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THE GENERAL HERMITIAN NONNEGATIVE-DEFINITE SOLUTION TO THE MATRIX EQUATION $AXA^* + BYB^* = C$

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Abstract. Consider the matrix equation

$$AXA^* + BYB^* = C.$$

A matrix pair (X_0, Y_0) is called a Hermitian nonnegative-definite solution to the matrix equation if X_0 and Y_0 are Hermitian nonnegative-definite and satisfy $AX_0A^* + BY_0B^* = C$. We give necessary and sufficient conditions for the existence of a Hermitian nonnegative-definite solution to the matrix equation, and further derive a representation of the general Hermitian nonnegative-definite solution to the equation when it has such solutions. An example shows these advantages of the proposed approach.

1. Introduction

Let $\mathbf{C}^{m\times n}$ be the set of all $m \times n$ complex matrices. For $X \in \mathbf{C}^{m\times n}$, let X^* be the conjugate transpose of X. We denote by I_n and O the $n \times n$ identity matrix and the zero matrix, respectively. For convenience, we present the following definition.

Definition 1. Given a matrix equation of the form

$$AXA^* + BYB^* = C \tag{1}$$

with known matrices $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{m\times p}$ and $C \in \mathbf{C}^{m\times m}$. A matrix pair (X_0, Y_0) is called a Hermitian nonnegative-definite solution to the matrix equation (1) if X_0 and Y_0 are Hermitian nonnegative-definite and satisfy $AX_0A^* + BY_0B^* = C$.

Solvability conditions and general solutions of the matrix equation (1) have been derived by Baksalary and Kala [2], Chu [6] and He [9]. Chang and Wang [5] have derived expressions for the general symmetric solution and the general minimum-2-norm symmetric solution to the matrix equation (1) within the real setting. Xu *at el.* [12] have obtained the general form of all least-squares Hermitian (skew-Hermitian) solution to the matrix equation (1). The general nonnegative-definite solution to the special case $B = O$ of (1) have been studied by Baksalary [1], Dai and Lancaster [10], Groß [7], Groß [8], Khatri and Mitra [11] and Zhang and Cheng [13].

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As the supplements of [5] and [12] and the extensions of [1], [10], [7], [8], [11] and [13], this paper establishes the following problem:

Problem 1. Given matrices $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{m\times p}$ and $C \in \mathbf{C}^{m\times m}$. Determine necessary and sufficient conditions for the existence of a Hermitian nonnegative-definite solution to the matrix equation (1). Furthermore, give a representation of the general Hermitian nonnegative-definite solution to the equation (1) when it has such solutions.

Now we make the following several notes about Problem 1.

- Since (1) can be write as

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} XA^* \\ YB^* \end{bmatrix} = \begin{bmatrix} AX & BY \end{bmatrix} \begin{bmatrix} A^* \\ B^* \end{bmatrix} = C,$$

which implies

$$\operatorname{rank} \begin{bmatrix} A & B & C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A & B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A & B & C^* \end{bmatrix} \tag{2}$$

is a necessary condition for the matrix equation (1) to have a solution.

- If $n + p < m$, then there exists a unitary matrix P such that

$$PA = \begin{bmatrix} A_1 \\ O \end{bmatrix}, \; PB = \begin{bmatrix} B_1 \\ O \end{bmatrix}, \tag{3}$$

where $A_1 \in \mathbf{C}^{(n+p)\times n}$ and $B_1 \in \mathbf{C}^{(n+p)\times p}$. This, together with (2), implies

$$PCP^* = \begin{bmatrix} C_1 & O \\ O & O \end{bmatrix}, \; C_1 \in \mathbf{C}^{(n+p)\times(n+p)}. \tag{4}$$

Substituting (3) and (4) into (1) gives

$$A_1XA_1^* + B_1YB_1^* = C_1. \tag{5}$$

Obviously, the matrix equations (1) and (5) have the same solutions.

- It is clear that C is Hermitian nonnegative-definite if the matrix equation (1) has a Hermitian nonnegative-definite solution. Thus, to ensure its solvability, we can write $C = DD^*$ for some $D \in \mathbf{C}^{m\times m}$.
- If $A = O$ or $B = O$, then (1) turns into $BYB^* = C$ or $AXA^* = C$. In this case Problem 1 has been solved by several authors. (see [1], [10], [7], [8], [11] and [13]).

Based on the above four notes, in order to solve Problem 1, it suffices to solve the following Problem 1′.

Problem 1′: Given matrices $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{m\times p}$ and $D \in \mathbf{C}^{m\times m}$ satisfying

$$m \le p + n, \; A \ne O, \; B \ne O. \tag{6}$$

Determine necessary and sufficient conditions for the existence of a Hermitian nonnegative-definite solution to the matrix equation

$$AXA^* + BYB^* = DD^*. \tag{7}$$

Furthermore, give a representation of the general Hermitian nonnegative-definite solution to the equation (7) when it has such solutions.

2. Solution to Problem 1′

This section solves Problem 1′ proposed in Section 1. We first introduce the following two lemmas. The first one can be easily derived from [3, Lemma 2.1] and the second is taken from [4, p. 270].

Lemma 1. *Given two positive integers n_1 and n_2 satisfying $n_1 \leq n_2$, and two matrices $F \in \mathbf{C}^{n_1 \times n_1}$ and $G \in \mathbf{C}^{n_1 \times n_2}$. Then $FF^* = GG^*$ if and only if $G = FT$ for some $T \in \mathbf{C}^{n_1 \times n_2}$ satisfying $TT^* = I_{n_1}$.*

Lemma 2. *Given matrices $M \in \mathbf{C}^{m \times p}$ and $N \in \mathbf{C}^{m \times n}$. Let M^- be an arbitrary but fixed generalized inverse of M. Then the matrix equation $MX = N$ has at least a solution if and only if $MM^-N = N$. When this condition is met, the general solution to the equation is given by*

$$X = M^-N + (I_p - M^-M)Y,$$

where Y is free to vary over $\mathbf{C}^{p \times n}$.

Combining Lemmas 1-2, the solution to Problem 1′ can be stated as follows.

Theorem 1. *Given matrices $A \in \mathbf{C}^{m \times n}$, $B \in \mathbf{C}^{m \times p}$ and $D \in \mathbf{C}^{m \times m}$ satisfying (6). Let A^- and B^- be arbitrary but fixed generalized inverses of A and B, respectively. Then*

(i) the matrix equation (7) has at least a Hermitian nonnegative-definite solution if and only if there exist $T_1 \in \mathbf{C}^{m \times n}$ and $T_2 \in \mathbf{C}^{m \times p}$ satisfying

$$T_1T_1^* + T_2T_2^* = I_m \tag{8}$$

and

$$AA^-DT_1 = DT_1,\ BB^-DT_2 = DT_2. \tag{9}$$

(ii) when (8) and (9) are met, a representation of the general Hermitian nonnegative-definite solution to the matrix equation (7) is given by

$$(X, Y) = (VV^*, WW^*) \tag{10}$$

with

$$V = A^-DT_1 + (I_n - A^-A)Z_1 \tag{11}$$

and

$$W = B^-DT_2 + (I_p - B^-B)Z_2, \tag{12}$$

where $Z_1 \in \mathbf{C}^{n \times n}$ and $Z_2 \in \mathbf{C}^{p \times p}$ are a pair of arbitrary parameter matrices, and T_1 and T_2 are a pair of parameter matrices satisfying (8) and (9).

Proof. (i) *The "if" part.* Suppose there exist $T_1 \in \mathbf{C}^{m \times n}$ and $T_2 \in \mathbf{C}^{m \times p}$ satisfying (8) and (9). It follows from (9) and Lemma 2 that

$$AV = DT_1,\ BW = DT_2 \tag{13}$$

for some $V \in \mathbf{C}^{n \times n}$ and $W \in \mathbf{C}^{p \times p}$. Thus,

$$A\left(VV^*\right)A^* + B\left(WW^*\right)B^* = DT_1T_1^*D^* + DT_2T_2^*D^* = D\left(T_1T_1^* + T_2T_2^*\right)D^*.$$

This, together with (8), implies that (VV^*, WW^*) is a Hermitian nonnegative-definite solution to the matrix equation (7).

The "only if" part. Suppose (VV^*, WW^*) is a Hermitian nonnegative-definite solution to the matrix equation (7), where $V \in \mathbf{C}^{n\times n}$ and $W \in \mathbf{C}^{p\times p}$. Then

$$AVV^*A^* + BWW^*B^* = DD^*,$$

i.e.,

$$\begin{bmatrix} AV & BW \end{bmatrix} \begin{bmatrix} AV & BW \end{bmatrix}^* = DD^*.$$

Noting (6) and applying Lemma 1 to $F = D$ and $G = \begin{bmatrix} AV & BW \end{bmatrix}$ yields

$$\begin{bmatrix} AV & BW \end{bmatrix} = DT \tag{14}$$

for some $T \in \mathbf{C}^{m\times(n+p)}$ satisfying

$$TT^* = I_m. \tag{15}$$

Let $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with $T_1 \in \mathbf{C}^{m\times n}$ and $T_2 \in \mathbf{C}^{m\times p}$. Then (15) turns into (8), and further (13) follows from (14). Combining (13) and Lemma 2 gives (9).

(ii) Firstly, if the matrix pair (X, Y) possesses the form (10) with (11) and (12), then

$$AV = AA^-DT_1,\ BW = BB^-DT_2,$$

and hence

$$\begin{aligned} AXA^* + BYB^* &= AV(AV)^* + BW(BW)^* \\ &= \left(AA^-DT_1\right)\left(AA^-DT_1\right)^* + \left(BB^-DT_2\right)\left(BB^-DT_2\right)^*. \end{aligned}$$

This, together with (8) and (9), implies

$$\begin{aligned} AXA^* + BYB^* &= (DT_1)(DT_1)^* + (DT_2)(DT_2)^* \\ &= D\left(T_1T_1^* + T_2T_2^*\right)D^* \\ &= DD^*, \end{aligned}$$

i.e., (X, Y) is a Hermitian nonnegative-definite solution to the matrix equation (7).

Secondly, for any fixed Hermitian nonnegative-definite solution $\left(\tilde{X}, \tilde{Y}\right)$ to the matrix equation (7), we can write

$$\tilde{X} = \tilde{V}\tilde{V}^*,\ \tilde{Y} = \tilde{W}\tilde{W}^*$$

for some $\tilde{V} \in \mathbf{C}^{n\times n}$ and $\tilde{W} \in \mathbf{C}^{p\times p}$. By a similar argument to the proof of the "only if" part of (i), we can obtain

$$A\tilde{V} = D\tilde{T}_1,\ B\tilde{W} = D\tilde{T}_2$$

for some $\tilde{T}_1 \in \mathbf{C}^{m\times n}$ and $\tilde{T}_2 \in \mathbf{C}^{m\times p}$ satisfying

$$\tilde{T}_1\tilde{T}_1^* + \tilde{T}_2\tilde{T}_2^* = I_m$$

and

$$AA^-D\tilde{T}_1 = D\tilde{T}_1,\ BB^-D\tilde{T}_2 = D\tilde{T}_2.$$

This, together with Lemma 2, derives that $\left(\tilde{X}, \tilde{Y}\right)$ possesses the form (10) with (11) and (12).

The proof is done. □

Using Theorem 1, we have the following corollary which gives a representation of the general Hermitian nonnegative-definite solution to the matrix equation

$$AXA^* + BYB^* = O \tag{16}$$

with known matrices $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{m\times p}$.

Corollary 1. *Given matrices $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{m\times p}$ satisfying (6). Let A^- and B^- be arbitrary but fixed generalized inverses of A and B, respectively. Then the general Hermitian nonnegative-definite solution to the matrix equation (16) is given by*

$$(X,Y) = (VV^*, WW^*)$$

with

$$V = (I_n - A^- A)Z_1,\ W = (I_p - B^- B)Z_2,$$

where Z_1 and Z_2 are a pair of arbitrary parameter matrices.

Theorem 1 shows that if the matrix equation (7) has a Hermitian nonnegative-definite solution, then its general Hermitian nonnegative-definite solution can be obtained once the general solution (T_1, T_2) to the pair of matrix equations (8) and (9) is derived. Note that the general solution (T_1, T_2) to the matrix equation (9) is given by

$$(T_1, T_2) = (E_1 U_1, E_2 U_2)\,, \tag{17}$$

with

$$E_1 = I_m - \left[\left(I_m - AA^-\right) D\right]^- \left[\left(I_m - AA^-\right) D\right]$$

and

$$E_2 = I_m - \left[\left(I_m - BB^-\right) D\right]^- \left[\left(I_m - BB^-\right) D\right],$$

where $[(I_m - AA^-)D]^-$ is an arbitrary but fixed generalized inverse of $(I_m - AA^-)D$, $[(I_m - BB^-)D]^-$ is an arbitrary but fixed generalized inverse of $(I_m - BB^-)D$, and $U_1 \in \mathbf{C}^{m\times n}$ and $U_2 \in \mathbf{C}^{m\times p}$ are a pair of arbitrary parameter matrices. Substituting (17) into (8) yields

$$E_1 U_1 U_1^* E_1^* + E_2 U_2 U_2^* E_2^* = I_m. \tag{18}$$

Therefore, to determine the general solution (T_1, T_2) to the pair of matrix equations (8) and (9), it suffices to give a representation of the general solution to the matrix equation (18). While this can be easily obtained by using singular value decompositions (for detail see the Appendix).

3. An Example

Consider the matrix equation (1) with the parameter matrices:

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 2 & 1 \\ 4 & 4 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 4 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously, $m = n = 3$ and $p = 2$. By choosing $D = C$, it is clear that (1) is of the form (7) satisfying (6). Choosing

$$A^- = \begin{bmatrix} 1.1765e-001 & 4.7059e-002 & 9.4118e-002 \\ -4.1176e-001 & 3.5294e-002 & 7.0588e-002 \\ 5.8824e-001 & 3.5294e-002 & 7.0588e-002 \end{bmatrix}$$

and

$$B^- = \begin{bmatrix} 6.6667e-002 & 6.6667e-001 & 1.3333e-001 \\ 6.6667e-002 & -3.3333e-001 & 1.3333e-001 \end{bmatrix},$$

we derive

$$\left(I_m - AA^-\right) D = \begin{bmatrix} 2.2204e-016 & -9.7145e-017 & 0 \\ -1.1102e-016 & 8.0000e-001 & 0 \\ -2.2204e-016 & -4.0000e-001 & 0 \end{bmatrix}$$

and

$$\left(I_m - BB^-\right) D = \begin{bmatrix} 8.0000e-001 & -2.2204e-016 & 0 \\ 8.3267e-017 & -2.2204e-016 & 0 \\ -4.0000e-001 & -4.4409e-016 & 0 \end{bmatrix}.$$

Again choosing

$$\left[\left(I_m - AA^-\right) D\right]^- = \begin{bmatrix} 2.2707e-047 & -9.4371e-032 & 4.7185e-032 \\ -2.4061e-016 & 1.0000e+000 & -5.0000e-001 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\left[\left(I_m - BB^-\right) D\right]^- = \begin{bmatrix} 1.0000e+000 & 1.0408e-016 & -5.0000e-001 \\ 2.4805e-031 & 2.5818e-047 & -1.2403e-031 \\ 0 & 0 & 0 \end{bmatrix},$$

we have

$$E_1 = \begin{bmatrix} 1.0000e+000 & 9.4371e-032 & 0 \\ 8.6282e-032 & 1.1102e-016 & 0 \\ 0 & 0 & 1.0000e+000 \end{bmatrix}$$

and

$$E_2 = \begin{bmatrix} 1.1102e-016 & -1.9722e-031 & 0 \\ -2.4805e-031 & 1.0000e+000 & 0 \\ 0 & 0 & 1.0000e+000 \end{bmatrix}.$$

Following the lines in Appendix, it is easy to see that a representation of the general solution to the matrix equation (18) is given by

$$U_1 = \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ b_7 & b_8 & b_9 \\ b_4 & b_5 & b_6 \end{bmatrix}, \; U_2 = \begin{bmatrix} a_3 & a_4 \\ a_1 & a_2 \\ a_5 & a_6 \end{bmatrix}, \tag{19}$$

where a_i, $i = 1, \ldots, 6$, and b_i, $i = 1, \ldots, 9$, are complex parameters satisfying

$$\begin{cases} |a_1|^2 + |a_2|^2 = 1 \\ \frac{a_3\overline{a_1} + a_4\overline{a_2}}{9007199254740992} = 0.1109 \times 10^{-30} \\ a_5\overline{a_1} + a_6\overline{a_2} = 0 \\ |b_1|^2 + |b_2|^2 + |b_3|^2 + \frac{|a_3|^2 + |a_4|^2}{81129638414606681695789005144064} = 1 \\ b_1\overline{b_4} + b_2\overline{b_5} + b_3\overline{b_6} - \frac{a_3\overline{a_5} + a_4\overline{a_6}}{9007199254740992} = 0 \\ |b_4|^2 + |b_5|^2 + |b_6|^2 + |a_5|^2 + |a_6|^2 = 1 \end{cases}. \tag{20}$$

Substituting (19) into (17) yields

$$T_1 = \begin{bmatrix} -b_1 + \frac{245 b_7}{2596148429267413814265248164610048} & -b_2 + \frac{245 b_8}{2596148429267413814265248164610048} & -b_3 + \frac{245 b_9}{2596148429267413814265248164610048} \\ -\frac{7 b_1}{811296384146066816957890051440 64} + \frac{b_7}{9007199254740992} & -\frac{7 b_2}{81129638414606681695789005144064} + \frac{b_8}{9007199254740992} & -\frac{7 b_3}{81129638414606681695789005144064} + \frac{b_7}{9007199254740992} \\ b_4 & b_5 & b_6 \end{bmatrix} \tag{21}$$

and

$$T_2 = \begin{bmatrix} \frac{a_3}{9007199254740992} - \frac{a_1}{5070602400912917605986812821504} & \frac{a_4}{9007199254740992} - \frac{a_2}{5070602400912917605986812821504} \\ -\frac{1416143659252943 a_3}{570899077082383952423314387779798054553098649 6} + a_1 & -\frac{1416143659252943 a_4}{5708990770823839524233143877797980545530986496} + a_2 \\ a_5 & a_6 \end{bmatrix}. \tag{22}$$

Using Theorem 1, we derive that the general Hermitian nonnegative-definite solution to the matrix equation (1) is given by (10) with

$$\begin{aligned} V &= \begin{bmatrix} 1.1765e-001 & 4.7059e-002 & 0 \\ -4.1176e-001 & 3.5294e-002 & 0 \\ 5.8824e-001 & 3.5294e-002 & 0 \end{bmatrix} T_1 \\ &+ \begin{bmatrix} 5.2941e-001 & -3.5294e-001 & -3.5294e-001 \\ -3.5294e-001 & 2.3529e-001 & 2.3529e-001 \\ -3.5294e-001 & 2.3529e-001 & 2.3529e-001 \end{bmatrix} Z_1 \end{aligned}$$

and

$$\begin{aligned} W &= \begin{bmatrix} 6.6667e-002 & 6.6667e-001 & 0 \\ 6.6667e-002 & -3.3333e-001 & 0 \end{bmatrix} T_2 \\ &+ \begin{bmatrix} 0 & 3.3307e-016 \\ -2.2204e-016 & -4.4409e-016 \end{bmatrix} Z_2, \end{aligned}$$

where $Z_1 \in \mathbf{C}^{3\times 3}$ and $Z_2 \in \mathbf{C}^{2\times 2}$ are a pair of arbitrary parameter matrices, and T_1 and T_2 are, respectively, given by (21) and (22) with complex parameters a_i, $i = 1, \ldots, 6$, and b_i, $i = 1, \ldots, 9$, satisfying (20).

Remark 1. The equations (20)-(22) seem at the surface to be complicated because they are computed using the symbolic operation of Maple — a tool software. In fact, they are very simple if they are computed by numerical method.

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5. Appendix: Solution to the matrix equation (18)

Since

$$\operatorname{rank}\begin{bmatrix} E_1 & E_2 \end{bmatrix} = m \tag{23}$$

is a necessary condition for (18) to have at least a solution, in this section we always assume that (23) is satisfied.

Let $r = \operatorname{rank} E_1$ and

$$E_1 = P_1 \begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix} Q_1^* \tag{24}$$

be a singular value decomposition of E_1, where $P_1, Q_1 \in \mathbf{C}^{m\times m}$ are unitary and $\Sigma_1 \in \mathbf{C}^{r\times r}$ is positive-definite and diagonal.

Case 1. Suppose $r = m$. Then (24) turns into

$$E_1 = P_1 \Sigma_1 Q_1^*. \tag{25}$$

Let

$$U_1 = Q_1 U_{11},\ U_{11} \in \mathbf{C}^{m\times n}. \tag{26}$$

Substituting (25) and (26) into (18) derives

$$\Sigma_1 U_{11} U_{11}^* \Sigma_1 + (P_1^* E_2) U_2 U_2^* (P_1^* E_2)^* = I_m.$$

Thus,

$$U_{11} U_{11}^* = \Sigma_1^{-1} \left[I_m - (P_1^* E_2) U_2 U_2^* (P_1^* E_2)^* \right] \Sigma_1^{-1}. \tag{27}$$

Let $s = \mathrm{rank} E_2$ and

$$P_1^* E_2 = P_2 \begin{bmatrix} \Sigma_2 & O \\ O & O \end{bmatrix} Q_2^* \tag{28}$$

be a singular value decomposition of $P_1^* E_2$, where $P_2, Q_2 \in \mathbf{C}^{m \times m}$ are unitary and $\Sigma_2 \in \mathbf{C}^{s \times s}$ is positive-definite and diagonal. Further, assume

$$U_2 = Q_2 \begin{bmatrix} U_{21} \\ U_{22} \end{bmatrix}, \; U_{21} \in \mathbf{C}^{s \times p}. \tag{29}$$

Substituting (28) and (29) into (27) yields

$$U_{11} U_{11}^* = \Sigma_1^{-1} P_2 \begin{bmatrix} I_s - \Sigma_2 U_{21} U_{21}^* \Sigma_2 & O \\ O & I_{m-s} \end{bmatrix} P_2^* \Sigma_1^{-1}. \tag{30}$$

This implies that $I_s - \Sigma_2 U_{21} U_{21}^* \Sigma_2$ is nonnegative definite, i.e., all singular values of $\Sigma_2 U_{21}$ are less than or equals to 1. Therefore, by s similar argument to [13, (23)], we can determine the general expression of U_{21}.

In summary, in this case a representation of the general solution to the matrix equation (18) is given by (26) and (29), where $U_{22} \in \mathbf{C}^{(m-s) \times p}$ is an arbitrary parameter matrix, $U_{21} \in \mathbf{C}^{s \times p}$ is a parameter matrix such that all singular values of $\Sigma_2 U_{21}$ are less than or equals to 1, and $U_{11} \in \mathbf{C}^{m \times n}$ is a parameter matrix satisfying (30). In particular,

$$(U_1, U_2) = \left(Q_1 \begin{bmatrix} \Sigma_1^{-1} & O \\ O & O \end{bmatrix}, O \right)$$

is a solution to the matrix equation (18).

Case 2. Suppose $r < m$. Let

$$U_1 = Q_1 \begin{bmatrix} U_{11} \\ U_{12} \end{bmatrix}, \; U_{11} \in \mathbf{C}^{r \times n} \tag{31}$$

and

$$P_1^* E_2 = \begin{bmatrix} E_{21} \\ E_{22} \end{bmatrix}, \; E_{21} \in \mathbf{C}^{r \times m}. \tag{32}$$

Substituting (24), (31) and (32) into (18) derives

$$\begin{bmatrix} \Sigma_1 U_{11} U_{11}^* \Sigma_1 + E_{21} U_2 U_2^* E_{21}^* & E_{21} U_2 U_2^* E_{22}^* \\ E_{22} U_2 U_2^* E_{21}^* & E_{22} U_2 U_2^* E_{22}^* \end{bmatrix} = I_m. \tag{33}$$

It is easy to see from (23), (24) and (32) that E_{22} is of full-row rank. Thus, we can write

$$E_{22} = P_3 \begin{bmatrix} \Sigma_3 & O \end{bmatrix} Q_3^*, \tag{34}$$

where $P_3 \in \mathbf{C}^{(m-r) \times (m-r)}$ and $Q_3 \in \mathbf{C}^{m \times m}$ are unitary and $\Sigma_3 \in \mathbf{C}^{(m-r) \times (m-r)}$ is positive-definite and diagonal. Further, assume

$$U_2 = Q_3 \begin{bmatrix} U_{31} \\ U_{32} \end{bmatrix}, \; U_{31} \in \mathbf{C}^{(m-r) \times p} \tag{35}$$

and

$$E_{21}Q_3 = \left[\begin{array}{cc} E_{31} & E_{32} \end{array} \right], \; E_{31} \in \mathbf{C}^{r\times(m-r)}. \tag{36}$$

Substituting (34), (35) and (36) into (33) gives

$$\left\{ \begin{array}{l} U_{31}U_{31}^* = \Sigma_3^{-2} \\ E_{32}U_{32}U_{31}^* = -E_{31}\Sigma_3^{-2} \\ \Sigma_1 U_{11}U_{11}^*\Sigma_1 + E_{32}U_{32}U_{32}^*E_{32}^* = I_m + E_{31}\Sigma_3^{-2}E_{31}^* \end{array} \right., \tag{37}$$

where $\Sigma_3^{-2} = \left(\Sigma_3^{-1}\right)^2$. When $U_{31} = \left[\begin{array}{cc} \Sigma_3^{-1} & O \end{array} \right]$, the relation (37) turns into

$$\left\{ \begin{array}{l} E_{32}U_{32} = \left[\begin{array}{cc} -E_{31}\Sigma_3^{-1} & Z \end{array} \right] \\ \Sigma_1 U_{11}U_{11}^*\Sigma_1 = I_m - ZZ^* \end{array} \right., \tag{38}$$

where $Z \in \mathbf{C}^{r\times(p-m+r)}$ is an arbitrary parameter matrix whose all singular value is less than or equal to 1.

Obviously,

$$\text{rank}\left[\begin{array}{cc} E_{31} & E_{32} \end{array} \right] = \text{rank}E_{32} \tag{39}$$

is a necessary and sufficient condition for (38) to hold for some Z, U_{11} and U_{32}. Thus, (39) is sufficient for (37) to hold for some U_{11}, U_{31} and U_{32}. Furthermore, it is easy to see that (39) is also necessary for (37) to hold for some U_{11}, U_{31} and U_{32}. Therefore, in this case a representation of the general solution (18) is given by (31) and (35) with (37) if (39) is met. In particular,

$$(U_1, U_2) = \left(Q_1 \left[\begin{array}{cc} \Sigma_1^{-1} & O \\ O & O \end{array} \right], \; Q_3 \left[\begin{array}{c} \left[\begin{array}{cc} \Sigma_3^{-1} & O \end{array} \right] \\ H \end{array} \right] \right), \; E_{32}H = \left[\begin{array}{cc} -E_{31}\Sigma_3^{-1} & O \end{array} \right]$$

is a solution to the matrix equation (18) if (39) is met.

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