Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
22 (2006), 171–177
www.emis.de/journals
ISSN 1786-0091

ON CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

ZHIGANG WANG, CHUNYI GAO AND SHAOMOU YUAN

Abstract. In the present paper, the authors introduce a new subclass $\mathcal{K}_s(\alpha, \beta)$ of close-to-convex functions. Several coefficient inequalities, growth, distortion and covering theorem for this class are provided.

1. Introduction

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic and univalent in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. Let $\mathcal{K}$ and $\mathcal{S}^*$ denote the usual subclasses of $\mathcal{S}$ whose members are close-to-convex and starlike in $\mathcal{U}$, respectively. Also let $\mathcal{S}^*(\alpha)$ denote the class of starlike functions of order α, $0 \leq \alpha < 1$.

Sakaguchi [5] once introduced a class $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points, it consists of functions $f(z) \in \mathcal{S}$ satisfying

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \quad (z \in \mathcal{U}).$$

In a later paper, Gao and Zhou [1] discussed a class $\mathcal{K}_s$ of analytic functions related to the starlike functions, that is the subclass of $f(z) \in \mathcal{S}$ satisfying the following inequality

$$\Re\left\{\frac{z^2 f'(z)}{g(z)g(-z)}\right\} < 0 \quad (z \in \mathcal{U}),$$

where $g(z) \in \mathcal{S}^*(\frac{1}{2})$.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Close-to-convex functions, differential subordination.

This work was supported by the Scientific Research Fund of Hunan Provincial Education Department and the Hunan Provincial Natural Science Foundation (No. 05JJ30013) of People's Republic of China.

Let $f(z)$ and $F(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathcal{U}$, if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$, denoted by $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$ (see [3]).

In the present paper, we introduce the following class of analytic functions, and obtain some interesting results.

Definition 1. Let $\mathcal{K}_s(\alpha, \beta)$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$\left|\frac{z^2 f'(z)}{g(z)g(-z)} + 1\right| < \beta \left|\frac{\alpha z^2 f'(z)}{g(z)g(-z)} - 1\right| \quad (z \in \mathcal{U}), \tag{1.1}$$

where $0 \leq \alpha \leq 1,\ 0 < \beta \leq 1$ and $g(z) \in \mathcal{S}^*(\frac{1}{2})$.

It is easy to know that $\mathcal{K}_s(1,1) = \mathcal{K}_s$, so $\mathcal{K}_s(\alpha, \beta)$ is a generalization of $\mathcal{K}_s$.

In the present paper, we shall provide several coefficient inequalities, growth, distortion and covering theorem for the class $\mathcal{K}_s(\alpha, \beta)$.

2. Coefficient estimate

First we give a meaningful conclusion about the class $\mathcal{K}_s(\alpha, \beta)$.

Theorem 1. *A function $f(z) \in \mathcal{K}_s(\alpha, \beta)$ if and only if*

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec \frac{1+\beta z}{1-\alpha\beta z} \quad (z \in \mathcal{U}). \tag{2.1}$$

Proof. Let $f(z) \in \mathcal{K}_s(\alpha, \beta)$, then from (1.1) we have

$$\left|\frac{z^2 f'(z)}{-g(z)g(-z)} - 1\right|^2 < \beta^2 \left|\frac{\alpha z^2 f'(z)}{-g(z)g(-z)} + 1\right|^2.$$

Expanding it we get

$$(1-\alpha^2\beta^2)\left|\frac{z^2 f'(z)}{-g(z)g(-z)}\right|^2 - 2(1+\alpha\beta^2)\Re\left\{\frac{z^2 f'(z)}{-g(z)g(-z)}\right\} < \beta^2 - 1.$$

If $\alpha \neq 1$ or $\beta \neq 1$, we have

$$\left|\frac{z^2 f'(z)}{-g(z)g(-z)}\right|^2 - \frac{2(1+\alpha\beta^2)}{1-\alpha^2\beta^2}\Re\left\{\frac{z^2 f'(z)}{-g(z)g(-z)}\right\} + \left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2 < \frac{\beta^2-1}{1-\alpha^2\beta^2} + \left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2,$$

that is,

$$\left|\frac{z^2 f'(z)}{-g(z)g(-z)} - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right|^2 < \frac{\beta^2(1+\alpha)^2}{(1-\alpha^2\beta^2)^2},$$

or equivalently,

$$\left|\frac{z^2f'(z)}{-g(z)g(-z)}-\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right|<\frac{\beta(1+\alpha)}{1-\alpha^2\beta^2}.$$

This tells us that the value region of $G(z)=(z^2f'(z))/(-g(z)g(-z))$ is contained in the disk whose center is $(1+\alpha\beta^2)/(1-\alpha^2\beta^2)$ and radius is $[\beta(1+\alpha)]/(1-\alpha^2\beta^2)$. And we know that the function $\omega=q(z)=(1+\beta z)/(1-\alpha\beta z)$ maps the unit disk to the disk:

$$\left|\omega-\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right|<\frac{\beta(1+\alpha)}{1-\alpha^2\beta^2}.$$

Notice that $G(0)=q(0)$, $G(\mathcal{U})\subset q(\mathcal{U})$, and $q(z)$ is univalent in $\mathcal{U}$, we obtain the following conclusion

$$-\frac{z^2f'(z)}{g(z)g(-z)}\prec q(z)=\frac{1+\beta z}{1-\alpha\beta z}.$$

Conversely, let

$$-\frac{z^2f'(z)}{g(z)g(-z)}\prec\frac{1+\beta z}{1-\alpha\beta z},$$

then

$$-\frac{z^2f'(z)}{g(z)g(-z)}=\frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)}, \tag{2.2}$$

where $\omega(z)$ is analytic in $\mathcal{U}$, and $\omega(0)=0$, $|\omega(z)|<1$. By calculation we can easily obtain from (2.2) that

$$\left|\frac{z^2f'(z)}{g(z)g(-z)}+1\right|<\beta\left|\frac{\alpha z^2f'(z)}{g(z)g(-z)}-1\right|,$$

that is $f(z)\in\mathcal{K}_s(\alpha,\beta)$.

If $\alpha=\beta=1$, inequality (1.1) becomes

$$\left|\frac{z^2f'(z)}{-g(z)g(-z)}-1\right|<\left|\frac{z^2f'(z)}{-g(z)g(-z)}+1\right|.$$

It is obvious that

$$-\frac{z^2f'(z)}{g(z)g(-z)}\prec\frac{1+z}{1-z}.$$

This completes the proof of Theorem 1. □

Remark 1. From Theorem 1 we know that

$$\Re\left\{\frac{zf'(z)}{(-g(z)g(-z))/z}\right\}>0\quad(z\in\mathcal{U}), \tag{2.3}$$

because of

$$\Re\left\{\frac{1+\beta z}{1-\alpha\beta z}\right\}>0\quad(z\in\mathcal{U}).$$

In order to give the coefficient estimate of functions belonging to the class $\mathcal{K}_s(\alpha,\beta)$, we shall require the following two lemmas.

Lemma 1 ([1]). *Let* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(\frac{1}{2})$, *then*

$$\frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^*,$$

where

(2.4)
$$B_{2n-1} = 2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + (-1)^n 2b_{n-1}b_{n+1} + (-1)^{n+1} b_n^2 \quad (n = 2, 3, \ldots).$$

Remark 2. From Lemma 1 and inequality (2.3), we know that if $f(z) \in \mathcal{K}_s(\alpha,\beta)$, then $f(z)$ is a close-to-convex function. So $\mathcal{K}_s(\alpha,\beta)$ is a subclass of the class of close-to-convex functions.

Lemma 2 ([6]). *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$, *and satisfy the inequality*

$$\left|\frac{zf'(z)}{g(z)} - 1\right| < \beta \left|\frac{\alpha z f'(z)}{g(z)} + 1\right| \quad (z \in \mathcal{U}),$$

where $0 \le \alpha \le 1$ *and* $0 < \beta \le 1$, *then for* $n \ge 2$, *we have*

$$|na_n - b_n|^2 \le 2(1+\alpha\beta^2) \sum_{k=1}^{n-1} k\,|a_k|\,|b_k| \quad (|a_1| = |b_1| = 1). \tag{2.5}$$

Now we give the following theorem.

Theorem 2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$, *and satisfy the inequality (1.1), then for* $n \ge 2$, *we have*

$$|na_n - B_{2n-1}|^2 \le 2(1+\alpha\beta^2) \sum_{k=1}^{n-1} k\,|a_k|\,|B_{2k-1}| \quad (|a_1| = |B_1| = 1), \tag{2.6}$$

where B_{2n-1} *is given by (2.4).*

Proof. It is easy to know that inequality (1.1) can be written as

$$\left|\frac{zf'(z)}{(-g(z)g(-z))/z} - 1\right| < \beta \left|\frac{\alpha z f'(z)}{(-g(z)g(-z))/z} + 1\right|. \tag{2.7}$$

By Lemma 1, we have

$$\frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^* \subset \mathcal{S}.$$

Now, suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ and satisfy (2.7), so $f(z)$ and $(-g(z)g(-z))/z$ satisfy the condition of Lemma 2, thus, from (2.5), we can get (2.6). □

3. Sufficient condition

In this section, we give the sufficient condition for functions belonging to the class $\mathcal{K}_s(\alpha,\beta)$.

Theorem 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *be analytic in* $\mathcal{U}$*, if for* $0 \le \alpha \le 1$ *and* $0 < \beta \le 1$*, we have*

$$\sum_{n=2}^{\infty} n(1+\alpha\beta)\,|a_n| + \sum_{n=2}^{\infty}(1+\beta)\,|B_{2n-1}| \le (1+\alpha)\beta, \tag{3.1}$$

where B_{2n-1} *is given by (2.4), then* $f(z) \in \mathcal{K}_s(\alpha,\beta)$.

Proof. By Lemma 1, we have

$$\frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^*.$$

Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then for $z \in \mathcal{U}$, we have

$$\begin{aligned} M &= \left| zf'(z) - \frac{-g(z)g(-z)}{z} \right| - \beta \left| \alpha z f'(z) + \frac{-g(z)g(-z)}{z} \right| \\ &= \left| z + \sum_{n=2}^{\infty} n a_n z^n - z - \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \right| \\ &\quad - \beta \left| \alpha z + \sum_{n=2}^{\infty} n\alpha a_n z^n + z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \right|. \end{aligned}$$

Now, for $|z| = r < 1$, we have

$$\begin{aligned} M &\le \sum_{n=2}^{\infty} n\,|a_n|\,r^n + \sum_{n=2}^{\infty} |B_{2n-1}|\,r^{2n-1} \\ &\quad - \beta \left[(1+\alpha)r - \sum_{n=2}^{\infty} n\alpha\,|a_n|\,r^n - \sum_{n=2}^{\infty} |B_{2n-1}|\,r^{2n-1} \right] \\ &< \left[-(1+\alpha)\beta + \sum_{n=2}^{\infty} n(1+\alpha\beta)\,|a_n| + \sum_{n=2}^{\infty}(1+\beta)\,|B_{2n-1}| \right] r. \end{aligned}$$

From (3.1) we know that $M < 0$, thus we have

$$\left| \frac{z^2 f'(z)}{g(z)g(-z)} + 1 \right| < \beta \left| \frac{\alpha z^2 f'(z)}{g(z)g(-z)} - 1 \right| \quad (z \in \mathcal{U}),$$

that is $f(z) \in \mathcal{K}_s(\alpha,\beta)$, and the proof is complete. □

4. Growth, distortion and covering theorem

Finally, we provide the growth, distortion and covering theorem for the class $\mathcal{K}_s(\alpha, \beta)$. For the purpose of this section, assume that the function $\phi(z)$ is an analytic function with positive real part in the unit disk $\mathcal{U}$, $\phi(\mathcal{U})$ is convex and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$.

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ (z \in \mathcal{U}),$$

which satisfy the condition $\Re\{p(z)\} > 0$. A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*(\phi)$ if

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $\phi(z) \in \mathcal{P}$. The class $\mathcal{S}^*(\phi)$ and a corresponding convex class $\mathcal{K}(\phi)$ were defined by Ma and Minda [2]. And the results about the convex class $\mathcal{K}(\phi)$ can be easily obtained from the corresponding results of functions in $\mathcal{S}^*(\phi)$. The functions $k_{\phi n}(z)$ $(n = 2, 3, \ldots)$ defined by $k_{\phi n}(0) = k'_{\phi n}(0) - 1 = 0$ and

$$1 + \frac{zk''_{\phi n}(z)}{k'_{\phi n}(z)} = \phi(z^{n-1})$$

are important examples of functions in $\mathcal{K}(\phi)$. The functions $h_{\phi n}(z)$ satisfying $h_{\phi n}(z) = zk'_{\phi n}(z)$ are examples of functions in $\mathcal{S}^*(\phi)$. Write $k_{\phi 2}(z)$ simply as $k_\phi(z)$ and $h_{\phi 2}(z)$ simply as $h_\phi(z)$.

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{K}_s(\phi)$ if

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $g(z) \in \mathcal{S}^*(\frac{1}{2})$ and $\phi(z) \in \mathcal{P}$.

In order to prove our next theorem, we shall require the following lemma. The proof of Lemma 3 below is much akin to that of Theorem 7 in [4], here we omit the details.

Lemma 3. *Let* $\min_{|z|=r} |\phi(z)| = \phi(-r)$, $\max_{|z|=r} |\phi(z)| = \phi(r)$, $|z| = r < 1$. *If* $f(z) \in \mathcal{K}_s(\phi) \subset \mathcal{K}$, *then we have*

$$h'_\phi(-r) \le |f'(z)| \le h'_\phi(r), \ -h_\phi(-r) \le |f(z)| \le h_\phi(r),$$

and

$$f(\mathcal{U}) \supset \{\omega : |\omega| \le -h(-1)\}.$$

These results are sharp.

Theorem 4. *Let* $\min_{|z|=r} \left|\frac{1+\beta z}{1-\alpha\beta z}\right| = \psi(-r)$, $\max_{|z|=r} \left|\frac{1+\beta z}{1-\alpha\beta z}\right| = \psi(r)$, $|z| = r < 1$. *If* $f(z) \in \mathcal{K}_s(\alpha, \beta)$, *then we have*

$$h'_\psi(-r) \le |f'(z)| \le h'_\psi(r), \ -h_\psi(-r) \le |f(z)| \le h_\psi(r),$$

and

$$f(\mathcal{U}) \supset \{\omega : |\omega| \leq -h(-1)\}.$$

These results are sharp.

Proof. Setting $\phi(z) = \frac{1+\beta z}{1-\alpha\beta z}$ in Lemma 3, we can get the assertion of Theorem 4 easily. □

References

[1] C. Gao and S. Zhou. On a class of analytic functions related to the starlike functions. *Kyungpook Math. J.*, 45(1):123–130, 2005.

[2] W. C. Ma and D. Minda. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, Conf. Proc. Lecture Notes Anal., I, pages 157–169, Cambridge, MA, 1994. Internat. Press.

[3] C. Pommerenke. *Univalent functions*. Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV.

[4] V. Ravichandran. Starlike and convex functions with respect to conjugate points. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 20(1):31–37 (electronic), 2004.

[5] K. Sakaguchi. On a certain univalent mapping. *J. Math. Soc. Japan*, 11:72–75, 1959.

[6] T. V. Sudharsan, P. Balasubrahmanyam, and K. G. Subramanian. On functions starlike with respect to symmetric and conjugate points. *Taiwanese J. Math.*, 2(1):57–68, 1998.

Received April 24, 2005; September 30, 2005 revised.

College of Mathematics and Computing Science,
Changsha University of Science and Technology,
Changsha, Hunan 410076,
People's Republic of China
E-mail address: zhigwang@163.com