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ON CERTAIN GENERALIZED CLASS OF NON-BAZILEVIČ FUNCTIONS

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Abstract. In this paper, a subclass $N(\lambda, \alpha, A, B)$ of analytic functions is introduced, which is a generalized class of non-Bazilevič functions. The subordination relations, inclusion relations, distortion theorems and inequality properties are discussed by applying differential subordination method.

1. Introduction

Let H denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Assume that $0 < \alpha < 1$, a function $f(z) \in N(\alpha)$ if and only if $f(z) \in H$ and

$$\operatorname{Re}\left\{ f'(z) \left(\frac{z}{f(z)} \right)^{1+\alpha} \right\} > 0, \quad z \in U. \tag{1.2}$$

$N(\alpha)$ was introduced by Obradović [5] recently, he called this class of functions to be of non-Bazilevič type. Until now, this class was studied in a direction of finding necessary conditions over α that embeds this class into the class of univalent functions or its subclasses, which is still an open problem.

Let $f(z)$ and $F(z)$ be analytic in U. Then we say that the function $f(z)$ is subordinate to $F(z)$ in U, if there exists an analytic function $\omega(z)$ in U such that $|\omega(z)| \le |z|$ and $f(z) = F(\omega(z))$, denoted $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in U, then the subordination is equivalent to $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [6]).

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Assume that $0 < \alpha < 1, \lambda \in C, -1 \leq B \leq 1, A \neq B, A \in R$, we define the following subclass of H:

$$N(\lambda,\alpha,A,B)=\left\{f(z)\in H:(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\prec\frac{1+Az}{1+Bz},\quad z\in U\right\}, \tag{1.3}$$

where all the powers are principal values, below we apply this agreement. Apparently, $f(z) \in N(\lambda, \alpha, \beta)$ if and only if $f(z) \in H$ and

$$\operatorname{Re}\left\{(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\right\}>\beta, 0\leq\beta<1,\quad z\in U. \tag{1.4}$$

If $\lambda = -1, A = 1, B = -1$, then the class $N(\lambda, \alpha, A, B)$ reduces to the class of non-Bazilevič functions. If $\lambda = -1, A = 1-2\beta, B = -1$, then the class $N(\lambda, \alpha, A, B)$ reduces to the class of non-Bazilevič functions of order β $(0 \leq \beta < 1)$. The Fekete-Szegö problem of the class $N(-1, \alpha, 1-2\beta, -1)$ were considered by N. Tuneski and M. Darus [8]. In this paper, we will discuss the subordination relations, inclusion relations, distortion theorems and inequality properties of $N(\lambda, \alpha, A, B)$.

2. Some Lemmas

Lemma 1 ([4]). *Let* $F(z) = 1 + b_1 z + b_2 z^2 + \cdots$ *be analytic in* U*,* $h(z)$ *be analytic and convex in* U*,* $h(0) = 1$*. If*

$$F(z)+\frac{1}{c}zF'(z)\prec h(z), \tag{2.1}$$

where $c \neq 0$ *and* $\operatorname{Re} c \geq 0$*, then*

$$F(z)\prec cz^{-c}\int_0^z t^{c-1}h(t)dt\prec h(z),$$

and $cz^{-c}\int_0^z t^{c-1}h(t)dt$ *is the best dominant for differential subordination (2.1).*

Lemma 2 ([1]). *Let* $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$*, then*

$$\frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}.$$

Lemma 3 ([3]). *Let* $F(z)$ *be analytic and convex in* U*,* $f(z) \in H, g(z) \in H$*,*

$$f(z)\prec F(z), g(z)\prec F(z), 0\leq\lambda\leq 1,$$

then

$$\lambda f(z)+(1-\lambda)g(z)\prec F(z).$$

Lemma 4 ([7]). *Let* $f(z) = \sum_{k=1}^{\infty} a_k z^k$ *be analytic in* U*,* $g(z) = \sum_{k=1}^{\infty} b_k z^k$ *be analytic and convex in* U*. If* $f(z) \prec g(z)$*, then* $|a_k| \leq |b_1|$*, for* $k = 1, 2, \ldots$*.*

3. Main Results and Their proofs

Theorem 1. *Let* $0 < \alpha < 1$*,* $\operatorname{Re}\lambda \geq 0$*,* $-1 \leq B \leq 1$*,* $A \neq B, A \in R$*,* $f(z) \in N(\lambda, \alpha, A, B)$*, then*

$$\left(\frac{z}{f(z)}\right)^{\alpha}\prec\frac{\alpha}{\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{\lambda}-1}du\prec\frac{1+Az}{1+Bz}.$$

Proof. Let

$$F(z) = \left(\frac{z}{f(z)}\right)^{\alpha}, \tag{3.1}$$

then $F(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is analytic in U. By taking the derivatives in the both sides of equation (3.1), we have

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} = F(z) + \frac{\lambda}{\alpha}zF'(z).$$

Since $f(z) \in N(\lambda, \alpha, A, B)$, we have

$$F(z) + \frac{\lambda}{\alpha}zF'(z) \prec \frac{1+Az}{1+Bz}.$$

It is obvious that $h(z) = (1+Az)/(1+Bz)$ is analytic and convex in U, $h(0) = 1$. Since $\alpha/\lambda \neq 0, \mathrm{Re}\{\alpha/\lambda\} \geq 0$, therefore it follows from Lemma 1 that

$$\left(\frac{z}{f(z)}\right)^{\alpha} = F(z) \prec \frac{\alpha}{\lambda}z^{-\frac{\alpha}{\lambda}}\int_0^z \frac{1+At}{1+Bt}t^{\frac{\alpha}{\lambda}-1}dt = \frac{\alpha}{\lambda}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{\lambda}-1}du \prec \frac{1+Az}{1+Bz}.$$

□

Corollary 1. *Let* $0 < \alpha < 1, \mathrm{Re}\,\lambda \geq 0, \beta \neq 1$. *If*

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+(1-2\beta)z}{1-z}, \quad z \in U,$$

then

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec \beta + \frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1+zu}{1-zu}u^{\frac{\alpha}{\lambda}-1}du, \quad z \in U.$$

Corollary 2. *Let* $0 < \alpha < 1, \mathrm{Re}\,\lambda \geq 0$, *then*

$$N(\lambda, \alpha, A, B) \subset N(0, \alpha, A, B).$$

Theorem 2. *Let* $0 < \alpha < 1, \lambda_2 \geq \lambda_1 \geq 0, -1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, *then*

$$N(\lambda_2, \alpha, A_2, B_2) \subset N(\lambda_1, \alpha, A_1, B_1).$$

Proof. Let $f(z) \in N(\lambda_2, \alpha, A_2, B_2)$, we have $f(z) \in H$ and

$$(1+\lambda_2)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda_2\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+A_2z}{1+B_2z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, therefore it follows from Lemma 2 that

$$(1+\lambda_2)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda_2\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+A_1z}{1+B_1z}, \tag{3.2}$$

that is $f(z) \in N(\lambda_2, \alpha, A_1, B_1)$. So Theorem 2 is proved when $\lambda_2 = \lambda_1 \geq 0$.

When $\lambda_2 > \lambda_1 \geq 0$, it follows from Corollary 2 that $f(z) \in N(0, \alpha, A_1, B_1)$, that is

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+A_1z}{1+B_1z}. \tag{3.3}$$

But

$$(1+\lambda_1)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda_1\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} = \left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{z}{f(z)}\right)^{\alpha}$$

$$+\frac{\lambda_1}{\lambda_2}\left[(1+\lambda_2)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda_2\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\right].$$

It is obvious that $h_1(z)=(1+A_1z)/(1+B_1z)$ is analytic and convex in U. So we obtain from Lemma 3 and differential subordinations (3.2) and (3.3) that

$$(1+\lambda_1)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda_1\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\prec\frac{1+A_1z}{1+B_1z},$$

that is $f(z)\in N(\lambda_1,\alpha,A_1,B_1)$. Thus we have

$$N(\lambda_2,\alpha,A_2,B_2)\subset N(\lambda_1,\alpha,A_1,B_1).$$

□

Corollary 3. *Let* $0<\alpha<1,\lambda_2\geq\lambda_1\geq 0, 1>\beta_2\geq\beta_1\geq 0$, *then*

$$N(\lambda_2,\alpha,\beta_2)\subset N(\lambda_1,\alpha,\beta_1).$$

Theorem 3. *Let* $0<\alpha<1,\operatorname{Re}\lambda\geq 0,-1\leq B<A\leq 1, f(z)\in N(\lambda,\alpha,A,B)$, *then*

$$\frac{\alpha}{\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du<\operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\}<\frac{\alpha}{\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du,\quad z\in U,\tag{3.4}$$

and inequality (3.4) is sharp, with the extremal function defined by

$$f_{\lambda,\alpha,A,B}(z)=z\left(\frac{\alpha}{\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{\lambda}-1}du\right)^{-\frac{1}{\alpha}}.\tag{3.5}$$

Proof. Since $f(z)\in N(\lambda,\alpha,A,B)$, according to Theorem 1 we have

$$\left(\frac{z}{f(z)}\right)^{\alpha}\prec\frac{\alpha}{\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{\lambda}-1}du.$$

Therefore it follows from the definition of the subordination and $A>B$ that

$$\begin{aligned}\operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\}&<\sup_{z\in U}\operatorname{Re}\left\{\frac{\alpha}{\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{\lambda}-1}du\right\}\\&\leq\frac{\alpha}{\lambda}\int_0^1\sup_{z\in U}\operatorname{Re}\left\{\frac{1+Azu}{1+Bzu}\right\}u^{\frac{\alpha}{\lambda}-1}du\\&<\frac{\alpha}{\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du;\\\operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\}&>\inf_{z\in U}\operatorname{Re}\left\{\frac{\alpha}{\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{\lambda}-1}du\right\}\\&\geq\frac{\alpha}{\lambda}\int_0^1\inf_{z\in U}\operatorname{Re}\left\{\frac{1+Azu}{1+Bzu}\right\}u^{\frac{\alpha}{\lambda}-1}du\\&>\frac{\alpha}{\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du.\end{aligned}$$

□

Note that $f_{\lambda,\alpha,A,B}(z)\in N(\lambda,\alpha,A,B)$, we obtain that inequality (3.4) is sharp.
By applying the similar method as in Theorem 3, we have

Theorem 4. *Let* $0<\alpha<1, \operatorname{Re}\lambda \geq 0, -1 \leq A < B \leq 1, f(z) \in N(\lambda,\alpha,A,B)$, *then*

$$\frac{\alpha}{\lambda}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du < \operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} < \frac{\alpha}{\lambda}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du, \quad z\in U, \tag{3.6}$$

and inequality (3.6) is sharp, with the extremal function defined by equation (3.5).

Corollary 4. *Let* $0<\alpha<1, \operatorname{Re}\lambda \geq 0, 0 \leq \beta < 1, f(z) \in N(\lambda,\alpha,\beta)$, *then*

$$\begin{aligned}\frac{\alpha}{\lambda}\int_0^1 \frac{1-(1-2\beta)u}{1+u}u^{\frac{\alpha}{\lambda}-1}du &< \operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\}\\ &< \frac{\alpha}{\lambda}\int_0^1 \frac{1+(1-2\beta)u}{1-u}u^{\frac{\alpha}{\lambda}-1}du, \quad z\in U,\end{aligned} \tag{3.7}$$

and inequality (3.7) is equivalent to

$$\begin{aligned}\beta+\frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1-u}{1+u}u^{\frac{\alpha}{\lambda}-1}du &< \operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\}\\ &< \beta+\frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1+u}{1-u}u^{\frac{\alpha}{\lambda}-1}du, \quad z\in U.\end{aligned}$$

Corollary 5. *Let* $0<\alpha<1, \operatorname{Re}\lambda \geq 0, \beta > 1, f(z) \in H$, *and*

$$\operatorname{Re}\left\{(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\right\} < \beta, \quad z\in U,$$

then

$$\begin{aligned}\frac{\alpha}{\lambda}\int_0^1 \frac{1+(1-2\beta)u}{1-u}u^{\frac{\alpha}{\lambda}-1}du &< \operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\}\\ &< \frac{\alpha}{\lambda}\int_0^1 \frac{1-(1-2\beta)u}{1+u}u^{\frac{\alpha}{\lambda}-1}du, \quad z\in U,\end{aligned} \tag{3.8}$$

and inequality (3.8) is equivalent to

$$\begin{aligned}\beta+\frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1+u}{1-u}u^{\frac{\alpha}{\lambda}-1}du &< \operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\}\\ &< \beta+\frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1-u}{1+u}u^{\frac{\alpha}{\lambda}-1}du, \quad z\in U.\end{aligned}$$

If $\operatorname{Re}\omega \geq 0$, then $(\operatorname{Re}\omega)^{\frac{1}{2}} \leq \operatorname{Re}\omega^{\frac{1}{2}} \leq |\omega(z)|^{\frac{1}{2}}$ (see [2]), so we have

Theorem 5. *Let* $0<\alpha<1, \operatorname{Re}\lambda \geq 0, -1 \leq B < A \leq 1, f(z) \in N(\lambda,\alpha,A,B)$, *then*

$$\begin{aligned}\left(\frac{\alpha}{\lambda}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du\right)^{\frac{1}{2}} &< \operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\frac{\alpha}{2}}\right\}\\ &< \left(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du\right)^{\frac{1}{2}}, \quad z\in U,\end{aligned} \tag{3.9}$$

and inequality (3.9) is sharp, with the extremal function defined by equation (3.5).

Proof. According to Theorem 1 we have

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz}.$$

Since $-1 \le B < A \le 1$, we have

$$0 \le \frac{1-A}{1-B} < \mathrm{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} < \frac{1+A}{1+B}.$$

Hence the result follows by Theorem 3. □

Note that $f_{\lambda,\alpha,A,B}(z) \in N(\lambda, \alpha, A, B)$, we obtain that inequality (3.9) is sharp.
By applying the similar method as in Theorem 5, we have

Theorem 6. *Let* $0 < \alpha < 1, \mathrm{Re}\,\lambda \ge 0, -1 \le A < B \le 1, f(z) \in N(\lambda, \alpha, A, B)$, *then*

$$\left(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du\right)^{\frac{1}{2}} < \mathrm{Re}\left\{\left(\frac{z}{f(z)}\right)^{\frac{\alpha}{2}}\right\} < \left(\frac{\alpha}{\lambda}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du\right)^{\frac{1}{2}}, \quad z \in U, \tag{3.10}$$

and inequality (3.10) is sharp, with the extremal function defined by equation (3.5).

Theorem 7. *Let* $0 < \alpha < 1$, $\mathrm{Re}\,\lambda \ge 0$, $-1 \le B < A \le 1$, $f(z) \in N(\lambda, \alpha, A, B)$.

(i) If $\lambda = 0$, *when* $|z| = r < 1$, *we have*

$$r\left(\frac{1+Br}{1+Ar}\right)^{\frac{1}{\alpha}} \le |f(z)| \le r\left(\frac{1-Br}{1-Ar}\right)^{\frac{1}{\alpha}}, \tag{3.11}$$

and inequality (3.11) is sharp, with the extremal function defined by

$$f(z) = z[(1+Bz)/(1+Az)]^{\frac{1}{\alpha}}.$$

(ii) If $\lambda \ne 0$, *when* $|z| = r < 1$, *we have*

$$r\left(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\alpha}{\lambda}-1}du\right)^{-\frac{1}{\alpha}} \le |f(z)| \le r\left(\frac{\alpha}{\lambda}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\alpha}{\lambda}-1}du\right)^{-\frac{1}{\alpha}}, \tag{3.12}$$

and inequality (3.12) is sharp, with the extremal function defined by equation (3.5).

Proof. (1) If $\lambda = 0$, since $f(z) \in N(\lambda, \alpha, A, B), -1 \le B < A \le 1$, we obtain from the definition of $N(\lambda, \alpha, A, B)$ that

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz}.$$

Therefore it follows from the definition of the subordination that

$$\left(\frac{z}{f(z)}\right)^{\alpha} = \frac{1+A\omega(z)}{1+B\omega(z)},$$

where $\omega(z)$ is analytic in U. By applying Schwarz Lemma we obtain that

$$\omega(z) = c_1 z + c_2 z^2 + \cdots$$

and $|\omega(z)| \le |z|$, so when $|z| = r < 1$, we have

$$\left|\frac{z}{f(z)}\right|^{\alpha} = \left|\frac{1+A\omega(z)}{1+B\omega(z)}\right| \le \frac{1+A\,|\omega(z)|}{1+B\,|w(z)|} \le \frac{1+Ar}{1+Br},$$

and

$$\left|\frac{z}{f(z)}\right|^{\alpha} \ge \mathrm{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} \ge \frac{1-Ar}{1-Br}.$$

It is obvious that inequality (3.11) is sharp, with the extremal function defined by $f(z) = z[(1+Bz)/(1+Az)]^{\frac{1}{\alpha}}$.

(2) If $\lambda \neq 0$, according to Theorem 1 we have

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{\alpha}{\lambda}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\alpha}{\lambda}-1} du.$$

Therefore it follows from the definition of the subordination that

$$\left(\frac{z}{f(z)}\right)^{\alpha} = \frac{\alpha}{\lambda}\int_0^1 \frac{1+Au\omega(z)}{1+Bu\omega(z)} u^{\frac{\alpha}{\lambda}-1} du,$$

where $\omega(z) = c_1 z + c_2 z^2 + \cdots$ is analytic in U and $|\omega(z)| \leq |z|$, so when $|z| = r < 1$, we have

$$\left|\frac{z}{f(z)}\right|^{\alpha} \leq \frac{\alpha}{\lambda}\int_0^1 \left|\frac{1+Au\omega(z)}{1+Bu\omega(z)}\right| u^{\frac{\alpha}{\lambda}-1} du \leq \frac{\alpha}{\lambda}\int_0^1 \frac{1+Au\,|\omega(z)|}{1+Bu\,|\omega(z)|} u^{\frac{\alpha}{\lambda}-1}$$
$$\leq \frac{\alpha}{\lambda}\int_0^1 \frac{1+Aur}{1+Bur} u^{\frac{\alpha}{\lambda}-1} du,$$

and

$$\left|\frac{z}{f(z)}\right|^{\alpha} \geq \mathrm{Re}\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} \geq \frac{\alpha}{\lambda}\int_0^1 \frac{1-Aur}{1-Bur} u^{\frac{\alpha}{\lambda}-1} du.$$

□

Note that $f_{\lambda,\alpha,A,B}(z) \in N(\lambda,\alpha,A,B)$, we obtain that inequality (3.12) is sharp.

By applying the similar method as in Theorem 7, we have

Theorem 8. *Let* $0 < \alpha < 1, \mathrm{Re}\,\lambda \geq 0, -1 \leq A < B \leq 1, f(z) \in N(\lambda,\alpha,A,B)$.

(i) If $\lambda = 0$, *when* $|z| = r < 1$, *we have*

$$r\left(\frac{1-Br}{1-Ar}\right)^{\frac{1}{\alpha}} \leq |f(z)| \leq r\left(\frac{1+Br}{1+Ar}\right)^{\frac{1}{\alpha}}, \tag{3.13}$$

and inequality (3.13) is sharp, with the extremal function defined by

$$f(z) = z[(1+Bz)/(1+Az)]^{\frac{1}{\alpha}}.$$

(ii) If $\lambda \neq 0$, *when* $|z| = r < 1$, *we have*

$$r\left(\frac{\alpha}{\lambda}\int_0^1 \frac{1-Aur}{1-Bur} u^{\frac{\alpha}{\lambda}-1} du\right)^{-\frac{1}{\alpha}} \leq |f(z)| \leq r\left(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Aur}{1+Bur} u^{\frac{\alpha}{\lambda}-1} du\right)^{-\frac{1}{\alpha}}.$$

and inequality (3.14) is sharp, with the extremal function defined by equation (3.5).

Theorem 9. *Let* $0 < \alpha < 1, \lambda \in C, -1 \leq B \leq 1, A \neq B, A \in R$,

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in N(\lambda,\alpha,A,B),$$

then

$$|a_{n+1}| \leq \frac{|A-B|}{|\lambda n + \alpha|},$$

and inequality (3.15) is sharp, with the extremal function defined by equation (3.5).

Proof. Since $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in N(\lambda, \alpha, A, B)$, we have

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} = 1 + (-\lambda n - \alpha)a_{n+1}z^n + \cdots \prec \frac{1+Az}{1+Bz}.$$

Hence it follows from Lemma 4 that $|(-\lambda n - \alpha)a_{n+1}| \le |A - B|$, so

$$|a_{n+1}| \le |A - B| \,/\, |\lambda n + \alpha|\,.$$

□

Note that $f(z) = z + [(A-B)/(\lambda n + \alpha)]z^{n+1} + \cdots \in N(\lambda, \alpha, A, B)$, we obtain that inequality (3.15) is sharp.

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