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ALMOST EVERYWHERE CONVERGENCE OF SUBSEQUENCE OF LOGARITHMIC MEANS OF WALSH-FOURIER SERIES

USHANGI GOGINAVA

ABSTRACT. In this paper we prove that the maximal operator of the subsequence of logarithmic means of Walsh-Fourier series is weak type (1,1). Moreover, the logarithmic means $t_{m_n}(f)$ of the function $f \in L$ converge a.e. to f as $n \to \infty$.

In the literature, it is known the notion of the Riesz's logarithmic means of a Fourier series. The n-th mean of the Fourier series of the integrable function f is defined by

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k}.$$

This Riesz's logarithmic means with respect to the trigonometric system has been studied by a lot of authors. We mention for instance the papers of Szász, and Yabuta ([Sz], [Ya]). This mean with respect to the Walsh, Vilenkin system is discussed by Simon, and Gát ([14], [2]).

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}S_k(f),$$

where $Q_n := \sum_{k=1}^{n-1} q_k$. If $q_k = \frac{1}{k}$, then we get the (Nörlund) logarithmic means:

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{n-k}.$$

Móricz and Siddiqi [11] investigates the approximation properties of some special Nörlund means of Walsh-Fourier series of L^p functions in norm. The case, when $q_k = \frac{1}{k}$ is excluded, since the methods of Móricz are not applicable for logarithmic means. In [7] we proved some convergence and divergence properties of the logarithmic means of functions in the class of continuous functions, and in the Lebesgue space L. Among others, we proved that the maximal norm convergence function space of this logarithmic means is $L\log^+ L$. On the other hand, with respect to

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approximation properties of logarithmic means of multiple Walsh-Fourier series see for instance the papers ([6, 4, 5]).

In this paper we discuss a.e. convergence of subsequence of logarithmic means of Walsh-Fourier series of functions from the space $f \in L$. In particular, we prove that the maximal operator of the subsequence of logarithmic means of Walsh-Fourier series is weak type (1,1). Moreover, the logarithmic means $t_{m_n}(f)$ of the function $f \in L$ converge a.e. to f as $n \to \infty$. For this we apply some Gát idea from [1], [3].

Let $r_0(x)$ be a function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2) \\ -1, & \text{if } x \in [1/2, 1) \end{cases}, \quad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geq 1 \text{ and } x \in [0, 1).$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e. $w_0(x) = 1$ and if

$$k = 2^{n_1} + \cdots + 2^{n_s}$$

is a positive integer with $n_1 > n_2 > \cdots > n_s \geq 0$, then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

The idea of using products of Rademacher's functions to define the Walsh system originated from Paley [12].

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 1/2^n), \\ 0, & \text{if } x \in [1/2^n, 1). \end{cases} \tag{1}$$

As usual, denote by $L(I)$ $(I := [0, 1))$ the set of all measurable functions defined on I, for which

$$\|f\|_1 = \int_0^1 |f(x)|\, dx < \infty.$$

The rectangular partial sums of Fourier series with respect to the Walsh system are defined by

$$S_n(f, x, y) = \sum_{m=0}^{n-1} \hat{f}(m)\, w_m(x),$$

where

$$\hat{f}(m) = \int_0^1 f(t)\, w_m(t)\, dt$$

is called the m-th Walsh-Fourier coefficient of function f.

The logarithmic means of Walsh-Fourier series is defined as follows

$$t_n(f, x) = \frac{1}{l_n} \sum_{i=1}^{n-1} \frac{S_i(f, x)}{n - i},$$

where

$$l_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

It is evident that

$$t_n(f,x,y) = \int_0^1 f(x \oplus t) F_n(t)\, dt,$$

where

$$F_n(t) = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k(t)}{n-k}$$

and $\oplus$ denotes the dyadic addition ([13, 9]).

For the maximal operator $t^*(f)$ we prove

Theorem 1. *Let $\{m_n : n \geq 1\}$ be sequence of positive integers for which*

$$\sum_{n=1}^{\infty} \frac{\log^2\left(m_n - 2^{[\log m_n]} + 1\right)}{\log m_n} < \infty.$$

Then the operator $t^*(f) := \sup_{n\geq 1} |t_{m_n}(f)|$ is weak type (1,1), i.e.

$$\|t^*(f)\|_{weak-L} := \sup_{\lambda} \lambda \operatorname{mes}(\{x : t^*(f,x) > \lambda\}) \leq c \|f\|_1 .$$

Corollary 1. *Let $\{m_n : n \geq 1\}$ be from Theorem 1. and $f \in L(I)$. Then*

$$t_{m_n}(f,x) \to f(x) \text{ a.e. as } n \to \infty.$$

Corollary 2. *Let $f \in L(I)$. Then*

$$t_{2^n}(f,x) \to f(x) \text{ a.e. as } n \to \infty.$$

Following the works of Gát [1, 3] the base of the proof of Theorem 1. is the following lemma.

Lemma 1. *Let $\{m_n : n \geq 1\}$ be from Theorem 1. Then*

$$\int_{2^{-k}}^1 \sup_{n \geq n(k)} |F_{m_n}(x)|\, dx < \infty,$$

where $n(k) = \min\{n : [\log m_n] \geq k\}$.

In order to prove Lemma 1., we shall need the following Lemmas.

Lemma 2 ([8]). *Let $1 \leq j < 2^n$. Then*

$$D_{2^n - j}(u) = D_{2^n}(u) - w_{2^n-1}(u) D_j(u).$$

Let us denote by K_j the jth Fejér kernel function, that is, $K_j = \frac{1}{j}\sum_{i=1}^{j} D_i$.

Lemma 3. *We have*

$$\int_{2^{-k}}^1 \sup_{n \geq 2^k} |K_n(x)|\, dx < \infty.$$

The proof can be found in work of Gát [1].

Lemma 4. *Let $2^n \le m < 2^{m+1}$. Then*

$$l_m F_m(x) = l_m D_{2^n}(x) - w_{2^n-1}(x) \sum_{j=1}^{2^n-2} \left(\frac{1}{m-2^n+j} - \frac{1}{m-2^n+j+1} \right) j K_j(x) - \frac{2^n-1}{m-1} w_{2^n-1}(x) K_{2^n-1}(x) + w_{2^n}(x) l_{m-2^n} F_{m-2^n}(x).$$

Proof of Lemma 4. It is evident that

$$l_m F_m(x) = \sum_{j=1}^{2^n} \frac{D_j(x)}{m-j} + \sum_{j=2^n+1}^{m-1} \frac{D_j(x)}{m-j} = I + II. \tag{2}$$

Using Abel transformation and Lemma 1. we have

$$\begin{aligned} I &= \sum_{j=0}^{2^n-1} \frac{D_{2^n-j}(x)}{m-2^n+j} = \frac{D_{2^n}(x)}{m-2^n} + \sum_{j=1}^{2^n-1} \frac{D_{2^n-j}(x)}{m-2^n+j} \\ &= \frac{D_{2^n}(x)}{m-2^n} + D_{2^n}(x) \left(\sum_{j=1}^{2^n-1} \frac{1}{m-2^n+j} \right) \\ &\quad - w_{2^n-1}(x) \sum_{j=1}^{2^n-1} \frac{D_j(x)}{m-2^n+j} = (l_m - l_{m-2^n}) D_{2^n}(x) \\ &\quad - w_{2^n-1}(x) \sum_{j=1}^{2^n-2} \left(\frac{1}{m-2^n+j} - \frac{1}{m-2^n+j+1} \right) j K_j(x) \\ &\quad - \frac{2^n-1}{m-1} w_{2^n-1}(x) K_{2^n-1}(x). \end{aligned} \tag{3}$$

Since

$$D_{j+2^n}(x) = D_{2^n}(x) + w_{2^n}(x) D_j(x), \quad j = 1, 2, \ldots, 2^n - 1,$$

for II we write

$$II = \sum_{j=1}^{m-2^n-1} \frac{D_{j+2^n}(x)}{m-2^n-j} = l_{m-2^n} D_{2^n}(x) + w_{2^n}(x) l_{m-2^n} F_{m-2^n}(x). \tag{4}$$

Combining (2)–(4) we complete the proof of Lemma 4. □

Lemma 5. *Let $\overline{\lim_{n\to\infty}} \frac{\log^2\left(m_n - 2^{[\log m_n]} + 1\right)}{\log m_n} < \infty$. Then*

$$\|F_{m_n}\|_1 \le c < \infty, \quad n = 1, 2, \ldots$$

Proof of Lemma 5. Since

$$\|F_n\|_1 \le \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{\|D_j\|_1}{n-j} \le \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{\ln(j+1)}{n-j} = O(l_n)$$

and

$$\|K_n\|_1 \le c < \infty, \quad n = 1, 2, \ldots \tag{5}$$

from Lemma 4. we have

$$\begin{aligned}
\|F_{m_n}\|_1 &\leq 1+\frac{1}{l_{m_n}}\sum_{j=1}^{2^{[\log m_n]}-2}\frac{\|K_j\|_1}{j} \\
&\quad +\left\|K_{2^{[\log m_n]}-1}\right\|_1+\frac{l_{m_n-2^{[\log m_n]}}}{l_{m_n}}\left\|F_{m_n-2^{[\log m_n]}}\right\|_1 \\
&= O\left(\frac{\log^2\left(m_n-2^{[\log m_n]}+1\right)}{\log m_n}\right)=O(1).
\end{aligned}$$

Lemma 5. is proved. □

Proof of Lemma 1. From Lemma 3. and by (1), (5) we have

$$\begin{aligned}
&\int_{2^{-k}}^{1}\sup_{n\geq n(k)}\left|F_{m_n}(x)\right|dx\leq c_1\int_{2^{-k}}^{1}\sup_{n\geq n(k)}\frac{1}{\log m_n}\sum_{j=1}^{2^{[\log m_n]}-2}\frac{\left|K_j(x)\right|}{j}dx \\
&\quad +c_2\int_{2^{-k}}^{1}\sup_{n\geq n(k)}\left|K_{2^{[\log m_n]}-1}(x)\right|dx \\
&\quad +\int_{2^{-k}}^{1}\sup_{n\geq n(k)}\frac{\log\left(m_n-2^{[\log m_n]}+1\right)}{\log m_n}\left|F_{m_n-2^{[\log m_n]}}(x)\right|dx \\
&\leq c_3+c_1\int_{2^{-k}}^{1}\sup_{n\geq n(k)}\frac{1}{\log m_n}\sum_{j=1}^{2^k-1}\frac{\left|K_j(x)\right|}{j}dx \\
&\quad +c_1\int_{2^{-k}}^{1}\sup_{n\geq n(k)}\frac{1}{\log m_n}\sum_{j=2^k}^{2^{[\log m_n]}-2}\frac{1}{j}\sup_{i\geq 2^k}\left|K_i(x)\right|dx \\
&\quad +\sum_{n=1}^{\infty}\frac{\log\left(m_n-2^{[\log m_n]}+1\right)}{\log m_n}\int_0^1\left|F_{m_n-2^{[\log m_n]}}(x)\right|dx \\
&\leq c_4+c_5\int_{2^{-k}}^{1}\sup_{i\geq 2^k}\left|K_i(x)\right|dx+\sum_{n=1}^{\infty}\frac{\log^2\left(m_n-2^{[\log m_n]}+1\right)}{\log m_n}\leq c_6<\infty.
\end{aligned}$$

Lemma 1. is proved. □

Given $u\in I$, let $I_k(u)$ denote a dyadic interval of length 2^{-k} which contains the point u.

In the sequel we prove that the maximal operator $t^*(f)$ is quasi-local. This reads as follows

Lemma 6. *Let* $f\in L(I)$, $\operatorname{supp} f\subset I_k(u)$ *and* $\int_{I_k(u)} f(x)\,dx=0$ *for some* $u\in I$. *Then*

$$\int_{2^{-k}}^{1} t^*(f,x)\,dx\leq c\left\|f\right\|_1.$$

Proof of Lemma 6. By the shift invariency of the Haar measure it can be supposed that $u = 0$. If $n \le n(k)$ then

$$t_{m_n}(f,x) = \int_I f(u) F_{m_n}(x \oplus u)\, du$$
$$= F_{m_n}(x) \int_0^{2^{-k}} f(u)\, du = 0.$$

Consequently, $n > n(k)$ can be supposed.

Then from Lemma 1 we have

$$\int_{2^{-k}}^{1} t^*(f,x)\, dx \le \int_0^{2^{-k}} |f(u)| \left(\int_{2^{-k}}^{1} \sup_{n \ge n(k)} |F_{m_n}(x \oplus u)|\, dx \right) du$$
$$\le c \|f\|_1 .$$

□

Proof of Theorem 1. As a consequence of Lemma 5. we have that the maximal operator $t^*(f)$ is of type (∞, ∞). Since the sublinear operator is quasi-local, then by standard argument [13] it follows that it is of weak type $(1,1)$. □

By making use of the well-known density argument due to Marcinkiewicz and Zygmund [10] we can show that Corollary 1. follows from Theorem 1.

References

[1] G. Gát. On the Fejér kernel function with respect to the Walsh-Paley system. *Acta Acad. Paed. Agriensis Sec. Matematicae*, 1987:105–110, 1987.

[2] G. Gát. Investigations of certain operators with respect to the Vilenkin system. *Acta Math. Hung.*, 61(1–2):131–149, 1993.

[3] G. Gát. On the almost everywhere convergence of Fejr means of functions on the group of 2-adic integers. *J. Approximation Theory*, 90(1):88–96, 1997.

[4] G. Gát and U. Goginava. Uniform and L-convergence of logarithmic means of cubical partial sums of double Walsh-Fourier series. *East. J. Approx.*, 10:1–22, 2004.

[5] G. Gát and U. Goginava. Uniform and L-convergence of logarithmic means of d-dimensional Walsh-Fourier series. *Bull. Georg. Acad. Sci.*, 170:234–236, 2004.

[6] G. Gát and U. Goginava. Uniform and L-convergence of logarithmic means of double Walsh-Fourier series. *Georgian Math. J.*, 12(1):75–88, 2005.

[7] Gát, G. and Goginava, U. Uniform and L-convergence of logarithmic means of Walsh-Fourier series. to appear in Acta. Math. Sinica.

[8] U. Goginava. Approximation properties of (C, α) means of double Walsh-Fourier series. *Anal. Theory Appl.*, 20(1):77–98, 2004.

[9] B. Golubov, A. Efimov, and V. Skvortsov. *Walsh series and transforms. Theory and applications. Transl. from the Russian by W.R. Wade.* Mathematics and Its Applications. Soviet Series. 64. Kluwer Academic Publishers Group, Dodrecht, etc., 1991.

[10] J. Marcinkiewicz and A. Zygmund. On the summability of double Fourier series. *Fundam. Math.*, 32:122–132, 1939.

[11] F. Móricz and A. Siddiqi. Approximation by Nrlund means of Walsh-Fourier series. *J. Approximation Theory*, 70(3):375–389, 1992.

[12] R. Paley. A remarkable series of orthogonal functions. I.–II. *Proc. Lond. Math. Soc., II. Ser.*, 34:241–264, 265–279, 1932.

[13] F. Schipp, W. Wade, and P. Simon. *Walsh series. An introduction to dyadic harmonic analysis. With the assistance from J. Pl.* Adam Hilger, Bristol, etc., 1990.

[14] P. Simon. Strong convergence of certain means with respect to the Walsh-Fourier series. *Acta Math. Hung.*, 49:425–431, 1987.

[15] O. Szász. On the logarithmic means of rearranged partial sums of a Fourier series. *Bull. Am. Math. Soc.*, 48:705–711, 1942.

[16] K. Yabuta. Quasi-Tauberian theorems, applied to the summability of Fourier series by Riesz's logarithmic means. *Tohoku Math. J., II. Ser.*, 22:117–129, 1970.

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Department of Mechanics and Mathematics,
Tbilisi State University,
Chavchavadze str. 1, Tbilisi 0128,
Georgia
E-mail address: z_goginava@hotmail.com