For any vector fields X, Y tangent to M we have the Gauss formula

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.1}$$

where h denotes the second fundamental form of M in R_1^n. Our second fundamental equation is the Weingarten formula

$$\overline{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{1.2}$$

where ξ is a normal vector field to M, A_ξ is the Weingarten map with respect to ξ and D is the normal connection of M [4]. It is also well-known that h and A are related by

$$< h(X, Y), \xi >=< A_\xi X, Y > \tag{1.3}$$

Let $\{\xi_1, \xi_2, \ldots, \xi_{n-m}\}$ be a local orthonormal frame field for $\chi^\perp(M)$. Then the mean curvature vector field H of M in R_1^n is given by [4]

$$H = \sum_{j=1}^{n-m} \frac{\operatorname{trace} A_{\xi_j}}{m} \xi_j. \tag{1.4}$$

Let ξ be a unit normal vector field to M. The Lipschitz-Killing curvature in the direction ξ at a point $p \in M$ is defined by

$$G(p, \xi) = \det A_\xi(p). \tag{1.5}$$

while the Gauss curvature at p is

$$G(p) := \sum_{j=1}^{n-m} G(p, \xi_j). \tag{1.6}$$

If $G(p) = 0$ for all $p \in M$, we say that M is developable [5].

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Let M be a n-dimensional Lorentzian manifold and $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_p(M)$, $p \in M$. The scalar curvature of M is defined by

$$\mathbf{r} = \sum_{i=1}^{n} \varepsilon_i \operatorname{Ric}(e_i, e_i) \tag{1.7}$$

where

$$\varepsilon_i =< e_i, e_i >, \qquad \varepsilon_i = \begin{cases} -1, & \text{if } e_i \text{ is timelike} \\ 1, & \text{if } e_i \text{ is spacelike} \end{cases}$$

and Ric is the Ricci curvature tensor field of M [4].

Now, suppose that M is a hypersurfaces in R_1^n and let A be the shape tensor of M.

The normal curvature of M along a unit tangent direction X_p in T_pM is defined by

$$k_n(X_p) =< A_p(X_p), X_p > \tag{1.8}$$

Let α be a null curve on the hypersurfaces M. According to [4], if the equality

$$< A(\alpha(t)), \alpha(t) >= 0 \tag{1.9}$$

is satisfied, then α is said to be an asymptotic curve on M. The null curve α is called a line of curvature on M if

$$Ao\alpha' = k.\alpha' \quad (k \in \mathbf{R}^*). \tag{1.10}$$

2. $(n-1)$-dimensional generalized null scrolls in R_1^n and their curvatures

We recall the notion of a generalized null scroll in R_1^n [1]. Let M be an $(n-1)$-dimensional generalized null scroll in R_1^n and suppose that the base null curve α is an pseudo-orthogonal trajectory of the generating space of M. Then we may parametrize M as

$$\varphi(t, u_0, u_1, \dots, u_{n-3}) = \alpha(t) + u_0 Y(t) + \sum_{i=1}^{n-3} u_i Z_i(t) \tag{2.1}$$

where $\{Y(t), Z_1(t), \dots, Z_{n-3}(t)\}$ is the null basis of the generating space and

$$\{X(t), Y(t), Z_1(t), \dots, Z_{n-3}(t)\}$$

with $X = \varphi_*(\frac{\partial}{\partial t})$ is pseudo-orthonormal basis of $T_{\varphi(t)}M$. We recall that a basis $\{X, Y, Z_1, \dots, Z_{n-2}\}$ of R_1^n is said to be pseudo-orthonormal if the following conditions are fulfilled [3]:

$$\begin{aligned} &< X, X >=< Y, Y >= 0; \qquad < X, Y >= -1 \\ &< X, Z_i >=< Y, Z_i >= 0; \qquad for\ 1 \le i \le n-2 \\ &< Z_i, Z_j >= \delta_{ij}, \qquad for\ 1 \le i \le n-2. \end{aligned}$$

The matrix of the shape operator $A_{\varphi(t)}$ with respect to this basis is of the form

$$M(A_{\varphi(t)}) = -\begin{bmatrix} b_{10} & b_{00} & c_{01} & c_{02} & \cdots & c_{0(n-3)} \\ 0 & b_{10} & 0 & 0 & \cdots & 0 \\ 0 & b_{11} & 0 & 0 & \cdots & 0 \\ 0 & b_{12} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{1(n-3)} & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2.2}$$

Thus, the characteristic equation of A is

$$\det(A-\lambda I_{n-1})=\det\begin{bmatrix} -b_{10}-\lambda & -b_{00} & -c_{01} & -c_{02} & \cdots & -c_{0(n-3)} \\ 0 & -b_{10}-\lambda & 0 & 0 & \cdots & 0 \\ 0 & -b_{11} & -\lambda & 0 & \cdots & 0 \\ 0 & -b_{12} & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -b_{1(n-3)} & 0 & 0 & \cdots & -\lambda \end{bmatrix}=0,$$

which leads to the relation

$$(b_{10}+\lambda)^2(-1)^{n-3}\lambda^{n-3}=0. \tag{2.3}$$

Since from equation (2.2) $\operatorname{rank} A=(n-3)$, the eigenvalues of A are

$$\lambda_3=\lambda_4=\cdots=\lambda_{n-1}=0, \tag{2.4}$$

hence we obtain

$$\lambda_1=\lambda_2=-b_{10} \tag{2.5}$$

and

$$\lambda_1+\lambda_2=2\lambda_1=2\lambda_2=-2b_{10},$$

therefore

$$\operatorname{trace} A=-2b_{10} \tag{2.6}$$

and

$$H=-\frac{2}{n-1}b_{10}. \tag{2.7}$$

If M is a minimal hypersurface, then $H=0$ and we get $b_{10}=0$, i.e., $\lambda_1=\lambda_2=0$. Thus we have the following

Corollary 2.1. *If an $(n-1)$-dimensional generalized null scroll M is minimal, then the principal curvatures of M vanish at any point.*

From the equation (2.5) we infer

Corollary 2.2. *For an $(n-1)$-dimensional generalized null scroll, the principal curvatures are equal.*

Theorem 2.1. *Let M be an $(n-1)$-dimensional generalized null scroll. Then the scalar curvature of M is*

$$r=-2\left\{\sum_{i=1}^{n-3}(b_{1i})^2+\lambda^2\right\},$$

where $\lambda_1=\lambda_2=\lambda$.

Proof. If we take $j=1$ in formula (4.16) of [1] we obtain

$$r=-2\sum_{i=1}^{n-3}(b_{1i})^2-2(b_{10})^2.$$

By equation (2.5), we have

$$r=-2\sum_{i=1}^{n-3}(b_{1i})^2-2\lambda_1^2$$

or

$$r=-2\sum_{i=1}^{n-3}(b_{1i})^2-2\lambda_2^2.$$

Also, since $\lambda_1 = \lambda_2$, we find

$$r = -2\left\{\sum_{i=1}^{n-3}(b_{1i})^2 + \lambda^2\right\},$$

as was to be shown. □

For $n \geq 4$, $\det A = 0$ by (2.2), so we have the following

Corollary 2.3. *The Gauss curvature of an $(n-1)$-dimensional generalized null scroll is identically zero if $n \geq 4$.*

Corollary 2.4. *The Lipschitz – Killing curvature of an $(n-1)$-dimensional generalized null scroll in the normal direction is equal to the Gauss curvature.*

Proof. If we take $j = 1$ in formula (4.11) of [1], the proof is clear. □

Now let $V_p \in T_pM$ and suppose that $V_p = (v, v_0, v_1, \ldots, v_{n-3})$. From equation (2.2) we obtain

$$A(V_p) = -\begin{bmatrix} b_{10} & b_{00} & c_{01} & c_{02} & \cdots & c_{0(n-3)} \\ 0 & b_{10} & 0 & 0 & \cdots & 0 \\ 0 & b_{11} & 0 & 0 & \cdots & 0 \\ 0 & b_{12} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{1(n-3)} & 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} v \\ v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{n-3} \end{bmatrix}$$

$$= (-b_{10}v - b_{00}v_0 - \sum_{i=1}^{n-3} c_{0i}v_i, -b_{10}v_0, -b_{11}v_0 \ldots, -b_{1(n-3)}v_0),$$

while by equation (1.8) we get

$$k_n(V_p) = (-\frac{n-1}{2}Hv + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i)v - (\sum_{i=0}^{n-3} b_{1i}v_i)v_0.$$

These relations lead to the following

Corollary 2.5. *Let M be an $(n-1)$-dimensional generalized null scroll. Then a direction V_p, whose first and second components are zero, is an asymptotic direction in M.*

Theorem 2.2. *Let M be an $(n-1)$-dimensional generalized null scroll. Then a direction $V_p = (v, v_0, v_1, \ldots, v_{n-3})$ of M is principal curvature direction for M at p if and only if*

$$-\frac{b_{1j}v_0}{v_j} + \frac{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}{v} = 0, \quad j = 0, 1, \ldots, n-3, \tag{2.8}$$

where b_{00}, b_{1j}, c_{0i}, are elements of the matrix of A.

Proof. If V_p is principal curvature direction for M at p, then from equation (1.10) we have

$$(-b_{10}v - b_{00}v_0 - \sum_{i=1}^{n-3} c_{0i}v_i, -b_{10}v_0, -b_{11}v_0, \ldots, -b_{1(n-3)}v_0) = k(v, v_0, v_1, \ldots, v_{n-3}).$$

Thus we obtain

$$k = -\frac{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}{v} \tag{2.9}$$

and

$$k = -\frac{b_{1j}v_0}{v_j}, \quad j = 0, 1, \ldots, n-3. \tag{2.10}$$

From equations (2.9) and (2.10), we get

$$-\frac{b_{1j}v_0}{v_j} + \frac{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}{v} = 0, \quad j = 0, 1, \ldots, n-3. \tag{2.11}$$

Conversely, let us assume that equation (2.8) is satisfied. Then we have

$$v_j = \frac{b_{1j}v_0 v}{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}, \quad j = 0, 1, \ldots, n-3; \tag{2.12}$$

and

$$A(V_p) = -\frac{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}{v} V_p$$

which concludes the proof. □

Corollary 2.6. *A curve β in an $(n-1)$-dimensional generalized null scroll M is a line of curvature if and only if it satisfies the following system of differential equations:*

$$-b_{1j}\frac{d\beta_0}{dt}\frac{d\beta_j}{dt} + \frac{d\beta_j}{dt}\left(b_{10}\frac{d\beta}{dt} + b_{00}\frac{d\beta_0}{dt} + \sum_{i=1}^{n-3} c_{0i}\frac{d\beta_i}{dt}\right) = 0, \quad j = 0, 1, \ldots, n-3. \tag{2.13}$$

Proof. This is clear by putting $v = \frac{d\beta}{dt}$, $v_k = \frac{d\beta_k}{dt}$, $\quad k = 0, 1, \ldots, n-3$, in Theorem 2.2. □

References

[1] H. Balgetir. *Generalized null scrolls in the Lorentzian space.* PhD thesis, Firat University, 2002.

[2] M. Bektaş. On the curvatures of $(r+1)$-dimensional generalized time-like ruled surface. *Acta Mathematica Academiae Paedagogicae Nyíregyháza*, 19(1):83–88, 2003.

[3] K.L. Duggal and A. Bejancu. *Lightlike Submanifolds of Semi-Riemannian Manifolds and Its Applications.* Kluwer, 1996.

[4] B. O'Neill. *Semi-Riemannian Geometry.* Academic Press, 1983.

[5] C. Thas. Properties of ruled surfaces in the euclidean space e^n. *Academia Sinica*, 6(1):133–142, 1978.

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