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CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS II

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Abstract. Let f be analytic in $D = \{z : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$ and $\frac{f(z)}{f(z)} f'(z) \neq 0$. Suppose $\delta \geq 0$ and $\gamma > 0$. For $0 < \beta < 1$, the largest $\alpha(\beta, \delta, \gamma)$ is found such that

$$\left(\delta\left(1 + \frac{zf''(z)}{f'(z)}\right) + (\gamma - \delta)\left(\frac{zf'(z)}{f(z)}\right)\right)\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\delta,\gamma)}$$
$$\implies \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

The result solves the inclusion problem for certain subclass of analytic functions involving starlike and convex functions defined in a sector. Further we investigate the inclusion problem involving addition of powers of convex and starlike functions.

1. Introduction

Let S denote the class of normalised analytic univalent functions f defined by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in D = \{z : |z| < 1\}$. It is well-known [7], [2] that $f \in C(\alpha)$ implies $f \in S^*(\beta)$ where

$$\beta = \begin{cases} \dfrac{1 - 2\alpha}{2^{2-2\alpha}(1 - 2^{2\alpha - 1})} & \text{if } \alpha \neq \dfrac{1}{2}, \\ \dfrac{1}{2\log 2} & \text{if } \alpha = \dfrac{1}{2} \end{cases}$$

and $C(\alpha)$ denotes the class of analytic convex functions satisfying

$$\operatorname{Re}\left(1 + \frac{z''z)}{f'(z)}\right) > \alpha$$

for $0 \leq \alpha < 1$ and $S(\beta)$ denotes the class of analytic starlike functions satisfying

$$\operatorname{Re}\left(\frac{zf'z)}{f(z)}\right) > \beta$$

for $0 \leq \beta < 1$, and that this result is best possible. Nunokawa and Thomas [5] proved the analogue of this result for function defined via a sector as follows:

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Theorem 1.1. *Let f be analytic in D, with $f(0) = f'(0) - 1 = 0$. Then for $0 < \beta \le 1$ and $z \in D$,*

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta)}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \tag{1.1}$$

where

$$\alpha(\beta) = \frac{2}{\pi} \arctan\left(\tan\frac{\beta\pi}{2} + \frac{\beta}{(1-\beta)^{\frac{1-\beta}{2}}(1+\beta)^{\frac{1+\beta}{2}}\cos\frac{\beta\pi}{2}}\right), \tag{1.2}$$

and $\alpha(\beta)$ given by (1.2) is the largest number such that (1.1) holds.

Subsequently, Marjono and Thomas [3] extended this and proved:

Theorem 1.2. *Let f be analytic in D, with $f(0) = f'(0)-1 = 0$ and $\frac{f(z)}{z}f'(z) \neq 0$, and, $0 < \beta \le 1$ be given. Then for $\delta > 0$ and $z \in D$,*

$$\delta\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\delta)\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\delta)}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \tag{1.3}$$

where

$$\alpha(\beta,\delta) = \frac{2}{\pi} \arctan\left(\tan\frac{\beta\pi}{2} + \frac{\beta\delta}{(1-\beta)^{\frac{1-\beta}{2}}(1+\beta)^{\frac{1+\beta}{2}}\cos\frac{\beta\pi}{2}}\right), \tag{1.4}$$

and $\alpha(\beta,\delta)$ given by (1.4) is the largest number such that (1.3) holds.

Recently, Darus [1] gave the following:

Theorem 1.3. *Let f be analytic in D, with $f(0) = f'(0)-1 = 0$ and $\frac{f(z)}{z}f'(z) \neq 0$. Suppose $\lambda \ge 0$ and $\lambda + \mu > 0$. Then for $0 < \beta \le 1$,*

$$\lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) + \mu\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\lambda,\mu)}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \tag{1.5}$$

for $z \in D$, where

(1.6)

$$\alpha(\beta,\lambda,\mu) = \frac{2}{\pi}\arctan\left(\tan\frac{\beta\pi}{2} + \frac{\beta\lambda}{(\lambda+\mu)(1-\beta)^{\frac{1-\beta}{2}}(1+\beta)^{\frac{1+\beta}{2}}\cos\frac{\beta\pi}{2}}\right),$$

and $\alpha(\beta,\lambda,\mu)$ given by (1.6) is the largest number such that (1.5) holds.

Next we consider a more general case involves convex and starlike functions.

2. Result

Theorem 2.1. *Let f be analytic in D, with $f(0) = f'(0)-1 = 0$ and $\frac{f(z)}{z}f'(z) \neq 0$. Suppose $\delta \geq 0$ and $\gamma > 0$. For $0 < \beta < 1$,*

$$\left(\delta\left(1 + \frac{zf''(z)}{f'(z)}\right) + (\gamma-\delta)\left(\frac{zf'(z)}{f(z)}\right)\right)\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\delta,\gamma)}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \tag{2.1}$$

for $z \in D$, where

$$\alpha(\beta,\delta,\gamma) = \frac{2}{\pi}\arctan\left(\tan\frac{\beta\pi}{2} + \frac{\beta\delta}{\gamma(1-\beta)^{\frac{1-\beta}{2}}(1+\beta)^{\frac{1+\beta}{2}}\cos\frac{\beta\pi}{2}}\right) + \beta, \tag{2.2}$$

and $\alpha(\beta,\delta,\gamma)$ given by (2.2) is the largest number such that (2.1) holds.

We shall need the following lemma.

Lemma 1 ([4])**.** *Let F be analytic in $\bar{D}$ and G be analytic and univalent in $\bar{D}$, with $F(0) = G(0)$. If $F \not\prec G$, then there is a point $z_0 \in D$ and $\zeta_0 \in \delta D$ such that $F(|z| < |z_0|) \subset G(D)$, $F(z_0) = G(\zeta_0)$ and $z_0F'(z_0) = m\zeta_0G'(\zeta_0)$ for $m \geq 1$.*

Proof of Theorem 2.1. Write $p(z) = \frac{zf'(z)}{f(z)}$, so that p is analytic in D and $p(0) = 1$. Thus we need to show that

$$\delta zp'(z) + \gamma p(z)^2 \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta},$$

whenever $\alpha = \alpha(\beta,\delta,\gamma)$.

Let $h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha(\beta)}$ and $q(z) = \left(\frac{1+z}{1-z}\right)^{\beta}$ so that $|\arg h(z)| < \frac{\alpha(\beta)\pi}{2}$ and $|\arg q(z)| < \frac{\beta\pi}{2}$. Suppose that $p \not\prec q$, then from Lemma 2.1, there exists $z_0 \in D$ and

$\zeta_0 \in \delta D$ such that $p(z_0) = q(\zeta_0)$ and $p(|z| < |z_0|) \subset q(D)$. Since $p(z_0) = q(\zeta_0) \neq 0$, it follows that $\zeta_0 \neq \underline{+}1$. Thus we can write $ri = \left(\frac{1+\zeta_0}{1-\zeta_0}\right)$ for $r \neq 0$. Next assume that $r > 0$, (if $r < 0$, the proof is similar) and Lemma 2.1 gives

$$\begin{aligned} \delta z p'(z) + \gamma p(z)^2 &= m\delta\zeta_0 q'(\zeta_0) + \gamma q(\zeta_0)^2 \\ &= \left(\gamma(ri)^\beta + \frac{m\beta\delta(1+r^2)i}{2r}\right)(ri)^\beta. \end{aligned} \tag{2.3}$$

Since $m \geq 1$, taking the arguments, we obtain

$$\begin{aligned} \arg\left(\delta z_0 p'(z_0) + \gamma p(z_0)^2\right) &= \arctan\left(\tan\frac{\beta\pi}{2} + \frac{m\beta\delta(1+r^2)}{\gamma r^{1+\beta}\cos\frac{\beta\pi}{2}}\right) + \frac{\beta\pi}{2}, \\ &\geq \arctan\left(\tan\frac{\beta\pi}{2} + \frac{\beta\delta(1+r^2)}{\gamma r^{1+\beta}\cos\frac{\beta\pi}{2}}\right) + \frac{\beta\pi}{2}, \\ &\geq \arctan\left(\tan\frac{\beta\pi}{2} + \frac{\beta\delta}{\gamma(1-\beta)^{\frac{1-\beta}{2}}(1+\beta)^{\frac{1+\beta}{2}}\cos\frac{\beta\pi}{2}}\right) + \frac{\beta\pi}{2}, \\ &= \frac{\alpha(\beta,\delta,\gamma)\pi}{2}, \end{aligned}$$

where a minimum is attained when $r = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$.

Hence combining the cases $r > 0$ and $r < 0$ we obtain

$$\frac{\alpha(\beta,\delta,\gamma)\pi}{2} \leq \left|\arg\left(\delta z_0 p'(z_0) + \gamma p(z_0)^2\right)\right| \leq \pi,$$

which contradicts the fact that $|\arg \quad h(z)| < \frac{\alpha(\beta,\delta,\gamma)\pi}{2}$, provided that (2.2) holds.

To show that $\alpha(\beta,\delta,\gamma)$ is exact, take $\alpha(\beta,\delta,\gamma) < \sigma < 1$ so that for some $\beta_0 > \beta$ we can write $\sigma = \alpha(\beta_0,\delta,\gamma)$. Now let $p(z) = \left(\frac{1+z}{1-z}\right)^{\beta_0}$. Then from the minimum principle for harmonic functions, it follows that

$$\inf_{|z|<1} \arg\left(\delta z p'(z) + \gamma p(z)^2\right)$$

is attained at some point $z = e^{i\theta}$ for $0 < \theta < 2\pi$. Thus

$$\delta z p'(z) + \gamma p(z)^2 = \left(\gamma\left(\frac{\sin\theta}{1-\cos\theta}\right)^{\beta_0} e^{\frac{\beta_0\pi i}{2}} + \frac{i\delta\beta_0}{\sin\theta}\right)\left(\frac{\sin\theta}{1-\cos\theta}\right)^{\beta_0} e^{\frac{\beta_0\pi i}{2}},$$

and so taking $t = \cos\theta$, we obtain

$$\arg\left(\delta z p'(z) + \gamma p(z)^2\right)$$
$$= \arctan\left(\tan\frac{\beta_0\pi}{2} + \frac{\beta_0\delta}{\gamma(1-t)^{\frac{1-\beta_0}{2}}(1+t)^{\frac{1+\beta_0}{2}}\cos\frac{\beta_0\pi}{2}}\right) + \frac{\beta_0\pi}{2},$$

and elementary calculation shows that the minimum of this expression is attained when $t = \beta_0$. Thus completes the proof of Theorem 2.1. □

Particular choices for δ and γ give the following interesting corollaries. First when $\delta = 1$ we have

Corollary 2.1. *Let f be analytic in D, with $f(0) = f'(0) - 1 = 0$ and $\frac{f(z)}{z}f'(z) \neq 0$. Then for $\gamma > 0$ and $0 < \beta < 1$,*

$$\left(1 + \frac{zf''(z)}{f'(z)} + (\gamma - 1)\left(\frac{zf'(z)}{f(z)}\right)\right)\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,1,\gamma)}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta},$$

for $z \in D$, where

$$\alpha(\beta, 1, \gamma) = \frac{2}{\pi}\arctan\left(\tan\frac{\beta\pi}{2} + \frac{\beta}{\gamma(1-\beta)^{\frac{1-\beta}{2}}(1+\beta)^{\frac{1+\beta}{2}}\cos\frac{\beta\pi}{2}}\right) + \beta.$$

Similarly, when $\gamma = 1$, we obtain

Corollary 2.2. *Let f be analytic in D, with $f(0) = f'(0) - 1 = 0$ and $\frac{f(z)}{z}f'(z) \neq 0$. Then for $\delta \geq 0$ and $0 < \beta < 1$,*

$$\left(\delta\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1 - \delta)\left(\frac{zf'(z)}{f(z)}\right)\right)\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\delta,1)}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta},$$

for $z \in D$, where

$$\alpha(\beta, \delta, 1) = \frac{2}{\pi}\arctan\left(\tan\frac{\beta\pi}{2} + \frac{\beta\delta}{(1-\beta)^{\frac{1-\beta}{2}}(1+\beta)^{\frac{1+\beta}{2}}\cos\frac{\beta\pi}{2}}\right) + \beta.$$

Finally, when $\delta = \gamma = 1$, we have the following interesting result.

Corollary 2.3. *Let f be analytic in D, with $f(0) = f'(0) - 1 = 0$ and $\frac{f(z)}{z} f'(z) \neq 0$. Then for $0 < \beta < 1$,*

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,1,1)}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta},$$

for $z \in D$, where

$$\alpha(\beta, 1, 1) = \frac{2}{\pi} \arctan \left(\tan \frac{\beta\pi}{2} + \frac{\beta}{(1-\beta)^{\frac{1-\beta}{2}} (1+\beta)^{\frac{1+\beta}{2}} \cos \frac{\beta\pi}{2}} \right) + \beta.$$

Remark 2.1. We note that $lim_{\beta \to 1} \alpha(\beta, 1, 1) = 2$, which suggests from Theorem 2.1 that

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{2}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}.$$

However if we let $\beta = 1$, the right hand side in (2.3) is real and the method of proof in Theorem 2.1 breaks down.

Next we give the following:

Theorem 2.2. *Let f be analytic in D, with $f(0) = f'(0) - 1 = 0$ and $\frac{f(z)}{z} f'(z) \neq 0$. Suppose $\lambda < \beta\mu$ and $0 < \lambda \leq 1$. Then for $0 < \beta \leq 1$,*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)^{\lambda} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\lambda,\mu)}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \tag{2.4}$$

for $z \in D$, where

(2.5)

$$\alpha(\beta, \lambda, \mu) = \frac{2}{\pi} \arctan \left(\frac{\tan \frac{\beta\mu\pi}{2} + \frac{(\lambda\beta)^{\lambda} \sin \frac{\lambda\pi}{2}}{(\lambda - \beta\mu)^{\frac{(\lambda-\beta\mu)}{2}} (\lambda + \beta\mu)^{\frac{(\lambda+\beta\mu)}{2}} \cos \frac{\beta\mu\pi}{2}}}{1 + \frac{(\lambda\beta)^{\lambda} \cos \frac{\lambda\pi}{2}}{(\lambda - \beta\mu)^{\frac{(\lambda-\beta\mu)}{2}} (\lambda + \beta\mu)^{\frac{(\lambda+\beta\mu)}{2}} \cos \frac{\beta\mu\pi}{2}}} \right),$$

and $\alpha(\beta, \lambda, \mu)$ given by (2.5) is the largest number such that (2.4) holds.

Proof. Write $p(z) = \dfrac{zf'(z)}{f(z)}$, so that p is analytic in D and $p(0) = 1$. Thus we need to show that

$$p(z)^{\mu} + \left(\frac{zp'(z)}{p(z)}\right)^{\lambda} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta},$$

whenever $\alpha = \alpha(\beta, \lambda, \mu)$.

As before, let $h(z) = \left(\dfrac{1+z}{1-z}\right)^{\alpha(\beta)}$ and $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\beta}$ so that

$$|\arg h(z)| < \frac{\alpha(\beta)\pi}{2}$$

and $|\arg q(z)| < \dfrac{\beta\pi}{2}$. Suppose that $p \not\prec q$, then from Lemma 2.1, there exists $z_0 \in D$ and $\zeta_0 \in \delta D$ such that $p(z_0) = q(\zeta_0)$ and $p(|z| < |z_0|) \subset q(D)$. Since $p(z_0) = q(\zeta_0) \neq 0$, it follows that $\zeta_0 \neq \underline{+}1$. Thus we can write $ri = \left(\dfrac{1+\zeta_0}{1-\zeta_0}\right)$ for $r \neq 0$. Next assume that $r > 0$, (if $r < 0$, the proof is similar) and Lemma 2.1 gives

$$\begin{aligned} p(z_0)^{\mu} + \left(\frac{z_0 p'(z_0)}{p(z_0)}\right) &= q(\zeta_0)^{\beta\mu} + \left(\frac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right)^{\lambda}, \\ &= (ri)^{\beta\mu} + \left(\frac{m\beta(1+r^2)i}{2r}\right)^{\lambda}. \end{aligned}$$

The result now follows by using the same arguments as before.

To show that $\alpha(\beta, \lambda, \mu)$ is exact, we argue as in the proof of Theorem 2.1 so that for some β_0, again choose $p(z) = \dfrac{zf'(z)}{f(z)} = \left(\frac{1+z}{1-z}\right)^{\beta_0}$ with $z = e^{i\theta}$ for $0 < \theta < 2\pi$. Thus with $t = \cos\theta$, we obtain

$$p(z)^{\mu} + \left(\frac{zp'(z)}{p(z)}\right)^{\lambda} = \left(\frac{1+t}{1-t}\right)^{\beta_0\mu} e^{\frac{\beta_0\mu\pi i}{2}} + \left(\frac{\beta_0}{\sqrt{1-t^2}}\right)^{\lambda} e^{\frac{\lambda\pi i}{2}}.$$

and taking arguments, we have

$$\arg\left(\left(\frac{zf'(z)}{f(z)}\right)^{\mu} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)^{\lambda}\right)$$

$$= \arctan\left(\frac{\tan\dfrac{\beta_0\mu\pi}{2} + \dfrac{\beta_0^{\lambda}\sin\dfrac{\lambda\pi}{2}}{(\lambda-\beta_0\mu)^{\frac{(\lambda-\beta_0\mu)}{2}}(\lambda+\beta_0\mu)^{\frac{(\lambda+\beta_0\mu)}{2}}\cos\frac{\beta_0\mu\pi}{2}}}{1 + \dfrac{\beta_0^{\lambda}\cos\dfrac{\lambda\pi}{2}}{(\lambda-\beta_0\mu)^{\frac{(\lambda-\beta_0\mu)}{2}}(\lambda+\beta_0\mu)^{\frac{(\lambda+\beta_0\mu)}{2}}\cos\frac{\beta_0\mu\pi}{2}}}\right),$$

and elementary calculation shows that the minimum of this expression is attained when $t = \dfrac{\beta_0\mu}{2}$. Thus the proof of Theorem 2.2 is complete. □

Remark 2.2. When $\lambda = \mu = 1$ we obtain Theorem 1.1.

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