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A SPECIAL NONLINEAR CONNECTION IN SECOND ORDER GEOMETRY

NICOLETA BRINZEI

Abstract. We show that, for mechanical system with external forces, the equations of deviations of solution curves of the corresponding Lagrange equations, determine a nonlinear connection on the second order tangent bundle. In particular, Jacobi equations in Finsler and Riemann spaces determine such a nonlinear connection.

1. Introduction

As shown in [27], nonlinear connections on bundles can be a powerful tool in integrating systems of differential equations. A way of obtaining them is that of deriving them from the respective systems of DE's, in particular, from variational principles, [2], [16], [15]. For instance, an ODE system of order 2 on a manifold M induces a nonlinear connection on its tangent bundle TM. A remarkable example is here the Cartan nonlinear connection of a Finsler space, which has the property that its autoparallel curves correspond to geodesics of the base manifold:

$$\frac{\delta y^i}{dt} := \frac{dy^i}{dt} + N^i{}_j y^j = 0.$$

Further, an ODE system of order three determines a nonlinear connection on the second order tangent (jet) bundle $T^2M = J_0^2(\mathbb{R}, M)$. For instance, Craig-Synge equations (R. Miron, [16])

$$\frac{d^3x^i}{dt^3} + 3!G^i(x, \dot{x}, \ddot{x}) = 0,$$

lead to:

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a) Miron's connection:

$$M_{(1)}{}^{i}_{j} = \frac{\partial G^i}{\partial y^{(2)j}}, \; M_{(2)}{}^{i}_{j} = \frac{1}{2}\left(S M_{(1)}{}^{i}_{j} + M_{(1)}{}^{i}_{m} M_{(1)}{}^{m}_{j} \right), \tag{1}$$

where $S = y^i \frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^i} - 3G^i \frac{\partial}{\partial y^{(2)i}}$ is a semispray on T^2M.

b) Bucătaru's connection

$$M_{(1)}{}^{i}{}_{j} = \frac{\partial G^i}{\partial y^{(2)j}}, M_{(2)}{}^{i}{}_{j} = \frac{\partial G^i}{\partial y^j}.$$

With respect to the last one, if G^i are the coefficients of a spray on T^2M (i.e., 3-homogeneous functions), then the Craig-Synge equations can be interpreted as:

$$\frac{\delta y^{(2)i}}{dt} = 0, \tag{2}$$

where $\frac{\delta y^{(2)i}}{dt} := \frac{dy^{(2)i}}{dt} + M_{(1)}{}^{i}{}_{j}\frac{dy^j}{dt} + M_{(2)}{}^{i}{}_{j}\frac{dx^j}{dt}$.

In Miron's and Bucătaru's approaches, nonlinear connections on T^2M are obtained from a Lagrangian of order 2, $L(x, \dot{x}, \ddot{x})$, by computing the first variation of its integral of action.

Here, we propose a different approach, which, we consider, could be at least as interesting as the above one from the point of view of Mechanics - namely, we start with a first order Lagrangian $L(x, \dot{x})$ and compute its second variation.

This way, for a mechanical system $(M, L(x, \dot{x}), F(x, \dot{x}))$ with external force field F, we obtain a nonlinear connection on T^2M, with respect to which the equations of deviations of evolution curves have a simple invariant form.

As a remark, our nonlinear connection is also suitable for modelling the solutions of a (globally defined) ODE system, not necessarily attached to a certain Lagrangian, together with the deviations of these solutions.

More precisely, in the following our aims are:

(1) to obtain the Jacobi equations for the trajectories

$$\frac{\delta y^i}{dt} = \frac{1}{2}F^i(x, y)$$

(for extremal curves of a 2-homogeneous Lagrangian $L(x, \dot{x})$ in presence of external forces).

(2) to build a nonlinear connection such that:

$$w \in \mathcal{X}(M) \text{ Jacobi field along } c \Leftrightarrow \frac{\delta w^{(2)i}}{dt} = 0,$$

where $\frac{d}{dt}$ denotes directional derivative with respect to $\dot{c}$ and

$$\frac{\delta w^{(2)i}}{dt} = \frac{1}{2}\frac{d^2 w^i}{dt^2} + \underset{(1)}{M}{}^i{}_j \frac{dw^j}{dt} + \underset{(2)}{M}{}^i{}_j w^j.$$

For $F = 0$, this nonlinear connection has as additional properties:

I. In Finsler spaces M, c is a geodesic of M if and only if its extension T^2M is horizontal.

II. A vector field w along a geodesic c on M is parallel along c if and only if $\frac{\delta w^i}{dt} = 0$.

Throughout the paper, by 'differentiable' or 'smooth' we mean $\mathcal{C}^\infty$-differentiable.

2. Tangent bundle of first and second order

Let M be a real differentiable manifold of dimension n and class $\mathcal{C}^\infty$; the coordinates of a point $x \in M$ in a local chart (U, ϕ) will be denoted by $\phi(x) = \left(x^i\right)$, $i = 1, \ldots, n$. Let (TM, π, M) be its tangent bundle and (x^i, y^i) the coordinates of a point in a local chart.

The *2-tangent bundle* (T^2M, π^2, M) is the space of jets of order two at 0 of all smooth functions $f\colon (-\varepsilon, \varepsilon) \to M$, $t \mapsto (f^i(t))$, on $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$, ([19]-[24], [16], [10]).

In a local chart, a point p of T^2M will have the coordinates $(x^i, y^i, y^{(2)i})$. This is,

$$x^i = f^i(0), \quad y^i = \dot{f}^i(0), \quad y^{(2)i} = \frac{1}{2}\ddot{f}^i(0), \quad i = 1, \ldots, n,$$

for some f as above. Then, $\left(T^2M, \pi^2, M\right)$ is a differentiable manifold of class $\mathcal{C}^\infty$ and dimension $3n$, and TM can be identified with a submanifold of T^2M. The local coordinate changes induced by local coordinate changes on M are, [16], [19]-[24],

$$\begin{aligned}
\widetilde{x}^i &= \widetilde{x}^i\left(x^1, \ldots, x^n\right), \ \det\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) \neq 0 \\
\widetilde{y}^i &= \frac{\partial \widetilde{x}^i}{\partial x^j} y^j \\
2\widetilde{y}^{(2)i} &= \frac{\partial \widetilde{y}^i}{\partial x^j} y^j + 2\frac{\partial \widetilde{y}^i}{\partial y^j} y^{(2)j}.
\end{aligned} \tag{3}$$

For a curve $c\colon [0,1] \to M$, $t \mapsto (x^i(t))$ on the base manifold M, let us denote:

- by $\widehat{c}$ its *extension* to the tangent bundle TM :

$$\widehat{c}\colon [0,1] \to M, t \mapsto (x^i(t), \dot{x}^i(t));$$

along $\widehat{c}$, there holds:

$$y^i = \dot{x}^i(t), \quad i = 1, \ldots, n;$$

- by $\widetilde{c}$ its *extension to* T^2M:

$$\widetilde{c}\colon [0,1] \to T^2M, \qquad t \mapsto (x^i(t), \dot{x}^i(t), \frac{1}{2}\ddot{x}^i(t));$$

along such an extension curve, there holds

$$y^i(t) = \dot{x}^i(t), \quad y^{(2)i}(t) = \frac{1}{2}\ddot{x}^i(t), \quad i = 1, \ldots, n.$$

A tensor field on TM (or T^2M) is called a *distinguished tensor field*, or simply, a *d-tensor field* if, under a change of local coordinates induced by a change of coordinates on the base manifold M, its components transform by the same rule as the components of a corresponding tensor field on M, [16].

3. Nonlinear connections on TM

Let (TM, π, M) be the tangent bundle of a differentiable manifold M as above and (x^i, y^i) the coordinates of a point $p \in TM$ in a local chart. For simplicity, we shall also denote $(x, y) = (x^i, y^i)_{i=\overline{1,n}}$.

Let $d\pi\colon T(TM) \to TM$ denote the tangent linear mapping of the projection $\pi\colon TM \to M$ and $V(TM) = \ker d\pi$, the *vertical subbundle* of $T(TM)$. Its fibres generate the *vertical distribution* V on TM of local dimension n, $V\colon p \in TM \mapsto V(p) \subset T_p(TM)$, locally spanned by $\{\frac{\partial}{\partial y^i}\}$.

A *nonlinear (Ehresmann) connection on* TM, [16], [18], is a distribution $N\colon p \in TM \mapsto N(p) \subset T_p(TM)$, which is supplementary to the vertical distribution:

$$T_p(TM) = N(p) \oplus V(p), \quad \forall p \in TM. \tag{4}$$

Let

$$B = \left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\},$$

where:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^j}, \quad i = 1, \ldots, n, \tag{5}$$

denote a local adapted basis to the direct decomposition (4). The quantities $N^i{}_j = N^i{}_j(x, y)$, [16], [18], are called the *coefficients* of the nonlinear connection N.

With respect to local coordinate changes on TM induced by changes of local coordinates $(x^i) \mapsto (\tilde{x}^i)$ on the base manifold M, $\frac{\delta}{\delta x^i}$ transform by the rule: $\frac{\delta}{\delta x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\delta}{\delta \widetilde{x}^j}$.

The dual basis of B is $B^* = \{dx^i, \delta y^i\}$, given by

(6) $$\delta y^i = dy^i + N^i{}_j dx^j.$$

With respect to changes of local coordinates on TM induced by local coordinate changes on M, there holds: $\delta\tilde{y}^i = \dfrac{\partial\tilde{x}^i}{\partial x^j}\delta y^j$.

Any vector field $X \in \mathcal{X}(TM)$ is represented in the local adapted basis as

(7) $$X = X^{(0)i}\frac{\delta}{\delta x^i} + X^{(1)i}\frac{\partial}{\partial y^i},$$

where the components $X^{(0)i}\dfrac{\delta}{\delta x^i}$ and $X^{(1)i}\dfrac{\partial}{\partial y^i}$ are d-vector fields.

Similarly, a 1-form $\omega \in \mathcal{X}^*(TM)$ will be decomposed as the sum of two d-1-forms:

(8) $$\omega = \omega_i^{(0)} dx^i + \omega_i^{(1)}\delta y^i.$$

In particular, if $\widehat{c}$: $t \to (x^i(t), y^i(t))$ is an extension curve to TM, then its tangent vector field is expressed in the adapted basis as

(9) $$\dot{\widehat{c}} = \frac{dx^i}{dt}\frac{\delta}{\delta x^i} + \frac{\delta y^i}{dt}\frac{\partial}{\partial y^i}.$$

In our further considerations, an important role will be played by the notions of *semispray* and *spray*, [25], [10]. A semispray $S \in \mathcal{X}(TM)$ is a vector field locally described in the natural basis by $S = y^i\dfrac{\partial}{\partial x^i} - 2G^i(x,y)\dfrac{\partial}{\partial y^i}$, where the functions G^i (called the *coefficients* of the semispray) obey, with respect to coordinate changes induced by a change of local coordinates $(x^i) \mapsto (\tilde{x}^i)$ on M, the rule: $2\tilde{G}^i = 2\dfrac{\partial\tilde{x}^i}{\partial x^j}G^j - \dfrac{\partial\tilde{y}^i}{\partial x^j}y^j$, $i = 1,\dots,n$. If G^i are 2-homogeneous functions in y, then the semispray is called a spray.

As shown by Grifone, [12], a semispray (in particular, a spray) on M determines a nonlinear connection on TM.

Also, evolution curves of mechanical systems with external forces, can be described in terms of semisprays on TM, (R. Miron, [15]):

Proposition 1. *Let $L = L(x,\dot{x})$ be a nondegenerate Lagrangian:*

$$\det\left(\frac{\partial^2 L}{\partial y^i \partial y^j}\right) \neq 0,$$

and $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^i\partial y^j}$, the induced (Lagrange) metric tensor. Then, the equations of evolution of a mechanical system with the Lagrangian L and the external force field $F = F_i(x,\dot{x})dx^i$ are

(10) $$\frac{d^2x^i}{dt^2} + 2G^i(x,\dot{x}) = \frac{1}{2}F^i(x,\dot{x}),$$

where

$$2G^i = \frac{1}{2} g^{is} \left(\frac{\partial^2 L}{\partial y^s \partial x^j} y^j - \frac{\partial L}{\partial x^s} \right),$$

yield a semispray (called the canonical semispray of the Lagrange space (M, L)*) and* $F^i = g^{ij} F_j$, $i = 1, \ldots, n$.

In the following, we shall use the above results in the case when G is a *spray*; this is, we shall have

$$2G^i = \frac{\partial G^i}{\partial y^j} y^j.$$

Then, [12], [2], [5], [18], the quantities

$$N^i_{\ j} = \frac{\partial G^i}{\partial y^j}$$

are the coefficients of a nonlinear connection on TM. Moreover, $N^i_{\ j} = N^i_{\ j}(x, y)$ are 1-homogeneous in y.

With respect to the above nonlinear connection, equations (10) take the form:

$$\frac{\delta y^i}{dt} = \frac{1}{2} F^i, \quad i = 1, \ldots, n. \tag{11}$$

In particular, if there are no external forces, this is, if $F^i = 0$, then the extremal curves $t \mapsto x^i(t)$ of the Lagrangian L have horizontal extensions and vice-versa: horizontal extension curves $\widehat{c}$ project onto solution curves of the Euler-Lagrange equations of L.

4. Nonlinear connections on T^2M

Let $d\pi^2 \colon T\left(T^2M\right) \to TM$ denote the tangent linear mapping of the projection $\pi^2 \colon T^2M \to M$ and $V\left(T^2M\right) = \ker d\pi^2$, the vertical subbundle of $T\left(T^2M\right)$. Its fibres generate the *vertical distribution* V on T^2M of local dimension $2n$, $V \colon p \in T^2M \mapsto V(p) \subset T_p\left(T^2M\right)$, locally spanned by $\left\{ \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^{(2)i}} \right\}$.

In the same way, if the projection $\pi^2_1 \colon T^2M \to TM$ is given by

$$\left(x^i, y^i, y^{(2)i}\right) \mapsto \left(x^i, y^i\right),$$

then $V_2 := \ker d\pi^2_1$ generates a distribution $V_2 \colon p \in T^2M \mapsto V_2(p) \subset T_p\left(T^2M\right)$ of local dimension n, locally spanned by $\left\{ \frac{\partial}{\partial y^{(2)i}} \right\}$.

Then, at any $p \in T^2M$, there exists a chain of vector spaces

$$V_2(p) \subset V(p) \subset T_p\left(T^2M\right).$$

Let us consider the $\mathcal{F}\left(T^2M\right)$-linear mapping $J \colon \mathcal{X}\left(T^2M\right) \to \mathcal{X}\left(T^2M\right)$,

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial y^{(2)i}}, \quad J\left(\frac{\partial}{\partial y^{(2)i}}\right) = 0, \tag{12}$$

called the *2-tangent structure* on T^2M. J is globally defined on T^2M and $\operatorname{Im} J = V$, $KerJ = V_2$, $J(V) = V_2$.

A *nonlinear connection on* T^2M, [16], is a distribution on T^2M, $N\colon p \in T^2M \to N(p) \subset T_p(T^2M)$, such that

$$T_p(T^2M) = N_0(p) \oplus V(p), \quad \forall p \in T^2M. \tag{13}$$

By setting $N_1(p) := J(N_0(p))$, $\forall p \in T^2M$, we get:

- the *horizontal distribution* $N_0\colon p \mapsto N(p)$;
- the v_1-*distribution* $N_1\colon p \mapsto N_1(p)$;
- the v_2-*distribution* $V_2\colon p \mapsto V_2(p)$, and there holds

$$T_p(T^2M) = N_0(p) \oplus N_1(p) \oplus V_2(p), \quad \forall p \in T^2M.$$

We denote by $h = v_0$, v_1 and v_2 the projectors corresponding to the above distributions.

Let $\mathcal{B}$ denote a local adapted basis to the decomposition (13):

$$\mathcal{B} = \left\{ \delta_{(0)i} := \frac{\delta}{\delta x^i}, \quad \delta_{(1)i} := \frac{\delta}{\delta y^i}, \quad \delta_{(2)i} := \frac{\delta}{\delta y^{(2)i}} \right\},$$

this is, $N_0 = \operatorname{Span}(\delta_{(0)i})$, $N_1 = \operatorname{Span}(\delta_{(1)i})$, $V_2 = Span(\delta_{(2)i})$. The elements of the adapted basis are locally expressed as

$$\begin{aligned} \delta_{(0)i} &= \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^j} - \underset{(2)}{N}{}^j_i \frac{\partial}{\partial y^{(2)j}} \\ \delta_{(1)i} &= \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^{(2)j}} \\ \delta_{(2)i} &= \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}. \end{aligned} \tag{14}$$

With respect to changes of local coordinates on T^2M, induced by changes $(x^i) \mapsto (\tilde{x}^i)$ of local coordinates on the base manifold M, for $\delta_{(\alpha)i}$, $\alpha = 0, 1, 2$, there holds: $\delta_{(\alpha)i} = \dfrac{\partial \tilde{x}^j}{\partial x^i} \widetilde{\delta}_{(\alpha)j}$.

The dual basis of $\mathcal{B}$ is $\mathcal{B}^* = \{dx^i, \delta y^i, \delta y^{(2)i}\}$, given by

$$\begin{aligned} \delta y^{(0)i} &= dx^i, \\ \delta y^i &= dy^i + \underset{(1)}{M}{}^i_j dx^j, \\ \delta y^{(2)i} &= dy^{(2)i} + \underset{(1)}{M}{}^i_j dy^j + \underset{(2)}{M}{}^i_j dx^j. \end{aligned} \tag{15}$$

The above $\delta y^{(\alpha)i}$, $\alpha = 0, 1, 2$, $i = 1, \ldots, n$, are d-1-forms on T^2M.

The quantities $\underset{(1)}{N}{}^j_i$, $\underset{(2)}{N}{}^j_i$ are called the *coefficients* of the nonlinear connection N, while $\underset{(1)}{M}{}^i_j$ and $\underset{(2)}{M}{}^i_j$ are called its *dual* coefficients. The link between the two sets of coefficients is, [16]:

$$M^i_{(1)\,j} = N^i_{(1)\,j},\ M^i_{(2)\,j} = N^i_{(2)\,j} + N^i_{(1)\,f} N^f_{(1)\,j}. \tag{16}$$

In the following, the next result will be very useful to us:

Theorem 2 ([16],[19]-[24]). *1. A transformation of coordinates (3) on the differentiable manifold T^2M implies the following transformation of the dual coefficients of a nonlinear connection*

$$\begin{aligned} \frac{\partial \widetilde{x}^i}{\partial x^k} M^k_{(1)\,j} &= \widetilde{M}^i_{(1)\,k} \frac{\partial \widetilde{x}^k}{\partial x^j} + \frac{\partial \widetilde{y}^i}{\partial x^j} \\ \frac{\partial \widetilde{x}^i}{\partial x^k} M^k_{(2)\,j} &= \widetilde{M}^i_{(2)\,k} \frac{\partial \widetilde{x}^k}{\partial x^j} + \widetilde{M}^i_{(1)\,k} \frac{\partial \widetilde{y}^k}{\partial x^j} + \frac{\partial \widetilde{y}^{(2)i}}{\partial x^j}. \end{aligned} \tag{17}$$

2. If on each domain of local chart on T^2M it is given a set of functions $\left(M^i_{(1)\,j}, M^i_{(2)\,j} \right)$, such that, with respect to (3), there hold the equalities (17), then there exists on T^2M a unique nonlinear connection N which has as dual coefficients the given set of functions.

In presence of a nonlinear connection, a vector field $X \in \mathcal{X}\left(T^2M\right)$ is represented in the local adapted basis as

$$X = X^{(0)i}\delta_{(0)i} + X^{(1)i}\delta_{(1)i} + X^{(2)i}\delta_{(2)i}, \tag{18}$$

with the three right terms (which are d-vector fields) belonging to the distributions N, N_1 and V_2 respectively.

A 1-form $\omega \in \mathcal{X}^*\left(T^2M\right)$ will be decomposed as

$$\omega = \omega^{(0)}_i dx^i + \omega^{(1)}_i \delta y^i + \omega^{(2)}_i \delta y^{(2)i}. \tag{19}$$

Similarly, a tensor field $T \in \mathcal{T}^r_s\left(T^2M\right)$ can be split with respect to (13) into components, which are d-tensor fields.

In particular, if $\widetilde{c}$: $t \to (x^i(t), y^i(t), y^{(2)i}(t))$ is an extension curve, then its tangent vector field is expressed in the adapted basis as

$$\dot{\widetilde{c}} = \frac{dx^i}{dt}\delta_{(0)i} + \frac{\delta y^i}{dt}\delta_{(1)i} + \frac{\delta y^{(2)i}}{dt}\delta_{(2)i}. \tag{20}$$

Our goal is to give a precise meaning to the equality $v_2(\dot{\widetilde{c}}) = 0$.

5. Berwald linear connection on T^2M

Let $G^i = G^i(x,y)$ be the coefficients of a spray on TM, and

$$N^i_{\ j}(x,y) = \frac{\partial G^i}{\partial y^j},$$

the coefficients of the induced nonlinear connection (on TM).

Let also

$$L^i{}_{jk}(x,y) = \frac{\partial N^i{}_j}{\partial y^k} = \frac{\partial^2 G^i}{\partial y^j \partial y^k},$$

the local coefficients of the induced Berwald linear connection on TM, [16].

Now, let on T^2M, a linear connection defined by $\underset{(1)}{N}{}^i{}_j = N^i{}_j(x, y^{(1)})$ as above, and arbitrary $\underset{(2)}{N}{}^i{}_j = \underset{(2)}{N}{}^i{}_j(x,y,y^{(2)})$. The *Berwald connection* on T^2M , [8], is the linear connection defined by

$$\text{(21)} \qquad \begin{aligned} &D_{\delta_{(0)k}}\delta_{(\alpha)j} = L^i{}_{jk}\delta_{(\alpha)i},\\ &D_{\delta_{(\beta)k}}\delta_{(\alpha)j} = 0, \quad \beta = 1,2, \ \alpha = 0,1,2. \end{aligned}$$

This is, with the notations in [16], the coefficients of the Berwald linear connection are $B\Gamma(N) = (L^i{}_{jk}, 0, 0)$.

For extensions $\widetilde{c}$ to T^2M of curves $c\colon [0.1] \to M$, we can express the v_1 component of the tangent vector field $\dot{\widetilde{c}}$, given by $\dfrac{\delta y^i}{dt}$ (the *geometric acceleration,* [13]) by means of the Berwald covariant derivative:

$$\text{(22)} \qquad \frac{Dy^i}{dt} := D_{\dot{\widetilde{c}}}\, y^i = \frac{\delta y^i}{dt}, \quad i = 1, \ldots, n.$$

Let $\mathbb{T}$ denote its torsion tensor, and:

$$R^i{}_{jk} = v_1\mathbb{T}(\delta_{(0)k}, \delta_{(0)j}) = \delta_{(0)k}N^i{}_j - \delta_{(0)j}N^i{}_k,$$

its $v_1(h,h)$ components.

Also, let $\mathbb{R}$ be the curvature tensor; then

$$\begin{aligned} R_j{}^i{}_{kl} &= \delta_{(0)l}L^i{}_{jk} - \delta_{(0)k}L^i{}_{jl} + L^m_{jk}L^i{}_{ml} - L^m_{jl}L^i{}_{mk},\\ P_j{}^i{}_{kl} &= \delta_{(1)l}L^i{}_{jk} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \end{aligned}$$

where $R_j{}^i{}_{kl}\delta_{(0)i} = h\mathbb{R}(\delta_{(0)l}, \delta_{(0)k})$, $P_j{}^i{}_{kl}\delta_{(0)i} = h\mathbb{R}(\delta_{(1)l}, \delta_{(0)k})$, define its only nonvanishing local components, [16].

Taking into account that $L^i{}_{jk}$ do not depend on $y^{(2)}$ and that $G^i = G^i(x,y)$ are 2-homogeneous in y, it follows:

$$\text{(23)} \qquad y^j R_j{}^i{}_{kl} = R^i{}_{kl}.$$

From the 2-homogeneity of G^i, we also have

$$\text{(24)} \qquad P_j{}^i{}_{kl}y^l = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}y^l = 0; \quad P_j{}^i{}_{kl}y^j = P_j{}^i{}_{kl}y^k = 0.$$

6. Jacobi equations for systems with external forces

Let us suppose that we know *a priori* a nonlinear connection on the first order tangent bundle TM, with (1-homogeneous) coefficients $N^i{}_j(x,y) = \dfrac{\partial G^i}{\partial y^j}$, coming from a spray on TM.

Let $c\colon [0,1] \to M$, $t \mapsto x^i(t)$ be a curve on M, such that x^i are solutions for the system of ODE's (10):

$$\frac{\delta \dot{x}^i}{dt} = \frac{1}{2} F^i(x, \dot{x}),$$

where F^i are the components of a d-vector field on M.

Let $\alpha\colon [0,1] \times (-\varepsilon, \varepsilon) \to M$, $(t,u) \mapsto (\alpha^i(t,u))$ denote a variation of c (not necessarily with fixed endpoints): $\alpha^i(t,0) = x^i(t)$, $\forall t \in [0,1]$,

$$y^i = \frac{\partial \alpha^i}{\partial t}|_{u=0} = \frac{dx^i}{dt}$$

the components of the tangent vector field of c and

$$w^i(t) = \frac{\partial \alpha^i}{\partial u}|_{u=0}$$

the components of the deviation vector field attached to the variation α. Let $\widetilde{\alpha}$ denote the following extension of α to the second order tangent bundle T^2M:

$$\widetilde{\alpha}\colon [0,1] \times (-\varepsilon, \varepsilon) \to T^2M, (t,u) \mapsto (\alpha^i(t,u), \frac{\partial \alpha^i}{\partial t}(t,u), \frac{1}{2}\frac{\partial^2 \alpha^i}{\partial t^2}(t,u)) \tag{25}$$

and

$$\alpha^i_t = \frac{\partial \alpha^i}{\partial t}, \quad \alpha^i_u = \frac{\partial \alpha^i}{\partial u}.$$

We have:

- $h\left(\dfrac{\partial \widetilde{\alpha}}{\partial t}\right) = \alpha^i_t \delta_{(0)i}$, $h\left(\dfrac{\partial \widetilde{\alpha}}{\partial u}\right) = \alpha^i_u \delta_{(0)i}$;
- $\alpha^i_t(t,0) = y^i(t)$, $\alpha^i_u(t,0) = w^i$, $\forall t \in [0,1]$.

Let us denote $\dfrac{D}{\partial t} = D_{\frac{\partial \widetilde{\alpha}}{\partial t}}$ and $\dfrac{D}{\partial u} = D_{\frac{\partial \widetilde{\alpha}}{\partial u}}$ the covariant derivations with respect to the Berwald connection on T^2M. Then:

$$\begin{aligned}
\frac{D\alpha^i_t}{\partial t} &= \frac{\partial \alpha^i_t}{\partial t} + N^i{}_j(\alpha, \alpha_t)\alpha^j_t, \\
\frac{D\alpha^i_t}{\partial u} &= \frac{\partial \alpha^i_t}{\partial u} + N^i{}_j(\alpha, \alpha_t)\alpha^j_u, \\
\frac{D\alpha^i_u}{\partial t} &= \frac{\partial \alpha^i_u}{\partial t} + N^i{}_j(\alpha, \alpha_t)\alpha^j_u;
\end{aligned} \tag{26}$$

(the covariant derivatives are taken 'with reference vector $\dfrac{\partial \widetilde{\alpha}}{\partial t}$', [5]).

By commuting partial derivatives of α^i, we have $\dfrac{\partial \alpha_t^i}{\partial u} = \dfrac{\partial \alpha_u^i}{\partial t}$, hence that the last two covariant derivatives (26) coincide:

$$\frac{D\alpha_t^i}{\partial u} = \frac{D\alpha_u^i}{\partial t},$$

which is,

$$\frac{D}{\partial u}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) = \frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial u}\right).$$

By applying $D_{\frac{\partial \widetilde{\alpha}}{\partial t}}$ again to the above equality, we get:

$$\frac{D}{\partial t}\frac{D}{\partial u}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) = \frac{D}{\partial t}\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial u}\right). \tag{27}$$

In the left hand side, we can commute covariant derivatives by means of the curvature tensor of D :

$$\frac{D}{\partial t}\frac{D}{\partial u}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) = R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) + \frac{D}{\partial u}\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right)$$
$$+ D_{\left[\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right]}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right).$$

But, $\left[\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right]$ is 0, hence the last term in the above relation vanishes and (27) becomes

$$\frac{D}{\partial t}\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial u}\right) = R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) + \frac{D}{\partial u}\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right). \tag{28}$$

Moreover, at $u = 0$, we have $h\dfrac{\partial \widetilde{\alpha}}{\partial t}|_{u=0} = \alpha_t^i(t,0)\delta_{(0)i} = y^i\delta_{(0)i}$, and by means of (11), we get

$$\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right)|_{u=0} = \frac{Dy^i}{\partial t}\delta_{(0)i} = \frac{1}{2}F^i\delta_{(0)i} =: \frac{1}{2}F$$

(where F is a d-vector field on T^2M). Then, (28) becomes

$$\frac{D^2}{\partial t^2}\left(h\frac{\partial \widetilde{\alpha}}{\partial u}|_{u=0}\right) = R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right)|_{u=0} + \frac{1}{2}D_uF. \tag{29}$$

At $u = 0$, we also have $h\dfrac{\partial \widetilde{\alpha}}{\partial u} = w^i\delta_{(0)i}$. In local writing, by evaluating

$$R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right)$$

and taking into account (24), we obtain

$$R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right)|_{u=0} = y^hy^kR_h{}^i{}_{jk}w^j\delta_{(0)i}.$$

We have thus proved

Proposition 3. *The components of the deviation vector field* $w^i = \frac{\partial \alpha^i}{\partial u}|_{u=0}$ *of the trajectories*

$$\frac{\delta y^i}{dt} = \frac{1}{2} F^i(x, y), \tag{30}$$

satisfy, with respect to the Berwald linear connection on T^2M*, the Jacobi-type equation*

$$\frac{D^2 w^i}{dt^2} = \frac{1}{2} \frac{DF^i}{\partial u}|_{u=0} + y^h y^k R_h{}^i{}_{jk} w^j. \tag{31}$$

The above generalizes the usual Jacobi equation, in the case of mechanical systems with external forces.

7. Nonlinear connection

In natural coordinates, (31) becomes:

$$\begin{aligned} &\frac{d^2 w^i}{dt^2} + \left(2N^i{}_j - \frac{1}{2} \frac{\partial F^i}{\partial y^j} \right) \frac{dw^j}{dt} \\ &\quad + \left(\frac{d}{dt}(N^i{}_j) + N^i{}_k N^k{}_j - y^h y^k R_h{}^i{}_{jk} + L^i{}_{kj} \frac{1}{2} F^k - \frac{1}{2} \frac{\partial F^i}{\partial x^j} \right) w^j = 0. \end{aligned} \tag{32}$$

Taking into account (23), we have $R^i{}_{hjk} y^h = R^i{}_{jk}$. Also, $L^i{}_{kj} = \frac{\partial N^i{}_k}{\partial y^j}$, hence the above equality can be seen as:

$$\begin{aligned} &\frac{d^2 w^i}{dt^2} + \left(2N^i{}_j - \frac{1}{2} \frac{\partial F^i}{\partial y^j} \right) \frac{dw^j}{dt} \\ &\quad + \left(\mathbb{C}(N^i{}_j) + N^i{}_k N^k{}_j - y^k R^{\ i}_{\ jk} + \frac{1}{2} \frac{\partial N^i{}_k}{\partial y^j} F^k - \frac{1}{2} \frac{\partial F^i}{\partial x^j} \right) w^j = 0, \end{aligned}$$

where

$$\mathbb{C} = y^k \frac{\partial}{\partial x^k} + 2y^{(2)k} \frac{\partial}{\partial y^k}.$$

There holds:

Theorem 4. (1) *The quantities*

$$\begin{aligned} \underset{(1)}{M}{}^i{}_j(x, y) &= \frac{1}{2} \left(2N^i{}_j - \frac{1}{2} \frac{\partial F^i}{\partial y^j} \right), \\ \underset{(2)}{M}{}^i{}_j(x, y, y^{(2)}) &= \frac{1}{2} \left(\mathbb{C}(N^i{}_j) + N^i{}_k N^k{}_j - y^k R^{\ i}_{\ jk} \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial N^i{}_k}{\partial y^j} F^k - \frac{1}{2} \frac{\partial F^i}{\partial x^j} \right) \end{aligned} \tag{33}$$

are the dual coefficients of a nonlinear connection on T^2M*.*

(2) *With respect to this nonlinear connection, the extensions of deviation vector fields attached to (10) have vanishing* v_2*-components:*

$$\frac{1}{2}\frac{d^2w^i}{dt^2} + \underset{(1)}{M}{}^i{}_j\frac{dw^j}{dt} + \underset{(2)}{M}{}^i{}_j w^j = 0.$$

Proof. 1): In the equation (31), both the left hand side and the right hand side are components of d-vector fields; by a direct computation, it follows that, with respect to local coordinate changes (3) on T^2M, the quantities $\underset{(1)}{M}{}^i{}_j$ and $\underset{(2)}{M}{}^i{}_j$ obey the rules of transformation (17) of the dual coefficients of a nonlinear connection on T^2M.

2): The deviation vector field attached to the variation $\tilde{\alpha}$ in (25) is

$$W = \frac{\partial\tilde{\alpha}}{\partial u}|_{u=0} \equiv \left\{\frac{\partial\alpha^i}{\partial u}\frac{\partial}{\partial x^i} + \frac{\partial}{\partial u}\left(\frac{\partial\alpha^i}{\partial t}\right)\frac{\partial}{\partial y^i} + \frac{1}{2}\frac{\partial}{\partial u}\left(\frac{\partial^2\alpha^i}{\partial t^2}\right)\frac{\partial}{\partial y^{(2)i}}\right\}|_{u=0}$$

$$= w^i\frac{\partial}{\partial x^i} + \frac{dw^i}{dt}\frac{\partial}{\partial y^i} + \frac{1}{2}\frac{d^2w^i}{dt^2}\frac{\partial}{\partial y^{(2)i}}.$$

In the adapted basis $(\delta_{(0)i}, \delta_{(1)i}, \delta_{(2)i})$, this yields:

$$W = w^i\delta_{(0)i} + \frac{\delta w^i}{dt}\delta_{(1)i} + \frac{\delta w^{(2)i}}{dt}\delta_{(2)i},$$

where $\frac{\delta w^i}{dt} = \frac{dw^i}{dt} + \underset{(1)}{M}{}^i{}_j(x,y)w^j$ and

$$\frac{\delta w^{(2)i}}{dt} = \frac{1}{2}\frac{d^2w^i}{dt^2} + \underset{(1)}{M}{}^i{}_j(x,y)\frac{dw^j}{dt} + \underset{(2)}{M}{}^i{}_j(x,y,y^{(2)})w^j.$$

Taking into account (33), the Jacobi equation (32) is re-expressed as: □

$$\frac{\delta w^{(2)i}}{dt} = 0.$$

In presence of the above nonlinear connection, the extension W to T^2M of any Jacobi field on M, corresponding to trajectories (10) in presence of external forces, belongs to the $N_0 \oplus N_1$ distribution.

8. Deviations of geodesics

Let us examine the particular case when $F = 0$. Let TM be endowed with a spray with coefficients $G^i = G^i(x,y)$ and $N^i{}_j = \frac{\partial G^i}{\partial y^j}$, the coefficients of the associated nonlinear connection on TM.

If $F = 0$, then we deal with deviations of autoparallel curves (called *geodesics*)

$$\frac{\delta y^i}{dt} = 0.$$

We get

$$\underset{(1)}{M}{}^i{}_j = N^i{}_j,$$

$$\underset{(2)}{M}{}^i{}_j = \frac{1}{2}(\mathbb{C}(N^i{}_j) + N^i{}_k N^k_j - y^j R^i{}_{jk});$$

taking into account that, in our approach, $\underset{(1)}{M}{}^i{}_j$ do not depend on $y^{(2)}$, we notice that, in the case $F = 0$, our nonlinear connection only differs by the term $-y^j R^i{}_{jk}$ from Miron's one (1), [16].

Remark 5. Along an extension curve $\widetilde{c}$: $[0,1] \to T^2M$, $t \mapsto (x^i(t), y^i(t) = \dot{x}^i(t), y^{(2)i}(t) = \frac{1}{2}\ddot{x}^i(t))$ there hold the equalities

$$\frac{\delta y^i}{dt} = \frac{Dy^i}{dt}, \quad \frac{\delta y^{(2)i}}{dt} = \frac{D^2 y^i}{dt^2},$$

where $\frac{D}{dt}$ denotes the covariant derivative associated to the Berwald connection on T^2M. For these curves, taking into account the equalities $y^j y^k R^i{}_{jk} = 0$ (which can be obtained by direct calculation), it follows that, with the assumptions made at the beginning of this section, $\frac{\delta y^i}{dt}$ and $\frac{\delta y^{(2)i}}{dt}$ have the same values as those obtained for the connection (1). Still, along general curves γ on T^2M, the value of $v_2(\dot{\gamma})$ does no longer coincide with that one obtained with respect to (1).

Remark 6. Also, for a vector field w along the projection c of $\widetilde{c}$ onto M, we have

$$\frac{\delta w^i}{dt} = \frac{Dw^i}{dt}.$$

Conclusions:

(1) c is a geodesic if and only if its extension to T^2M is horizontal.
(2) For a vector field w along a geodesic c on M, we have:
   (a) $\frac{\delta w^i}{dt} = 0$, if and only if w is parallel along $\dot{c} = y$.
   (b) $\frac{\delta w^{(2)i}}{dt} = 0$ if and only if w is a Jacobi field along c.

In the case $F = 0$, we should mention some related results and approaches:

In the geometry of TM: In the case when the base manifold M is endowed with a linear connection ∇, a linear connection on the tangent bundle TM, with similar properties to those of (33) is given by the *complete lift* ∇^C *of* ∇ (cf. [28] and [10]). Namely, in the two cited monographs, it is shown that, if a curve $\bar{\sigma}$: $[0,1] \to TM$, $t \mapsto (x^i(t), w^i(t))$ is a geodesic with respect to ∇^C, then its projection $\sigma\colon t \mapsto (x^i(t))$ onto M is a geodesic with respect to ∇ and $X(t) = w^i(t)\dfrac{\partial}{\partial x^i}$ is a Jacobi field along σ.

In the geometry of T^2M: In presence of a linear connection ∇ on M, C. Dodson and M. Radivoiovici, [11] built a covariant derivation law $\bar{\nabla} : \mathcal{X}(M) \times \Gamma(T^2M) \to \Gamma(T^2M)$ for sections of the second order tangent bundle (regarded as a vector bundle over M) and used it in order to define a nonlinear connection in the frame bundle of order 2 $L^{(2)}M$. In the case when ∇ is torsion-free, the covariant derivative $\bar{\nabla}_v X$, where $v = \dfrac{\partial \alpha}{\partial u}|_{u=0}$, and $X \equiv \left(\dfrac{\partial \alpha}{\partial t}, \dfrac{D}{dt}\dfrac{\partial \alpha}{\partial t}\right)$ (with our notations in Section 6) would yield our $\left(\dfrac{\delta w^i}{dt}, \dfrac{\delta w^{(2)i}}{dt}\right)$. Still, in the cited paper, it is not established any link between the defined connection and the Jacobi equation on M.

The novelty of our approach consists in relating the v_2-distribution on T^2M to deviations of geodesics of the base manifold.

9. External forces in Finsler-locally Minkowskian spaces

Another interesting particular case is that of Finsler-locally Minkowskian spaces (whose geodesics are straight lines). Let $(M, L(y))$ be a Finsler-locally Minkowskian space, [2], [5].

Then, $N^i{}_j = 0$, $L^i{}_{jk} = 0$ (for the Berwald connection), [2], [5]. In presence of an external force field, the evolution equations of a mechanical system will take the form

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}F^i(x, \dot{x}). \tag{34}$$

In this case, with the above notations, our nonlinear connection is given by

$$\underset{(1)}{M}{}^i{}_j = -\frac{1}{4}\frac{\partial F^i}{\partial y^j},$$
$$\underset{(2)}{M}{}^i{}_j = -\frac{1}{4}\frac{\partial F^i}{\partial x^j}.$$

This is, deviations of the evolution curves (34) can be written simply:

$$2\frac{\delta w^{(2)i}}{dt} \equiv \frac{d^2w^i}{dt^2} - \frac{1}{2}\frac{\partial F^i}{\partial y^j}\frac{dw^j}{dt} - \frac{1}{2}\frac{\partial F^i}{\partial x^j}w^j = 0.$$

The result holds valid for any globally defined system of ordinary differential equations of order 2 on M, of the form (34).

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Transilvania University,
Brasov, Romania
E-mail address: nico.brinzei@rdslink.ro