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MAXIMAL (C, α, β) OPERATORS OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

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Abstract. The main aim of this paper is to prove that for the boundedness of the maximal operator $\sigma_*^{\alpha,\beta}$ from the Hardy space $H_p\left(I^2\right)$ to the space $L_p\left(I^2\right)$ the assumption $p > \max\{1/(\alpha+1), 1/(\beta+1)\}$ is essential.

We denote the set of non-negative integers by $\mathbf{N}$. For a set $X \neq \emptyset$ let X^2 be its Cartesian product $X \times X$ taken with itself. By a dyadic interval in $I := [0,1)$ we mean one of the form $\left[l2^{-k}, (l+1)\,2^{-k}\right)$ for some $k \in \mathbf{N}$, $0 \leq l < 2^k$. Given $k \in \mathbf{N}$ and $x \in [0,1)$, let $I_k(x)$ denote the dyadic interval of length 2^{-k} which contains the point x. The Cartesian product of two dyadic intervals is said to be a rectangle. Clearly, the dyadic rectangle of area $2^{-n} \times 2^{-m}$ containing $(x^1, x^2) \in I^2$ is given by $I_{n,m}(x^1, x^2) := I_n(x^1) \times I_m(x^2)$. We also use the notation $\operatorname{mes}(A)$ for the Lebesgue measure of any measurable set A.

Let $r_0(x)$ be a function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2), \\ -1, & \text{if } x \in [1/2, 1), \end{cases}$$
$$r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \ n \geq 1 \text{ and } x \in [0,1).$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e. $w_0(x) = 1$ and if

$$n = 2^{n_1} + \cdots + 2^{n_r}$$

is a positive integer with $n_1 > n_2 > \cdots > n_r$ then

$$w_n(x) = r_{n_1}(x) \cdots r_{n_r}(x).$$

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The Walsh-Dirichlet kernel is defined by

$$D_n(x)=\sum_{k=0}^{n-1}w_k(x).$$

Recall that

$$D_{2^n}(x)=\begin{cases}2^n, & \text{if } x\in[0,2^{-n}),\\ 0, & \text{if } x\in[2^{-n},1).\end{cases}$$

The Kronecker product $(w_{n,m}:n,m\in\mathbf{N})$ of two Walsh systems is said to be the two-dimensional Walsh system. Thus

$$w_{n,m}\left(x^1,x^2\right):=w_n\left(x^1\right)w_m\left(x^2\right).$$

The partial sums of the two-dimensional Walsh-Fourier series are defined as follows:

$$S_{n,m}f\left(x^1,x^2\right)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\widehat{f}(i,j)\,w_{i,j}\left(x^1,x^2\right),$$

where the number

$$\widehat{f}(i,j)=\int_I f\left(u^1,u^2\right)w_{i,j}\left(u^1,u^2\right)du^1du^2$$

is said to be the (i,j)th Walsh-Fourier coefficient of the function f.

The norm (or quasinorm) of the space $L_p(I^2)$ is defined by

$$\|f\|_p:=\left(\int_{I^2}\left|f\left(x^1,x^2\right)\right|^p dx^1dx^2\right)^{1/p}\qquad(0<p<+\infty).$$

The σ-algebra generated by the dyadic rectangles $\{I_{n,m}(x^1,x^2):x,y\in I\}$ will be denoted by $F_{n,m}\,(n,m\in\mathbf{N})$, more precisely,

$$F_{n,m}=\sigma\left\{\left[k2^{-n},(k+1)2^{-n}\right)\times\left[l2^{-m},(l+1)2^{-m}\right):0\le k<2^n,0\le l<2^m\right\},$$

where $\sigma(A)$ denotes the σ-algebra generated by an arbitrary set system A.

Denote by $f=\left(f^{(n,m)},n\in\mathbf{N}\right)$ two-parameter martingale with respect to $(F_{n,m},n,m\in\mathbf{N})$ (for details see, e.g. [6, 9]). The maximal function of a martingale f is defined by

$$f^*=\sup_{n,m\in\mathbf{N}}\left|f^{(n,m)}\right|.$$

In case $f\in L_1(I^2)$, the maximal function can also be given by

$$f^*\left(x^1,x^2\right)=\sup_{n,m\in N}\frac{1}{\operatorname{mes}\left(I_n(x^1)\times I_m(x^2)\right)}\left|\int_{I_n(x^1)\times I_m(x^2)}f(u,v)\,dudv\right|,$$

$(x^1,x^2)\in I^2$.

For $0 < p < \infty$ the Hardy martingale space $H_p(I^2)$ consists all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(I^2)$ then it is easy to show that the sequence $(S_{2^n,2^m}(f) : n, m \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n,m)} : n, m \in \mathbf{N})$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i,j) = \lim_{k,l\to\infty} \int\limits_{I^2} f^{(k,l)}\left(x^1, x^2\right) w_i(x^1) w_j(x^2) dx^1 dx^2.$$

The Walsh-Fourier coefficients of the function $f \in L_1(I^2)$ are the same as the ones of the martingale $(S_{2^n,2^m}(f) : n, m \in \mathbf{N})$ obtained from the function f.

The (C, α, β) means of the two-dimensional Walsh-Fourier series of the martingale f is given by

$$\sigma_{n,m}^{\alpha,\beta}(f, x^1, x^2) = \frac{1}{A_{n-1}^{\alpha}} \frac{1}{A_{m-1}^{\beta}} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j} f\left(x^1, x^2\right),$$

where

$$A_n^{\alpha} := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$$

for any $n \in \mathbf{N}, \alpha \neq -1, -2, \dots$. It is known ([10]) that $A_n^{\alpha} \sim n^{\alpha}$.

For the martingale f we consider the maximal operator

$$\sigma_*^{\alpha,\beta} f = \sup_{n,m} |\sigma_{n,m}^{\alpha,\beta}(f, x^1, x^2)|.$$

The (C, α) kernel defined by

$$K_n^{\alpha}(x) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{n} A_{n-j}^{\alpha-1} D_k(x).$$

In the one-dimensional case, Fine [1] proved that the (C, α) means $\sigma_n^{\alpha} f$ of a function $f \in L(I)$ converge a.e. to f as $n \to \infty$. The maximal operator $\sigma_*^{\alpha} f := \sup_n |\sigma_n^{\alpha} f|$ $(0 < \alpha < 1)$ of the (C, α) means of the Walsh-Paley Fourier series was investigated by Weisz [8]. In his paper Weisz proved the boundedness of $\sigma_*^{\alpha} : H_p \to L_p$ when $p > 1/(1+\alpha)$. The author [3] showed that in Theorem of Weisz the assumption $p > 1/(\alpha+1)$ is essential. In particular, we proved that the maximal operator σ_*^{α} of the (C, α) means of the Walsh-Paley Fourier series is not bounded from the Hardy space $H_{1/(\alpha+1)}(I)$ to the space $L_{1/(\alpha+1)}(I)$.

For double Walsh-Fourier series it is known [5] that the (C, α, β) means $\sigma_{n,m}^{\alpha,\beta} f \to f$ in L_p norm as $n, m \to \infty$ whenever $f \in L_p(I^2)$ for some $1 \leq p < \infty$.

On the other hand, in 1992 Móricz, Schipp and Wade [4] proved with respect to the Walsh-Paley system that

$$\sigma_{n,m}f = \frac{1}{nm}\sum_{i=1}^{n}\sum_{k=1}^{m} S_{i,k}(f) \to f$$

a.e. for each $f \in L\log^+ L([0,1)^2)$, when $\min\{n,m\} \to \infty$. In 2000 Gát proved [2] that the theorem of Móricz, Schipp and Wade above can not be improved. Namely, let $\delta : [0,+\infty) \to [0,+\infty)$ be a measurable function with property $\lim_{t\to\infty}\delta(t) = 0$. Gát proved [2] the existence of a function $f \in L^1(I^2)$ such that $f \in L\log^+ L\delta(L)$, and $\sigma_{n,m}f$ does not converge to f a.e. as $\min\{n,m\} \to \infty$. That is, the maximal convergence space for the $(C,1)$ means of two-dimensional partial sums is $L\log^+ L(I^2)$. Weisz [7] investigated the maximal operator of (C,α,β) means of two-dimensional Walsh-Fourier series and proved that the maximal operator $\sigma_*^{\alpha,\beta}f$ is bounded from $H_p(I^2)$ to $L_p(I^2)$ if $1/(1+\alpha), 1/(1+\beta) < p < \infty$. In [7] Weisz conjectured that for the boundedness of the maximal operator $\sigma_*^{\alpha,\beta}$ from the Hardy space $H_p(I)$ to the space $L_p(I)$ the assumption $p > 1/(\alpha+1), 1/(1+\beta)$ is essential. We give answer to the question and prove that the maximal operator $\sigma_*^{\alpha,\beta}$ of the (C,α,β) $(0 < \alpha \leq \beta \leq 1)$ means of the two-dimensional Walsh-Fourier series is not bounded from the Hardy space $H_{1/(\alpha+1)}(I^2)$ to the space $L_{1/(\alpha+1)}(I^2)$. The following is true.

Theorem 1. *Let* $0 < \alpha \leq \beta \leq 1$. *Then the maximal operator* $\sigma_*^{\alpha,\beta}$ *of the* (C,α,β) *means of the two-dimensional Walsh-Fourier series is not bounded from the Hardy space* $H_{1/(\alpha+1)}(I^2)$ *to the space* $L_{1/(\alpha+1)}(I^2)$.

In order to prove Theorem 1 we need the following lemma.

Lemma 1. *(*[3]*) Let* $n \in \mathbf{N}$ *and* $0 < \alpha \leq 1$. *Then*

$$\int_I \max_{1\leq N\leq 2^n}\left(A_{N-1}^{\alpha}\left|K_N^{\alpha}(x)\right|\right)^{1/(\alpha+1)} dx \geq c(\alpha)\frac{n}{\log(n+2)}.$$

Proof of Theorem 1. Let

$$f_n\left(x^1,x^2\right) := \left[D_{2^{n+1}}\left(x^1\right) - D_{2^n}\left(x^1\right)\right] w_{2^n-1}\left(x^2\right).$$

Since

$$\widehat{f}_n(\nu,\mu) = \int_I \left[D_{2^{n+1}}\left(u^1\right) - D_{2^n}\left(u^1\right)\right] w_{\nu}\left(u^1\right) du^1 \int_I w_{2^n-1}\left(u^2\right) w_{\mu}\left(u^2\right) du^2$$

$$= \begin{cases} 1, & \text{if } \nu = 2^n,\ldots,2^{n+1}-1, \mu = 2^n-1, \\ 0, & \text{otherwise,} \end{cases}$$

we can write

$$(1)\quad S_{i,j}f_n\left(x^1,x^2\right) = \sum_{\nu=0}^{i-1}\widehat{f}_n\left(\nu,2^n-1\right) w_{\nu}\left(x^1\right) w_{2^n-1}\left(x^2\right)$$

$$= \begin{cases} \left[D_i\left(x^1\right) - D_{2^n}\left(x^1\right)\right] w_{2^n-1}\left(x^2\right), & \text{if } i = 2^n+1, \ldots, 2^{n+1}-1, j \geq 2^n, \\ f_n\left(x^1, x^2\right), & \text{if } i \geq 2^{n+1},\ j \geq 2^n, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$f_n^*\left(x^1, x^2\right) = \sup_{i,j} \left|S_{2^i,2^j} f_n\left(x^1, x^2\right)\right| = \left|f_n\left(x^1, x^2\right)\right|,$$

$$\left\|f_n\right\|_{H_p} = \left\|f_n^*\right\|_p = \left\|D_{2^n}\right\|_p = 2^{n(1-1/p)}. \tag{2}$$

Let $1 \leq N < 2^n$. Then from (1) we obtain

$$\begin{aligned} &\sigma^{\alpha}_{2^n+N,2^{n+1}} f_n\left(x^1, x^2\right) = \\ &= \frac{1}{A^{\alpha}_{2^n+N-1}} \frac{1}{A^{\beta}_{2^{n+1}-1}} \left|\sum_{i=1}^{2^n+N} \sum_{j=1}^{2^{n+1}} A^{\alpha-1}_{2^n+N-i} A^{\beta-1}_{2^{n+1}-j} S_{i,j} f_n\left(x^1, x^2\right)\right| \\ &= \frac{1}{A^{\alpha}_{2^n+N-1}} \frac{1}{A^{\beta}_{2^{n+1}-1}} \left|\sum_{i=2^n+1}^{2^n+N} \sum_{j=2^n}^{2^{n+1}} A^{\alpha-1}_{2^n+N-i} A^{\beta-1}_{2^{n+1}-j} S_{i,j} f_n\left(x^1, x^2\right)\right| \\ &= \frac{1}{A^{\alpha}_{2^n+N-1}} \frac{1}{A^{\beta}_{2^{n+1}-1}} \times \\ &\quad \times \left|\sum_{i=2^n+1}^{2^n+N} \sum_{j=2^n}^{2^{n+1}} A^{\alpha-1}_{2^n+N-i} A^{\beta-1}_{2^{n+1}-j} \left[D_i\left(x^1\right) - D_{2^n}\left(x^1\right)\right] w_{2^n-1}\left(x^2\right)\right| \\ &\geq \frac{c(\alpha,\beta)}{2^{n\alpha} 2^{n\beta}} \left|\sum_{i=1}^{N} A^{\alpha-1}_{N-i} \left[D_{i+2^n}\left(x^1\right) - D_{2^n}\left(x^1\right)\right]\right| \left|\sum_{j=0}^{2^n} A^{\beta-1}_{2^n-j}\right| \\ &\geq \frac{c(\alpha,\beta)}{2^{n\alpha}} \left|\sum_{i=1}^{N} A^{\alpha-1}_{N-i} \left[D_{i+2^n}\left(x^1\right) - D_{2^n}\left(x^1\right)\right]\right| \\ &= \frac{c(\alpha,\beta)}{2^{n\alpha}} \left|\sum_{i=1}^{N} A^{\alpha-1}_{N-i} D_i\left(x^1\right)\right|. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma^{\alpha,\beta}_* f_n\left(x^1, x^2\right) &\geq \sup_{1 \leq N \leq 2^n} \left|\sigma^{\alpha}_{2^n+N,2^{n+1}} f_n\left(x^1, x^2\right)\right| \\ &\geq \frac{c(\alpha,\beta)}{2^{n\alpha}} \sup_{1 \leq N \leq 2^n} \left|\sum_{i=1}^{N} A^{\alpha-1}_{N-i} D_i\left(x^1\right)\right|. \end{aligned}$$

Then from Lemma 1 and (2) we get

$$\frac{\left\|\sigma^{\alpha,\beta}_* f_n\right\|_{1/(\alpha+1)}}{\left\|f_n\right\|_{H_{1/(\alpha+1)}}} \geq \frac{c(\alpha,\beta)}{2^{n\alpha} 2^{-n\alpha}} \left(\int_I \sup_{1 \leq N \leq 2^n} \left(A^{\alpha}_{N-1} \left|K^{\alpha}_N(x)\right|\right)^{1/(\alpha+1)} dx\right)^{\alpha+1}$$

$$\geq c\,(\alpha,\beta)\left(\frac{n}{\log\,(n+2)}\right)^{\alpha+1}\to\infty \text{ as } n\to\infty.$$

Theorem 1 is proved. □

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