



# Computational complexity of necessary envy-freeness

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## ABSTRACT

We consider the fundamental problem of fairly allocating indivisible items when agents have strict ordinal preferences over individual items. We focus on the well-studied fairness criterion of necessary envy-freeness. For a constant number of agents, the computational complexity of the deciding whether there exists an allocation that satisfies necessary envy-freeness has been open for several years. We settle this question by showing that the problem is NP-complete even for three agents. Considering that the problem is polynomial-time solvable for the case of two agents, we provide a clear understanding of the complexity of the problem with respect to the number of agents.

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## 1. Introduction

When allocating items among agents, a natural and fundamental concern is fairness (Aziz, 2020a; Bouveret et al., 2016; Brams and Taylor, 1996; Moulin, 2003). We consider the setting in which agents have strict ordinal preferences over the items. The fairness concept we focus on is necessary envy-freeness (Aziz et al., 2015; Bouveret et al., 2010). An allocation satisfies necessary envy-freeness if for any two agents  $i$  and  $j$  with allocations  $I_i$  and  $I_j$ , there exists an injection  $f$  from  $I_j$  to  $I_i$  such that for each item  $x \in I_j$ , agent  $i$  prefers the item  $f(x)$  over  $x$ . This requirement has been referred to by different terms in the literature including responsive-set (RS) envy-freeness (Aziz et al., 2015), stochastic-dominance (sd) envy-freeness (Aziz et al., 2015), not possible envy-freeness (Brams et al., 2001) and itemwise envy-freeness (Brams et al., 2014).

Bouveret et al. (2010) considered the computational complexity of checking whether a complete necessary envy-free allocation exists or not; we will call this problem EXISTSNEF (precise definitions follow in Section 2). Bouveret et al. (2010) proved that EXISTSNEF is NP-complete even if the number of items is twice as much as the number of agents. They also showed that the problem is polynomial-time solvable when the number of agents is two. Since the work of Bouveret et al. (2010) in 2010, the complexity of the problem has been open for constant number of agents (Aziz et al., 2015; Bouveret et al., 2010), even though scenarios where the task is to find a fair allocation of items

among a small, fixed number of agents is of interest, and has been the focus of considerable research in the area (see, e.g., the papers Aziz, 2016; Brams et al., 2014; Brederbeck et al., 2021, 2019; Goldman and Procaccia, 2014).

In this paper, we resolve this open problem by showing that EXISTSNEF is NP-complete if the number of agents is a constant at least three. We first prove NP-completeness for the case of three agents in Section 3, and then for the case when the number of agents is a fixed integer at least three in Section 4. This completes our understanding of the computational complexity of EXISTSNEF as a function of the number of agents involved; see Table 1. We remark that our result for the case where the number of agents is exactly three was announced in conference paper (Aziz et al., 2016).

## 2. Preliminaries

Formally, an instance of our problem is a triple  $(N, I, L)$ , where  $N$  is a set of agents,  $I$  a set of indivisible items, and  $L$  is a collection of preference lists  $L^A$  for each agent  $A \in N$ . Each preference list  $L^A$  is a strict linear ordering over the set  $I$  of items.

An assignment  $\pi$  of items to agents is an allocation, and  $\pi$  is complete if it assigns each item of  $I$  to some agent. A complete allocation can be viewed as a partitioning of the items into  $|N|$  bundles with each bundle corresponding to an agent's allocation.

When reasoning about preferences over bundles of items, an agent may be required to express preferences over an exponential number of bundles. A compact way of expressing preferences over bundles is for agents to express preferences over individual items and then extend them over bundles of items with respect to the responsive set extension. In this notion, we say that an agent  $A$  prefers a set  $I_1$  of items over a set  $I_2$  of items if there exists an

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**Table 1**Complexity of EXISTSNEF on a set  $N$  of agents. Our result is in bold font.

	$ N  = 2$	Fixed $ N  \geq 3$	Unbounded $ N $
EXISTSNEF	in P (Bouveret et al., 2010)	<b>NP-complete</b> (Theorem 2)	NP-complete (Bouveret et al., 2010)

injection  $f$  from  $I_2$  to  $I_1$  such that for each item  $x \in I_2$ , agent  $A$  prefers the item  $f(x)$  over  $x$ . An allocation is *necessarily envy-free* (NEF) if each agent prefers its own set of items over any set of items allocated to some other agent. Note that a necessarily envy-free allocation is envy-free for all additive valuations consistent with the ordinal preferences.

**Example 1.** Suppose we have four items 1, 2, 3, and 4, and two agents  $A$  and  $B$  with the following preferences over the items.

$A: 1 \succ 2 \succ 3 \succ 4$

$B: 2 \succ 1 \succ 4 \succ 3$

In that case, the unique complete NEF allocation is the one in which  $A$  gets 1 and 3, while agent  $B$  gets 2 and 4.

The central problem we consider in this paper is the EXISTSNEF problem whose task is to decide whether a complete NEF allocation exists.

#### EXISTSNEF

**Input:** A triple  $(N, I, L)$  where  $N$  is a set of agents,  $I$  a set of items, and  $L$  is a collection of preference lists for each agent in  $N$ .

**Question:** Does there exist a complete NEF allocation for  $(N, I, L)$ ?

**Notation.** We let  $[h] = \{1, 2, \dots, h\}$  for any positive integer  $h$ . For a linear ordering  $L = (s_1, \dots, s_m)$  over a set  $S = \cup_{i \in [m]} \{s_i\}$  of items, for any indices  $i$  and  $j$  with  $1 \leq i \leq j \leq m$  we define  $L(i : j) = (s_i, s_{i+1}, \dots, s_j)$ . For  $X \subseteq S$ , we let  $L|_X$  be the restriction of  $L$  to  $X$ , and we write  $[L|_X]$  for the set of elements in  $L|_X$ , that is,  $[L|_X] = X$ .

The definition of necessary envy-freeness can be reformulated using Hall's theorem into the following equivalent form, which we will use throughout the paper. The characterization below is well known in the literature, see e.g., Aziz (2016, Lemma 1); we present a proof for completeness.

**Proposition 1.** For a given set  $N$  of agents, a set  $I$  of items, and a preference list  $L^A$  for each agent  $A \in N$ , an allocation  $\pi : I \rightarrow N$  is NEF if and only if for each pair of distinct agents  $A$  and  $B$ , and index  $i \in [|I|]$  we have:

$$|[L^A(1 : i)] \cap \pi^{-1}(A)| \geq |[L^A(1 : i)] \cap \pi^{-1}(B)|. \quad (1)$$

**Proof.** We prove the two directions of the claim separately. Assume first that  $\pi$  is a necessarily envy-free allocation for  $(N, I, L)$ . Consider two distinct agents  $A$  and  $B$ , and the preference list of  $A$  until the  $i$ th item, that is,  $L^A(1 : i)$ . Since  $\pi$  is NEF, there exists an injection  $f$  from  $\pi^{-1}(B)$  to  $\pi^{-1}(A)$  that to each item  $x$  allocated to  $B$  assigns an item  $f(x)$  allocated to  $A$  that  $A$  prefers to  $x$ . Consider any item  $x$  in  $L^A(1 : i)$  that  $\pi$  allocates to  $B$ . Since  $A$  prefers  $f(x)$  to  $x$ , we know that  $f(x)$ , allocated to  $A$  by  $\pi$ , is also in  $L^A(1 : i)$ . Since  $f$  is an injection, the number of items in  $L^A(1 : i)$  allocated to  $B$  by  $\pi$  is therefore at most the number of items in  $L^A(1 : i)$  allocated to  $A$  by  $\pi$ . Hence, Inequality (1) holds.

Assume now that  $\pi$  satisfies Inequality (1) for each two agents  $A$  and  $B$  and index  $i \in [|I|]$ ; we will prove that  $\pi$  is NEF. Consider two agents  $A$  and  $B$ , and the set of items they obtain under  $\pi$ ; our aim is to construct an injection  $f$  that to each item  $x \in \pi^{-1}(B)$  assigns an item  $f(x) \in \pi^{-1}(A)$  so that  $A$  prefers  $f(x)$  to  $x$ . We

construct a bipartite graph  $G$  over the vertex set  $\pi^{-1}(B) \cup \pi^{-1}(A)$  by connecting each item  $x \in \pi^{-1}(B)$  with all items in  $\pi^{-1}(A)$  that  $A$  prefers to  $x$ .

Clearly, if there is a complete matching  $M$  in  $G$ , then we can obtain an injection with the desired properties by assigning to each  $x \in \pi^{-1}(B)$  the item  $x'$  for which  $\{x, x'\} \in M$ . Hence, it suffices to prove that  $G$  admits a complete matching. To do so, we use Hall's theorem, and prove the existence of a complete matching by showing that  $|N_G(I')| \geq |I'|$  for each  $I' \subseteq \pi^{-1}(B)$  where  $N_G(I')$  denotes the neighborhood of  $I'$  in  $G$ . Let  $x^*$  be the least preferred item in  $I'$ , and suppose that  $x^*$  is the  $i^*$ th item in  $L^A$ . Observe that  $N_G(I')$  is then exactly the set  $\pi^{-1}(A) \cap L^A(1 : i^*)$ . Inequality (1) implies

$$|N_G(I')| = |\pi^{-1}(A) \cap L^A(1 : i^*)| \geq |\pi^{-1}(B) \cap L^A(1 : i^*)| \geq |I'|$$

where the last inequality follows from  $I' \subseteq \pi^{-1}(B) \cap L^A(1 : i^*)$ , implied by our definition of  $x^*$  and  $i^*$ . This proves the existence of a complete matching in  $G$  and, in turn, the necessary envy-freeness of  $\pi$ .  $\square$

Proposition 1 shows that a complete NEF allocation  $\pi$  must, for each  $i \in [|I|]$ , allocate at least as many items to  $A$  as to  $B$  from among  $A$ 's top  $i$  items (that is, from the set  $[L^A(1 : i)]$ ), for any two agents  $A$  and  $B$ . In particular, taking  $i = 1$  yields that  $\pi$  must assign each agent its most preferred item.

### 3. Result for exactly three agents

We start by determining the computational complexity of EXISTSNEF in the case when there are exactly three agents.

Before stating and proving our main result, Theorem 1, let us first give some intuition why a complete NEF allocation may be hard to find. By Proposition 1, each agent must be allocated its top-choice item in any complete NEF allocation. Hence, a natural approach would be to consider the requirements of Proposition 1 in an iterative manner, starting with the top-choice items and considering longer and longer prefixes of the preference lists at each step, maintaining throughout a “representative” set of allocations of the items appearing in the current prefixes. If we could keep the size of such a representative set small and, simultaneously, guarantee that at least one allocation in our representative set can be completed into a complete NEF allocation (assuming that such an allocation exists), then such an incremental algorithm would yield a possibility for solving EXISTSNEF efficiently.

Theorem 1 shows, however, that EXISTSNEF is unlikely to be polynomial-time solvable even for three agents. The intuition behind our NP-hardness proof builds on the flaw in the approach described in the above paragraph. First, there may be several partial allocations that respect the requirements of Proposition 1 for some index  $i$ , and in fact, it is not hard to see that the number of such allocations can grow exponentially in  $i$ . Second, as our reduction shows, selecting a relatively small subset of partial solutions among those that satisfy the requirements of Proposition 1 for some index  $i$ , so that we can safely disregard the remaining ones when considering larger indices, is not possible. It turns out that if such an approach were viable, then it could be used for determining the value of certain variables in a given Boolean formula (or to narrow down the set of possible truth assignments on them) without even knowing the formula itself. So the hardness of the problem lies in deciding how to allocate

those items that appear early in the agents' preference lists in a way that we will not regret our choices later on, when we allocate the less-desired items.

**Theorem 1.** EXISTSNEF, the problem of deciding whether a complete NEF allocation exists, is NP-complete for instances with three agents.

Containment in NP is trivial due to Proposition 1. We dedicate the rest of this section to showing the NP-hardness of our problem by a reduction from the NP-complete NOT-ALL-EQUAL 3SAT problem (Schaefer, 1978). The input for NOT-ALL-EQUAL 3SAT is a Boolean formula  $\varphi = c_1 \wedge \dots \wedge c_m$  in conjunctive normal form with variables  $x_1, \dots, x_n$ , where each clause contains three literals. The task is to find a truth assignment for  $\varphi$  such that each clause contains at least one true literal and at least one false literal; such an assignment is *valid*.

NOT-ALL-EQUAL 3SAT	
Input:	A Boolean formula $\varphi = c_1 \wedge \dots \wedge c_m$ in conjunctive normal form with variables $x_1, \dots, x_n$ , where each clause contains three literals.
Question:	Does there exist a valid truth assignment for $\varphi$ ?

We construct an instance  $(N, I, L)$  of EXISTSNEF with agent set  $N = \{A, B, C\}$  such that  $(N, I, L)$  admits a complete NEF allocation if and only if  $\varphi$  has a valid assignment.

**Construction.** Let  $\mu_i$  denote the number of occurrences of variable  $x_i$  in  $\varphi$  as a positive or negative literal; note  $\sum_{i=1}^n \mu_i = 3m$ . Without loss of generality we may assume that each  $\mu_i$  is an even number; this can be achieved by adding the clause  $(x_i \vee x_i \vee \neg x_i)$  for each variable  $x_i$  with an odd number of occurrences.

The set  $I$  of items is defined as follows; one can verify that  $|I| = 66m + 3$ . We will provide more information on the various type of items later on, when we define the preferences of agents; the definition below simply serves to provide a concise description of  $I$ .

$$I = (\{a_{1,0}^3, b_{1,0}^3, c_{1,0}^3\} \cup \bigcup_{k \in [m]} \{v_k, v'_k, v''_k\} \\ \cup \bigcup_{i \in [n], j \in [\mu_i]} \left( \{a_{i,j}, \beta_{i,j}, \gamma_{i,j}\} \cup \{a_{i,j}^h, b_{i,j}^h, c_{i,j}^h : h \in \{1, 2, 3\}\} \right. \\ \left. \cup \{\bar{a}_{i,j}, \bar{\beta}_{i,j}, \bar{\gamma}_{i,j}, \bar{b}_{i,j}, \bar{c}_{i,j}, \bar{c}_{i,j}, \bar{c}_{i,j}, \bar{c}_{i,j}, \bar{c}_{i,j}, \bar{c}_{i,j}\} \right)$$

We will define the preferences of agents through several types of “building blocks”. A *block* is a triple of lists where each *list* is a linearly ordered subset of  $I$ . Given two blocks  $L = (L_1, L_2, L_3)$  and  $L' = (L'_1, L'_2, L'_3)$  such that  $L_i$  and  $L'_i$  are disjoint for each  $i \in [3]$ , we define the *concatenation* of  $L$  and  $L'$  as  $L + L' = (L_1 + L'_1, L_2 + L'_2, L_3 + L'_3)$ , where  $L_i + L'_i$  denotes the (standard) concatenation of lists.

**Preference lists: a high-level view.** We begin with a single *initial block*  $I_0$ . Then, for each variable  $x_i$ ,  $i \in [n]$ , we define the following blocks. For each occurrence of  $x_i$  in  $\varphi$ , we construct a *literal block*: for some  $j \in [\mu_i]$ , we denote the literal block corresponding to the  $j$ th occurrence of variable  $x_i$  by  $X_{i,j}$ . We denote the concatenation  $X_{i,1} + \dots + X_{i,\mu_i}$  by  $Y_i$ . We also construct  $\mu_i/2$  *equivalence blocks*  $E_{i,2j}$  where  $j \in [\mu_i/2]$ , and we denote their concatenation  $E_{i,2} + \dots + E_{i,\mu_i}$  by  $F_i$ . See Fig. 1 for an illustration of the “super-blocks”  $Y_i$  and  $F_i$ .

Each literal block will represent the choice of a truth assignment for the given occurrence of a variable, as there will be two possible ways to allocate the items appearing in a given literal block to the agents. The equivalence blocks will ensure that these choices are consistent for a given variable  $x_i$ . Thus, the blocks

in  $Y_i$  and in  $F_i$  together represent the choice of a truth assignment for the variable  $x_i$ . The initial block  $I_0$  will be followed first by  $Y_1 + \dots + Y_n$  and then by  $F_1 + \dots + F_n$ .

Next, for each clause  $c_k$  of  $\varphi$ , we define a *validity block*  $V_k$ ; this block will make sure that any complete NEF allocation corresponds to a truth assignment that is valid for the clause  $c_k$ . Finally, we define a *closing block*  $Z$  whose sole function is to ensure that each preference list contains all items in  $I$ . The full preference lists of the agents, as illustrated in Fig. 2, are obtained by the concatenation

$$I_0 + Y_1 + \dots + Y_n + F_1 + \dots + F_n + V_1 + \dots + V_m + Z.$$

**Details of the blocks.** We give the definitions of the building blocks below. For better readability, we give each block as subsequences of the preference lists of the agents in  $N = \{A, B, C\}$ ; hence, a block  $(L_1, L_2, L_3)$  will be presented below in the form

$$\begin{aligned} A: & L_1 \\ B: & L_2 \\ C: & L_3. \end{aligned}$$

Whenever a block contains some list that does not fit into one row, it should be read row by row, proceeding from left to right within each row.

We further refine our blocks as follows: we define a *triad* as a group of three items contained in some list  $L^X[3t + 2 : 3t + 4]$  for some  $t \in \mathbb{Z}$  and  $X \in N$ . That is, disregarding the top item as well as the last two items in any of the preference lists, we divide the remainder into disjoint segments, each containing three consecutive items. As we have  $|I| = 66m + 3$  items, we obtain that each preference list contains  $22m = |I|/3 - 1$  triads; namely, the preference list  $L^X$  for some agent  $X \in N$  contains the triads  $L^X[3t + 2 : 3t + 4]$  for indices  $t \in \{0, \dots, 22m - 1\}$ . In the arguments below, it will be crucial to view the list contained in some block (other than the short blocks  $I_0$  and  $Z$ ) as sequences of triads.

*Initial block  $I_0$ :*

$$\begin{aligned} A: & a_{1,0}^3 \\ B: & b_{1,0}^3 \\ C: & c_{1,0}^3 \end{aligned}$$

Recall that as a direct consequence of Proposition 1, any complete NEF allocation gives each agent its most preferred item. Hence, we immediately obtain the following.

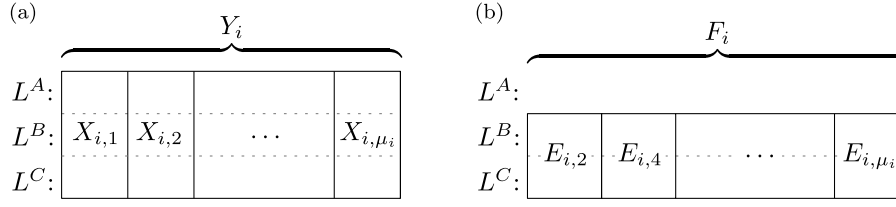
**Proposition 2.** Suppose that  $\pi$  is a complete NEF allocation for  $(N, I, L)$ . Then  $\pi(a_{1,0}^3) = A$ ,  $\pi(b_{1,0}^3) = B$ , and  $\pi(c_{1,0}^3) = C$ .

Let us now proceed with the details of literal blocks.

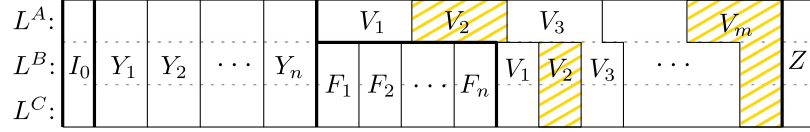
*Literal block  $X_{i,j}$  for some  $i \in [n]$  and  $j \in [\mu_i]$ :*

$$\begin{aligned} A: & b_{i,j-1}^3, c_{i,j-1}^3, a_{i,j}^1, & b_{i,j}^1, \bar{a}_{i,j}, \bar{\bar{a}}_{i,j}, & c_{i,j}^1, \beta_{i,j}, a_{i,j}^2, \\ & c_{i,j}^2, \bar{a}_{i,j}, \bar{\bar{a}}_{i,j}, & b_{i,j}^2, \gamma_{i,j}, a_{i,j}^3, & \bar{b}_{i,j}, \bar{a}_{i,j}, \bar{c}_{i,j} \\ B: & a_{i,j-1}^3, \bar{b}_{i,j}, \bar{\bar{b}}_{i,j}, & c_{i,j-1}^3, \alpha_{i,j}, b_{i,j}^1, & a_{i,j}^1, c_{i,j}^1, b_{i,j}^2, \\ & c_{i,j}^2, \bar{a}_{i,j}, \bar{\bar{a}}_{i,j}, & a_{i,j}^2, \gamma_{i,j}, b_{i,j}^3, & \bar{\bar{a}}_{i,j}, \bar{a}_{i,j}, \bar{b}_{i,j} \\ C: & a_{i,j-1}^3, \bar{b}_{i,j}, \bar{\bar{b}}_{i,j}, & b_{i,j-1}^3, \alpha_{i,j}, c_{i,j}^1, & b_{i,j}^1, \bar{a}_{i,j}, \bar{\bar{a}}_{i,j}, \\ & a_{i,j}^1, \beta_{i,j}, c_{i,j}^2, & a_{i,j}^2, b_{i,j}^2, c_{i,j}^3, & \bar{a}_{i,j}, \bar{c}_{i,j}, \bar{b}_{i,j} \end{aligned}$$

Here, we set  $a_{i,0}^3 = a_{i-1,\mu_{i-1}}^3$ ,  $b_{i,0}^3 = b_{i-1,\mu_{i-1}}^3$ , and  $c_{i,0}^3 = c_{i-1,\mu_{i-1}}^3$  for any index  $i \geq 2$ ; we only have duplicate names for these items



**Fig. 1.** Illustration of super-blocks  $Y_i$  and  $F_i$  for some  $i \in [n]$ , depicted in subfigures (a) and (b), respectively. Each literal block within  $Y_i$  contains equal-length sublists from each agent's preference list, but equivalence blocks in  $F_i$  contain only empty sublists from  $L^A$ .



**Fig. 2.** A high-level overview of the blocks constituting the preference lists in the constructed instance  $(N, I, L)$ . Note that the lengths of the blocks as shown on the figure are not proportional to their real sizes. For better visibility, we tiled every other validity block with a (yellow) striped pattern. (For colored figures, the reader is referred to the web version of this article.)

to ease the formalization. We now provide some intuition on the items appearing in a literal block; more detailed arguments will follow later in Lemma 1.

First, the items  $a_{1,0}^3$ ,  $b_{1,0}^3$ , and  $c_{1,0}^3$  must be assigned by any complete NEF allocation to agents  $A$ ,  $B$ , and  $C$ , respectively, due to Proposition 2. More generally, since the items  $a_{i,j-1}^3$ ,  $b_{i,j-1}^3$ , and  $c_{i,j-1}^3$  already appear in the previous block, by an inductive argument we will be able to deduce that any complete NEF allocation must assign them to agents  $A$ ,  $B$ , and  $C$ , respectively. Using this as a starting point, a careful observation of the block (presented later in Lemma 1) will also reveal that all items  $a_{i,j}^h$ ,  $b_{i,j}^h$ , and  $c_{i,j}^h$  for some  $h \in [3]$  need to be allocated to agents  $A$ ,  $B$ , and  $C$ , respectively. Each of these items, except for  $a_{i,j}^3$ ,  $b_{i,j}^3$ , and  $c_{i,j}^3$  appear three times in the block  $X_{i,j}$ , and thus do not appear in any other block. By contrast, the items  $a_{i,j}^3$ ,  $b_{i,j}^3$ , and  $c_{i,j}^3$  each appear once in the block  $X_{i,j}$ , and twice in the next block, which is either the block  $X_{i,j+1}$  (if  $j < \mu_i$ ) or the block  $X_{i+1,1}$  (if  $j = \mu_i$ ). An exception to this is the last literal block  $X_{n,\mu_n}$ , because the items  $a_{n,\mu_n}^3$ ,  $b_{n,\mu_n}^3$ , and  $c_{n,\mu_n}^3$  each appear once in  $X_{n,\mu_n}$  and twice in the closing block  $Z$ .

The items  $\alpha_{i,j}$ ,  $\beta_{i,j}$ , and  $\gamma_{i,j}$  will also have the property that any complete NEF allocation assigns them to agents  $A$ ,  $B$ , and  $C$ , respectively. However, each of these items only appears twice in the literal block  $X_{i,j}$ . Notably, although we will be able to infer from the structure of  $X_{i,j}$  that  $\alpha_{i,j}$  must be allocated to  $A$ , it does not appear in the preferences of  $A$  within  $X_{i,j}$ . Instead, we will use the appearance of  $\alpha_{i,j}$  in the preferences of  $A$  later on, namely in the validity block belonging to the clause that contains the  $j$ th occurrence of variable  $x_i$ . Similarly, we will be able to infer that the items  $\beta_{i,j}$  and  $\gamma_{i,j}$  must be allocated to  $B$  and to  $C$ , respectively, but they do not appear in the preference list of agent  $B$  and of agent  $C$ , respectively, within the block  $X_{i,j}$ . Instead, we will use the appearance of  $\beta_{i,j}$  and  $\gamma_{i,j}$  in the preference list of  $B$  and  $C$ , respectively, within some equivalence block  $E_{i,j}$ .

The three items  $\mathbf{a}_{i,j}$ ,  $\overline{\mathbf{a}}_{i,j}$ , and  $\overline{\overline{\mathbf{a}}}_{i,j}$  will have the property that each of them are assigned to agents  $A$  or  $B$  by any complete NEF allocation, as we will argue later on. Similarly, items  $\mathbf{b}_{i,j}$ ,  $\overline{\mathbf{b}}_{i,j}$ , and  $\overline{\overline{\mathbf{b}}}_{i,j}$  must be allocated to agents  $B$  and  $C$ , and items  $\mathbf{c}_{i,j}$ ,  $\overline{\mathbf{c}}_{i,j}$ , and  $\overline{\overline{\mathbf{c}}}_{i,j}$  must be allocated to agents  $C$  and  $A$ . We will call the set of these nine items the *choice items* for  $(i, j)$ , since their purpose is to enable a possibility of choice in our instance: the choice of a complete NEF allocation for assigning the item  $\mathbf{a}_{i,j}$ , either to  $A$  or to  $B$ , will correspond to a truth assignment for variable  $x_i$ . Importantly, the choice for assigning  $\mathbf{a}_{i,j}$  will also

determine how each of the remaining choice items for  $(i, j)$  are assigned to agents; moreover, we will ensure that these choices are consistent over all literal blocks corresponding to variable  $x_i$ . Note that each of  $\overline{\mathbf{a}}_{i,j}$ ,  $\overline{\mathbf{b}}_{i,j}$ , and  $\overline{\overline{\mathbf{c}}}_{i,j}$  appears three times in the block  $X_{i,j}$ , and each of the remaining six items under consideration appears twice in  $X_{i,j}$ . The remaining appearances of items  $\mathbf{a}_{i,j}$ ,  $\overline{\mathbf{a}}_{i,j}$ ,  $\mathbf{c}_{i,j}$ , and  $\overline{\overline{\mathbf{c}}}_{i,j}$  will later be used in equivalence blocks, and the remaining appearances of  $\mathbf{b}_{i,j}$  and  $\overline{\mathbf{b}}_{i,j}$  will later be used in validity blocks.

*Equivalence block  $E_{i,2j}$  for some  $i \in [n]$  and  $j \in [\mu_i/2]$ :*

$$\begin{array}{ll} A: & - \\ B: & \overline{\mathbf{a}}_{i,2j-1}, \mathbf{c}_{i,2j}, \beta_{i,2j-1}, \mathbf{c}_{i,2j-1}, \overline{\mathbf{a}}_{i,2j}, \beta_{i,2j} \\ C: & \overline{\mathbf{a}}_{i,2j}, \mathbf{a}_{i,2j+1}, \gamma_{i,2j}, \mathbf{a}_{i,2j}, \overline{\mathbf{a}}_{i,2j+1}, \gamma_{i,2j+1} \end{array}$$

Here, we let  $\mathbf{a}_{i,\mu_i+1} = \mathbf{a}_{i,1}$ ,  $\overline{\mathbf{a}}_{i,\mu_i+1} = \overline{\mathbf{a}}_{i,1}$ , and  $\gamma_{i,\mu_i+1} = \gamma_{i,1}$  for each  $i \in [n]$ , so indices are taken modulo  $\mu_i$ .

These blocks contain only items that have already appeared in previous blocks. The purpose of equivalence blocks is to ensure that any complete NEF allocation assigns all items  $\mathbf{a}_{i,j}$  for some  $j \in [\mu_i]$  to the same agent,  $A$  or  $B$ ; this will ensure also that the allocation of all choice items for  $(i, j)$  is independent from the value of  $j$  and depends only on  $i$ . This observation will be crucial for constructing a valid truth assignment for  $\varphi$  from a given complete NEF allocation for  $(N, I, L)$ .

*Validity block  $V_k$  for some  $k \in [m]$ :* The definition of  $V_k$  depends on clause  $c_k$ . Let  $c_k$  contain the  $j_u$ th,  $j_v$ th, and  $j_z$ th occurrence of variables  $x_u$ ,  $x_v$ , and  $x_z$ , respectively, in the formula  $\varphi$ . If  $x_u$  appears in  $c_k$  as a positive literal, then we define  $\ell_u$  as  $\ell_u = \mathbf{b}_{u,j_u}$ , otherwise we set  $\ell_u = \overline{\mathbf{b}}_{u,j_u}$ . Furthermore, we denote by  $\overline{\ell}_u$  the item corresponding to the negated form of the literal of  $x_i$  contained by  $c_k$ , that is, if  $\ell_u = \mathbf{b}_{u,j_u}$ , then  $\overline{\ell}_u = \overline{\mathbf{b}}_{u,j_u}$ , and vice versa, if  $\ell_u = \overline{\mathbf{b}}_{u,j_u}$ , then  $\overline{\ell}_u = \mathbf{b}_{u,j_u}$ . Observe that in either case  $\{\ell_u, \overline{\ell}_u\} = \{\mathbf{b}_{u,j_u}, \overline{\mathbf{b}}_{u,j_u}\}$ . We define  $\ell_v$ ,  $\ell_z$ ,  $\overline{\ell}_v$ , and  $\overline{\ell}_z$  analogously.

Now we are ready to describe the validity block  $V_k$ .

$$\begin{array}{ll} A: & \alpha_{u,j_u}, v_k, \ell_u, \ell_v, \ell_z, v'_k, \alpha_{v,j_v}, \alpha_{z,j_z}, \overline{\ell}_u, \overline{\ell}_v, \overline{\ell}_z, v''_k \\ B: & v_k, v'_k, v''_k \\ C: & v_k, v'_k, v''_k \end{array}$$

Observe that apart from the choice items in  $V_k$ , each of which must be allocated to either  $B$  or to  $C$  by any complete NEF



allocation, the validity block also contains three items that must be allocated to  $A$  (the items  $\alpha_{u,j_u}$ ,  $\alpha_{v,j_v}$ , and  $\alpha_{z,j_z}$ ). Further, it contains items  $v_k$ ,  $v'_k$ , and  $v''_k$ , each appearing three times in  $V_k$ ; we will be able to show that each of our three agents must obtain exactly one of these three items (for a given  $k \in [m]$ ) in any complete NEF allocation. It will always be “safe” to allocate  $v_k$  to  $A$ , while the allocation of  $v'_k$  and  $v''_k$  needs more circumspection.

**Example 2.** Let the third clause of  $\varphi$  be  $c_3 = (x_1 \vee \neg x_4 \vee \neg x_5)$ , and suppose that  $c_3$  contains the first occurrences of variables  $x_1$  and  $x_4$ , and the second occurrence of variable  $x_5$  in  $\varphi$ . Then the validity block  $V_3$  corresponding to  $c_3$  is as follows:

$$\begin{aligned} A: & \alpha_{1,1}, v_3, \overline{bc}_{1,1}, \overline{bc}_{4,1}, \overline{bc}_{5,2}, v'_3, \alpha_{4,1}, \alpha_{5,2}, \overline{bc}_{1,1}, \overline{bc}_{4,1}, \overline{bc}_{5,2}, v''_3 \\ B: & v_3, v'_3, v''_3 \\ C: & v_3, v'_3, v''_3 \end{aligned}$$

Closing block  $Z$ :

$$\begin{aligned} A: & b_{n,\mu_n}^3, c_{n,\mu_n}^3 \\ B: & a_{n,\mu_n}^3, c_{n,\mu_n}^3 \\ C: & a_{n,\mu_n}^3, b_{n,\mu_n}^3 \end{aligned}$$

**Well-formed instance.** It is clear that the construction takes polynomial time. It is, however, not so obvious to see that the concatenation of the constructed blocks yields a well-formed instance; one has to check that each preference list contains each item exactly once.

Items  $a_{1,0}^3$ ,  $b_{1,0}^3$ , and  $c_{1,0}^3$  appear in the initial block  $I_0$  as the top item for agent  $A$ ,  $B$ , and  $C$ , respectively, and they appear twice more within the first literal block  $X_{1,1}$  in the preferences of the remaining two agents. Items of the form  $a_{i,j}^h$ ,  $b_{i,j}^h$  and  $c_{i,j}^h$  with  $i \in [n]$ ,  $j \in [\mu_i]$  and  $h \in [3]$  appear in the literal block  $X_{i,j}$  for each agent, with two exceptions: in agent  $A$ 's preference list, items  $b_{i,j}^3$  and  $c_{i,j}^3$  only appear in the literal block following  $X_{i,j}$  (or, for  $i = n$  and  $j = \mu_n$ , in the closing block  $Z$ ); the same happens in the preference list of agents  $B$  and  $C$  regarding the items of  $\{a_{i,j}^3, c_{i,j}^3\}$  and  $\{a_{i,j}^3, b_{i,j}^3\}$ , respectively.

Items of  $\{bc_{i,j}, \overline{bc}_{i,j}, \alpha_{i,j}\}$  for some  $i \in [n]$ ,  $j \in [\mu_i]$  appear in the preferences of agents  $B$  and  $C$  within the literal block  $X_{i,j}$ . They further appear in  $L^A$  within the validity block  $V_k$  corresponding to the clause  $c_k$  that contains the  $j$ th occurrence of variable  $x_i$  in  $\varphi$ : this is easy to check for  $\alpha_{i,j}$ , but needs some attention in the case of items  $bc_{i,j}$  and  $\overline{bc}_{i,j}$ , since they appear under an alias of the form  $\ell_w$  and  $\overline{\ell}_w$  for some appropriate  $w$  in the definition of  $V_k$ . To check these details, recall that for each variable  $x_u$  occurring the  $j_u$ th time in a given clause  $c_k$ , we list both items of  $\{\ell_u, \overline{\ell}_u\} = \{bc_{u,j_u}, \overline{bc}_{u,j_u}\}$  in the preferences of  $A$  within the validity block  $V_k$ . Hence, we list both items of  $\{bc_{i,j}, \overline{bc}_{i,j}\}$  in the block  $V_k$  if  $c_k$  is the clause containing the  $j$ th occurrence of variable  $x_i$ .

Items of  $\{\alpha_{i,j}, \overline{\alpha}_{i,j}, \beta_{i,j}\}$  appear within the literal block  $X_{i,j}$  for agents  $A$  and  $C$ , and in the equivalence block  $E_{i,j'}$  for agent  $B$ , where  $j' = 2\lceil j/2 \rceil$ . Similarly, items of  $\{\alpha_{i,j}, \overline{\alpha}_{i,j}, \gamma_{i,j}\}$  appear within  $X_{i,j}$  for agents  $A$  and  $B$ , and in  $E_{i,j'}$  for agent  $C$ , where  $j' = 2\lceil j/2 \rceil$  if  $j > 1$ , and  $j' = \mu_i$  if  $j = 1$ . Each of the remaining choice items  $\overline{\alpha}_{i,j}$ ,  $\overline{\beta}_{i,j}$ , or  $\overline{\gamma}_{i,j}$  for some  $i \in [n]$  and  $j \in [\mu_i]$  appears three times within the literal block  $X_{i,j}$ , once in each agent's preferences. This leaves us with the items  $v_k$ ,  $v'_k$ , and  $v''_k$  for some  $k \in [m]$ , each of them appearing once in the preferences of each agent within the validity block  $V_k$ .

We can thus conclude that each constructed preference list is indeed a strict linear order over  $I$ .

**Correctness.** To verify the correctness of our reduction, we state a series of observations and lemmas that describe how a complete

NEF allocation can assign items to agents in the constructed instance.

Let a *triadic prefix* be a prefix of some preference list of length  $3t + 1$  for some integer  $t$  with  $3t + 1 \leq |I|$ . Our arguments will often rely on the following notion: we say that an allocation  $\pi$  is *smooth* on the triadic prefix  $L^X(1 : 3t + 1)$  for some agent  $X \in N$  and  $t \in [|I|/3 - 1]$ , if  $\pi$  assigns to each agent exactly one item from each triad contained in  $[L^X(1 : 3t + 1)]$ , and additionally, assigns to  $X$  its most preferred item. Intuitively, an allocation is smooth on a triadic prefix, if on each of its prefixes fulfills the conditions of Proposition 1 tightly. We also say that  $\pi$  is *smooth until* a given block  $B$  in  $(N, I, L)$ , if for each agent  $X \in N$  it is smooth on the prefix of  $L^X$  ending right before the block  $B$ . Note that every complete NEF allocation is smooth until the first literal block  $X_{1,1}$  due to Proposition 2. The following proposition explains why smoothness is an important property.

**Proposition 3.** Let  $\pi$  be a complete NEF allocation for  $(N, I, L)$  that is smooth on a triadic prefix  $P$  of  $L^X$  for some agent  $X \in N$ . Then  $\pi$  allocates at least one item to  $X$  from the triad following  $P$  in  $L^X$ . More generally,  $\pi$  allocates at least  $t$  items to  $X$  from the  $t$  consecutive triads following  $P$  in  $L^X$  for each  $t$  with  $|P| + 3t \leq |I|$ .

**Proof.** By our assumption on the smoothness of  $\pi$ , we know that  $\pi$  assigns exactly  $(|P| - 1)/3 + 1$  items from  $P$  to  $X$  and assigns exactly  $(|P| - 1)/3$  items to every other agent. For the sake of contradiction, assume that  $\pi$  assigns fewer than  $t$  items to  $X$  from the  $t$  triads following  $P$  in  $L^X$ . By the pigeon-hole principle,  $\pi$  must assign at least  $t + 1$  items from these triads to some other agent  $X'$ . Thus,  $\pi$  assigns at most  $(|P| - 1)/3 + t$  items to  $X$ , but at least  $(|P| - 1)/3 + t + 1$  items to  $X'$  from the triadic prefix containing  $P$  and the next  $t$  triads in  $L^X$ . By Proposition 1 this contradicts the envy-freeness of  $\pi$ .  $\square$

Our next lemma explores the key properties of how a complete NEF and smooth allocation can allocate the items appearing in a literal block.

**Lemma 1.** Let  $\pi$  be a complete NEF allocation for  $(N, I, L)$ , and let  $i \in [n]$  and  $j \in [\mu_i]$ . If  $\pi$  is smooth until the literal block  $X_{i,j}$ , then:

(i) For each  $h \in [3]$  we have

$$\begin{aligned} \pi(a_{i,j}^h) &= A, & \pi(\alpha_{i,j}) &= A, \\ \pi(b_{i,j}^h) &= B, & \pi(\beta_{i,j}) &= B, \\ \pi(c_{i,j}^h) &= C, & \pi(\gamma_{i,j}) &= C. \end{aligned}$$

(ii) One of the followings hold:

(C1)  $X_{i,j}$  is of type 1, meaning

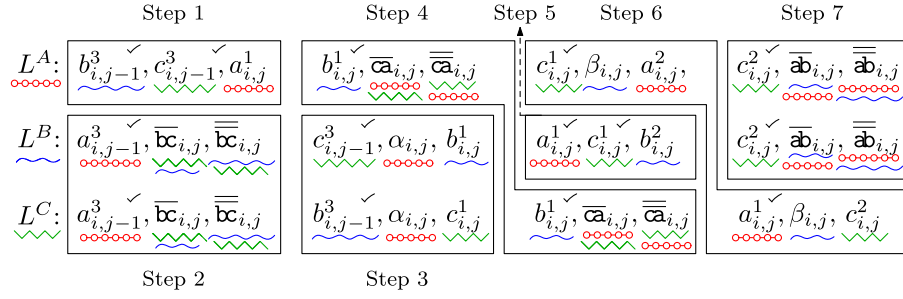
$$\begin{aligned} \pi(\overline{\alpha}_{i,j}) &= B, & \pi(\overline{\alpha}_{i,j}) &= A, & \pi(\alpha_{i,j}) &= A, \\ \pi(\overline{\beta}_{i,j}) &= C, & \pi(\overline{\beta}_{i,j}) &= B, & \pi(\beta_{i,j}) &= B, \\ \pi(\overline{\gamma}_{i,j}) &= A, & \pi(\overline{\gamma}_{i,j}) &= C, & \pi(\gamma_{i,j}) &= C. \end{aligned}$$

(C2)  $X_{i,j}$  is of type 2, meaning

$$\begin{aligned} \pi(\overline{\alpha}_{i,j}) &= A, & \pi(\overline{\alpha}_{i,j}) &= B, & \pi(\alpha_{i,j}) &= B, \\ \pi(\overline{\beta}_{i,j}) &= B, & \pi(\overline{\beta}_{i,j}) &= C, & \pi(\beta_{i,j}) &= C, \\ \pi(\overline{\gamma}_{i,j}) &= C, & \pi(\overline{\gamma}_{i,j}) &= A, & \pi(\gamma_{i,j}) &= A. \end{aligned}$$

(iii)  $\pi$  remains smooth until the block following  $X_{i,j}$ .

**Proof.** We prove the lemma by induction on  $i$  and  $j$ . Fix some  $i$  and  $j$ , and consider the literal block  $X_{i,j}$ . We will assume that either  $i = j = 1$ , or the lemma holds for the last literal block  $X_{i',j'}$  preceding  $X_{i,j}$ , i.e., either for  $i' = i$  and  $j' = j - 1$ , or (in the case  $j = 1$ ) for  $i' = i - 1$  and  $j' = \mu_{i-1}$ .



**Fig. 3.** Illustration of the proof of Lemma 1 for some literal block  $X_{i,j}$ , describing the steps of our arguments for the first four triads. The figure groups the triads involved in each step, displaying also the order of these steps. Items allocated to agents A, B, and C by  $\pi$  are underlined with a (red) chained, (blue) wavy, and (green) zigzagged line, respectively, as shown for  $L^A$ ,  $L^B$ , and  $L^C$ . In those cases where there are two possibilities for  $\pi$  to allocate a given item, we underline the item using both symbols representing the two possibilities. Furthermore, we mark each item by a checkmark ( $\checkmark$ ) whose allocation by  $\pi$  is known at the moment when we reach the given triad in our proof.

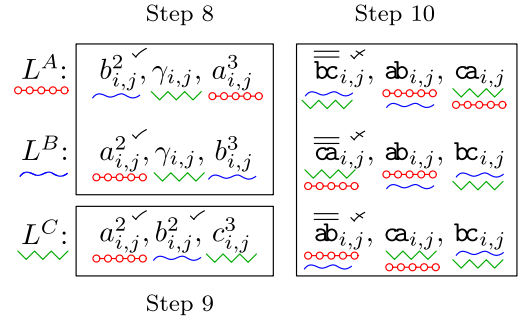
We claim  $\pi(a_{i,j-1}^3) = A$ ,  $\pi(b_{i,j-1}^3) = B$  and  $\pi(c_{i,j-1}^3) = C$ . For  $i = j = 1$  this follows from Proposition 2. Otherwise, recall that  $X_{i',j'}$  is the last literal block preceding  $X_{i,j}$ . Observe also that since  $\pi$  is smooth until  $X_{i,j}$ , it is also smooth until  $X_{i',j'}$ . Hence, the claim follows from our induction hypothesis for  $X_{i',j'}$ ; recall that if  $i > 1$ , then  $a_{i,0}^3 = a_{i-1,\mu_{i-1}}^3$ ,  $b_{i,0}^3 = b_{i-1,\mu_{i-1}}^3$  and  $c_{i,0}^3 = c_{i-1,\mu_{i-1}}^3$ . We will move forward within the literal block  $X_{i,j}$  triad by triad. At each step, when we consider a given set of triads, we will rely on the smoothness of  $\pi$  on the triadic prefixes preceding the given triads. Using Proposition 3 and the structure of  $X_{i,j}$  we will then prove that  $\pi$  remains smooth also on the triadic prefixes ending with the given triads. This argument will not be made explicit each time, in order to facilitate focusing more on the allocation of the items. See Figs. 3 and 4 for an illustration.

Since  $\pi$  is smooth until  $X_{i,j}$ , by Proposition 3 we know that each agent has to obtain at least one item from its three most preferred items in  $X_{i,j}$  to ensure necessary envy-freeness. Looking at the first triad for A within  $X_{i,j}$ , this implies that  $\pi$  must allocate  $a_{i,j}^1$  to A (see Step 1 in Fig. 3). The first triads for agents B and C show that one of  $\bar{b}_{i,j}$  and  $\bar{c}_{i,j}$  must be allocated to B, and the other to C (see Step 2 in Fig. 3).

Observing the second triads for B and C in  $X_{i,j}$  as shown in Step 3 of Fig. 3, we get that  $\alpha_{i,j}$  can only be allocated to A, so as not to create too many items in the preference list of B allocated to C, or vice versa: indeed, assuming  $\pi(\alpha_{i,j}) = B$  implies that  $\pi$  assigns fewer items from the prefix of  $L^C$  ending with  $\alpha_{i,j}$  to B than to C, contradicting the envy-freeness of  $\pi$  by Proposition 1; assuming  $\pi(\alpha_{i,j}) = C$  leads to a similar contradiction. From this, we also obtain  $\pi(b_{i,j}^1) = B$  and  $\pi(c_{i,j}^1) = C$ . Now, considering agents A and C and their second and third triads in  $X_{i,j}$ , respectively, as shown in Step 4 of Fig. 3, we get that one of  $\bar{a}_{i,j}$  and  $\bar{a}_{i,j}$  must be allocated to A, and the other to C. Considering the third triad for agent B,  $\pi(b_{i,j}^2) = B$  follows (Step 5).

Next, looking at the third triad for A and the fourth triad for C, as shown in Step 6 of Fig. 3, we can observe that  $\beta_{i,j}$  must be allocated to B so as not to allocate too many items from  $L^A$  to C, or from  $L^C$  to A; then  $\pi(a_{i,j}^2) = A$  and  $\pi(c_{i,j}^2) = C$  follow as well. By the fourth triads for A and B, one of  $\bar{a}_{i,j}$  and  $\bar{a}_{i,j}$  must be allocated to A, and the other to B (Step 7). Considering the fifth triads, depicted in Fig. 4 (Steps 8 and 9), arguing as above we get  $\pi(a_{i,j}^3) = A$ ,  $\pi(b_{i,j}^3) = B$  and  $\pi(c_{i,j}^3) = \pi(\gamma_{i,j}) = C$ . This shows that statement (i) holds for  $X_{i,j}$ .

Now, consider the last triads of  $X_{i,j}$ , as shown in Step 10 of Fig. 4. Clearly, each agent has to be allocated at least one item from his or her triad, and there are exactly three items ( $\bar{b}_{i,j}$ ,  $\bar{c}_{i,j}$ , and  $\bar{a}_{i,j}$ ) that they can get. Supposing that  $\pi$  allocates both  $\bar{b}_{i,j}$  and  $\bar{c}_{i,j}$  to C, one can see that neither  $\bar{b}_{i,j}$ , nor  $\bar{c}_{i,j}$  can be



**Fig. 4.** Illustration of the proof of Lemma 1 for some literal block  $X_{i,j}$ , describing the steps of our arguments for the last two triads. The notation is the same as for Fig. 3 with the addition of a “half-checkmark” ( $\checkmark$ ) for each item which is known to be allocated by  $\pi$  to one of two agents at the moment when we reach the given triad in our proof (e.g., when considering the last triads within  $X_{i,j}$ , we know that  $\pi$  assigns  $\bar{b}_{i,j}$  to either B or C).

allocated to C, as that would create too many items allocated by  $\pi$  to C in the preference list of either A or B. Analogously, we obtain that neither  $\pi(\bar{b}_{i,j}) = \pi(\bar{a}_{i,j}) = B$ , nor  $\pi(\bar{a}_{i,j}) = \pi(\bar{a}_{i,j}) = A$  is possible. Hence, we must have that either  $\pi(\bar{b}_{i,j}) = C$ ,  $\pi(\bar{a}_{i,j}) = A$  and  $\pi(\bar{a}_{i,j}) = B$ , or  $\pi(\bar{b}_{i,j}) = B$ ,  $\pi(\bar{a}_{i,j}) = C$  and  $\pi(\bar{a}_{i,j}) = A$ . In the former case, we quickly get that A cannot have  $\bar{a}_{i,j}$  (as then B would have two items in his last triad of  $X_{i,j}$  allocated to A), yielding  $\pi(\bar{a}_{i,j}) = B$ . Similarly, we get  $\pi(\bar{b}_{i,j}) = C$  and  $\pi(\bar{c}_{i,j}) = A$  as well. In the latter case, analogous arguments prove  $\pi(\bar{b}_{i,j}) = B$ ,  $\pi(\bar{c}_{i,j}) = C$  and  $\pi(\bar{a}_{i,j}) = A$ . Recalling our observations in the previous paragraph on items  $\bar{a}_{i,j}$ ,  $\bar{b}_{i,j}$  and  $\bar{c}_{i,j}$ , we get that  $X_{i,j}$  is either of type 1 or of type 2. This proves statement (ii).

Finally, notice that (iii) follows directly from (i) and (ii).  $\square$

Next, we turn our attention to equivalence blocks.

**Lemma 2.** Let  $\pi$  be a complete NEF allocation for  $(N, I, L)$ , and let  $i \in [n]$  and  $j \in [\mu_i/2]$ . If  $\pi$  is smooth until the equivalence block  $E_{i,2j}$ , then:

- (i) The literal blocks  $X_{i,2j-1}$ ,  $X_{i,2j}$ , and  $X_{i,2j+1}$  all have the same type (where indices are taken modulo  $\mu_i$  so that  $X_{i,2\mu_i+1} = X_{i,1}$ ).
- (ii)  $\pi$  remains smooth until the block following  $E_{i,2j}$ .

**Proof.** By our assumption on the smoothness of  $\pi$  we can apply Lemma 1 for index  $i$  and each  $j' \in [\mu_i]$ , since all literal blocks  $X_{i,j'}$ ,  $j' \in [\mu_i]$ , precede  $E_{i,2j}$ . Thus, Lemma 1 yields  $\{\pi(\bar{a}_{i,j'}), \pi(\bar{b}_{i,j'})\} = \{A, B\}$  and  $\pi(\gamma_{i,j'}) = C$ , and similarly,  $\{\pi(\bar{c}_{i,j'}), \pi(\bar{a}_{i,j'})\} = \{A, C\}$

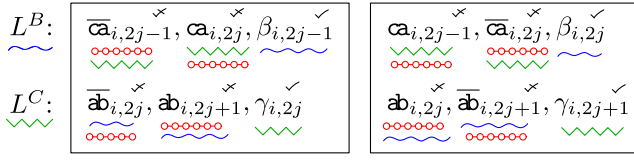


Fig. 5. Illustration of the proof of Lemma 2 for some equivalence block  $E_{i,2j}$ . The notation is the same as for Figs. 3 and 4.

and  $\pi(\beta_{i,j'}) = B$  for each  $j' \in [\mu_i]$ . See Fig. 5 for an illustration of these facts.

Since  $\pi$  is smooth until  $E_{i,2j}$ , Proposition 1 implies that  $\pi$  cannot assign two items to agent A (or, similarly, to agent C) from the first triad of  $L^B$  within  $E_{i,2j}$ : indeed, assigning both  $\bar{\alpha}_{i,2j-1}$  and  $\alpha_{i,2j}$  to A (or to C) would mean that A (or C) would obtain  $t+2$  items while B would obtain only  $t+1$  items from the prefix  $L^B(1 : 3t+3)$ , where  $t$  denotes the number of triads preceding  $E_{i,2j}$ . Therefore, either  $\pi(\bar{\alpha}_{i,2j-1}) = A$  and  $\pi(\alpha_{i,2j}) = C$ , or vice versa. By Lemma 1, this means exactly that  $X_{i,2j-1}$  and  $X_{i,2j}$  must be of the same type. Note also that each agent obtains exactly one item from both triads of  $L^B$  within the block.

Applying the same reasoning to the first triad of  $L^C$  within  $E_{i,2j}$ , we get that either  $\pi(\bar{\alpha}_{i,2j}) = A$  and  $\pi(\bar{\alpha}_{i,2j+1}) = C$ , or vice versa, showing that  $X_{i,2j}$  and  $X_{i,2j+1}$  have the same type,<sup>1</sup> and that  $\pi$  allocates an item from both triads of  $L^C$  within  $E_{i,j}$  to each agent. This finishes the proof of both statements of the lemma.  $\square$

The next lemma shows that a complete NEF allocation is, roughly speaking, smooth on most parts of the constructed instance, and it is *almost* smooth on the remaining parts. Formally, we say that an allocation  $\pi$  is *half-smooth* on  $L^X(1 : 6t+1)$ , or equivalently, on the triadic prefix containing the first  $2t$  triads for some agent  $X \in N$  and  $t \in [(|I|-1)/6]$ , if  $\pi$  assigns to each agent exactly two items from the two triads in  $L^X(6(t'-1)+2 : 6t'+1)$  for each  $t' \in [t]$ , and additionally, assigns to  $X$  its most preferred item.

Before stating Lemma 3, we observe the following implications of half-smoothness; since the proof uses exactly the same arguments as the proof of Proposition 3, we omit it.

**Proposition 4.** *Let  $\pi$  be a complete NEF allocation for  $(N, I, L)$  that is half-smooth on a triadic prefix  $P$  of  $L^X$  for some agent  $X \in N$  such that  $P$  contains an even number of triads. Then  $\pi$  allocates at least one item to  $X$  from the triad following  $P$  in  $L^X$ . More generally,  $\pi$  allocates at least  $t$  items to  $X$  from the  $t$  consecutive triads following  $P$  in  $L^X$  for each  $t$  with  $|P| + 3t \leq |I|$ .*

**Lemma 3.** *Let  $\pi$  be a complete NEF allocation for  $(N, I, L)$ . Then  $\pi$  is smooth on  $L^B$  and on  $L^C$  until the closing block, and it is smooth on  $L^A$  until the first validity block. Furthermore,  $\pi$  is half-smooth on  $L^A$  until the closing block.*

**Proof.** We start by showing that  $\pi$  is smooth until the first validity block, using Lemmas 1 and 2. Note that  $\pi$  is smooth until the first literal block  $X_{1,1}$ , due to Proposition 2 and since there are no triads preceding  $X_{1,1}$ . Now, assuming that  $\pi$  is smooth until a given literal or equivalence block  $B$ , Lemma 1 (if  $B$  is a literal block) or Lemma 2 (if  $B$  is an equivalence block) shows that  $\pi$  is smooth also until the block following  $B$ . Hence,  $\pi$  is smooth until the first validity block.

Now, we will show that if  $\pi$  is smooth on  $L^B$  and on  $L^C$  until the validity block  $V_k$ ,  $k \in [m]$ , and half-smooth on  $L^A$  until  $V_k$ ,

then it retains these properties until the block following  $V_k$ . This suffices to prove the lemma.

Consider some  $k \in [m]$ , and see Fig. 6 for an illustration. Since  $\pi$  is a complete NEF allocation that is smooth until  $V_k$ , Proposition 3 applied to  $L^B$  and to  $L^C$  shows that  $\pi$  allocates at least one item to each agent from its first triad within  $V_k$ . Hence, both  $B$  and  $C$  are allocated at least one item from  $\{v_k, v'_k, v''_k\}$ . Proposition 4 for agent A also yields that  $\pi$  must allocate to A at least two items from the first two triads of  $L^A$  within  $V_k$ , containing the items  $\{\alpha_{u,j_u}, v_k, \ell_u, \ell_v, \ell_z, v'_k\}$ . By Lemma 1, each of the choice items present in  $L^A$  within  $V_k$  (i.e.,  $\ell_u, \ell_v, \ell_z, \bar{\ell}_u, \bar{\ell}_v$ , and  $\bar{\ell}_z$ ) is allocated to  $B$  or to  $C$  by  $\pi$ . Thus, we obtain that  $\pi$  assigns at least one item from  $\{v_k, v'_k\}$  to A. Hence, each agent obtains exactly one item from the set  $\{v_k, v'_k, v''_k\}$ .

Consequently,  $\pi$  remains smooth on  $L^B$  and on  $L^C$  until the block following  $V_k$ . Moreover, looking at the first two and the last two triads of  $L^A$  within  $V_k$ , we can also observe that  $\pi$  can assign at most, and hence must assign *exactly*, two items to A from each of these six-item sets. Thus,  $\pi$  remains half-smooth on  $L^A$  until the block following  $V_k$ .  $\square$

The next lemma is a direct consequence of Lemmas 2 and 3.

**Lemma 4.** *Let  $\pi$  be a complete NEF allocation for  $(N, I, L)$ , and let  $i \in [n]$ . Then all literal blocks in  $Y_i$  have the same type; we call this the type of  $Y_i$ .*

**Proof.** By Lemma 3 we know that  $\pi$  is smooth for each agent until the first validity block. From this, Lemma 2 yields that for any  $j \in \mu_i$ , the literal blocks  $X_{i,2j-1}$ ,  $X_{i,2j}$ , and  $X_{i,2j+1}$ , with the indices taken modulo  $\mu_i$  (so that  $X_{i,2\mu_i+1} = X_{i,1}$ ), have the same type. This means that all literal blocks  $X_{i,j}$  where  $j \in [\mu_i]$  must have the same type, as required.  $\square$

We are now ready to show the correctness of our reduction, which proves Theorem 1.

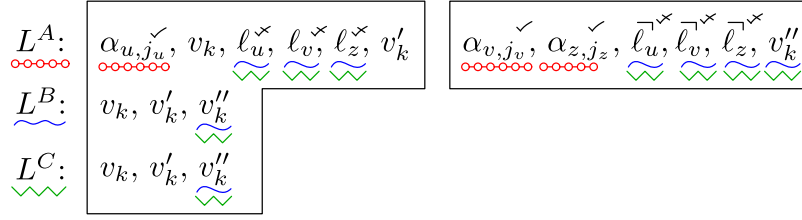
**Lemma 5.** *The constructed instance  $(N, I, L)$  admits a complete NEF allocation if and only if there exists a valid truth assignment for the input formula  $\varphi$ .*

**Proof** (Direction “ $\Rightarrow$ ”). Let us first suppose that  $\pi : I \rightarrow N$  is a complete NEF allocation. We construct a valid truth assignment for  $\varphi$  based on allocation  $\pi$ . Namely, we set  $x_i$  to true if and only if the literal blocks in  $Y_i$  are of type 1; by Lemma 4,  $\pi$  is well defined.

Consider the validity block  $V_k$  for some  $k \in [m]$ , involving the  $j_u$ th,  $j_v$ th, and  $j_z$ th occurrence of the variables  $x_u$ ,  $x_v$ , and  $x_z$ , respectively. By Lemma 3 we know that  $\pi$  is half-smooth until the closing block  $Z$ , and thus by definition it allocates to each agent exactly two items from the first two triads of  $L^A$  within  $V_k$ , that is, from the item set  $\{\alpha_{u,j_u}, v_k, \ell_u, \ell_v, \ell_z, v'_k\}$ . By Lemma 1, we know  $\pi(\alpha_{u,j_u}) = A$ , and from claim (ii) we get that each of  $\ell_u, \ell_v$ , and  $\ell_z$  is allocated to one of the agents  $B$  or  $C$ . Thus, either  $v_k$  or  $v'_k$  is allocated to A. Therefore we obtain that  $\pi$  allocates either 1 or 2 among the items  $\ell_u, \ell_v$ , and  $\ell_z$  to B.

Recall now the definition of  $\ell_u$ : if  $x_u$  appears as a positive literal in  $c_k$ , then  $\ell_u = \text{bc}_{u,j_u}$ , otherwise  $\ell_u = \text{bc}_{u,j_u}$ . Now,  $\text{bc}_{u,j_u}$  is assigned to agent B exactly if  $Y_u$  has type 1, and  $\text{bc}_{u,j_u}$  is assigned to agent B exactly if  $Y_u$  has type 2. Hence,  $x_u$  becomes a true literal in  $c_k$  exactly if the item  $\ell_u$  is assigned to B by  $\pi$ . As the analogous statements hold for  $x_v$  and  $x_z$  as well, we obtain that the number of true literals in the clause  $c_k$  equals the number of items in  $\{\ell_u, \ell_v, \ell_z\}$  allocated to B by  $\pi$ . Since this value must be either 1 or 2 (as argued above), we get that  $c_k$  contains at least 1

<sup>1</sup> Recall that indices within the equivalence block  $E_{i,2j}$  are taken modulo  $\mu_i$ , so for  $j = \mu_i/2$  we obtain that  $X_{i,\mu(i)}$  and  $X_{i,1}$  have the same type.



**Fig. 6.** Illustration of the proof of [Lemma 3](#) depicting some validity block  $V_k$ . We use the same notation as for [Figs. 3–5](#) with the addition that items that  $\pi$  may assign to any of the three agents are not underlined in any way.

but at most 2 true literals. Hence, our truth assignment is indeed valid for  $\varphi$ .

**Direction “ $\Leftarrow$ ”:** For the converse direction, suppose that we are given a valid truth assignment  $\sigma$  for  $\varphi$ . We construct an allocation  $\pi$  as follows. First, we allocate all items appearing in claim (i) of [Lemma 1](#) as described there. Next, for each variable  $x_i$ , we let  $Y_i$  have type 1 exactly if  $\sigma$  sets  $x_i$  to true, and we let  $Y_i$  have type 2 otherwise (yielding the allocations as given in claim (ii) of [Lemma 1](#)). We also set  $\pi(v_k) = A$  for each clause  $c_k$ . Finally, we set  $\pi(v'_k) = C$  and  $\pi(v''_k) = B$  if there are 2 true literals in the clause  $c_k$  according to  $\sigma$ , and we set  $\pi(v'_k) = B$  and  $\pi(v''_k) = C$  otherwise.

It is clear that  $\pi$  is complete. To verify that it is NEF, we use the characterization given in [Proposition 1](#). Note that  $\pi$  allocates each agent its most-preferred item. Therefore, if  $\pi$  is smooth on a triadic prefix  $P$ , then it fulfills the requirements of [Proposition 1](#) for prefixes of  $P$ , and is therefore a complete NEF allocation. First, it is easy to verify that  $\pi$  allocates exactly one item to each agent from each triad of any preference list, except for the triads of  $L^A$  contained in a validity block; in other words,  $\pi$  is smooth on  $L^B$  and on  $L^C$  until the closing block  $Z$ , and on  $L^A$  until the first validity block.

Regarding  $L^A$  within some validity block  $V_k$  for some  $k \in [m]$ , by our definitions, the number of true literals in  $c_k$  equals the number of items in  $\{\ell_u, \ell_v, \ell_z\}$  allocated to  $B$  by  $\pi$  (where  $\ell_u, \ell_v$ , and  $\ell_z$  are the three choice items in the first two triads of  $V_k$ ). Hence,  $\pi$  assigns exactly two items from  $\{\ell_u, \ell_v, \ell_z, v'_k\}$  to  $B$ , and assigns the remaining two items to  $C$ . Consequently, both from the first two triads, and also from the last two triads of  $L^A$  within  $V_k$ ,  $\pi$  always assigns the first two items to agent  $A$ , followed by four items distributed among  $B$  and  $C$  evenly. This means that  $\pi$  fulfills the requirements of [Proposition 1](#) for each prefix of  $L^A$  as well. We can conclude that  $\pi$  is a complete NEF allocation.  $\square$

#### 4. Result for at least three agents

In this section we generalize [Theorem 1](#) to the case where the number of agents is a constant integer at least three.

**Theorem 2.** *For every fixed integer  $q \geq 3$ , EXISTSNEF, the problem of deciding whether a complete NEF allocation exists, is NP-complete on instances with  $q$  agents.*

Again, EXISTSNEF is clearly in NP, so we need to show its NP-hardness. To this end, we are going to modify the reduction given in the proof of [Theorem 1](#). We will re-use most of the notation defined in Section 3. The reduction is from the same variant of NOT-ALL-EQUAL 3SAT as in the proof of [Theorem 1](#), meaning that we again assume that  $\mu_i$ , the number of occurrences of variable  $x_i$ , is an even integer for each  $i \in [n]$ .

**Dummy agents and items.** We construct an instance  $(\tilde{N}, \tilde{I}, \tilde{L})$  of EXISTSNEF that contains agents  $A, B, C$  and  $q-3$  additional dummy agents denoted as  $D^1, \dots, D^{q-3}$ . We will keep the set  $I$  of items

used in the proof of [Theorem 1](#), and we define our current set of items as

$$\tilde{I} = I \cup \{d_\tau^r \mid r \in [q-3], 0 \leq \tau \leq |I|/3 - 1\}.$$

The dummy item  $d_\tau^r$  will appear at the  $(\tau q + 1)$ st position in the preference list of agent  $D^r$ ; we will make sure that any NEF allocation assigns  $d_\tau^r$  to  $D^r$ . For brevity, we let  $\langle d_\tau \rangle$  denote the sequence  $d_\tau^1, \dots, d_\tau^{q-3}$ , and we let  $\langle d_\tau \rangle^{-r}$  denote the sequence obtained from  $\langle d_\tau \rangle$  by removing the item  $d_\tau^r$ .

**Preferences.** We define the preferences  $\tilde{L}$  using the preference lists  $L^A, L^B$ , and  $L^C$  defined in the proof of [Theorem 1](#) for agents  $A, B$ , and  $C$ . Instead of considering triads (i.e., sequences of three items in the preference lists) we now decompose each preference list within a block (except for the initial and closing blocks) into sequences of  $q$  items which we will call  $q$ -ads. Formally, a  $q$ -ad is a sublist of a preference list  $\tilde{L}^X$  of some agent  $X \in \tilde{N}$  that is of the form  $\tilde{L}^X(q \cdot t + 2 : q(t+1) + 1)$  for some  $t \in \mathbb{N}$ . The number of  $q$ -ads in each preference list is then  $|\tilde{I}|/q - 1 = (|I| + (q-3)|I|/3)/q - 1 = |I|/3 - 1 = 22m$ .

First, we deal with the agents  $A, B$ , and  $C$ . To construct the new preference list  $\tilde{L}^X$  for some agent  $X \in \{A, B, C\}$ , for each  $\tau \in [22m]$  we insert  $\langle d_{\tau-1} \rangle$  at the beginning of the  $\tau$ th triad in  $L^X$ , that is, the triad  $L^X[3\tau - 1 : 3\tau + 1]$ . This way, the  $\tau$ th triad of  $L^X$  becomes the  $\tau$ th  $q$ -ad of  $\tilde{L}^X$ . To construct the closing block for some agent  $X \in \{A, B, C\}$ , we append  $\langle d_{22m} \rangle$  to the end of the preference list.

**Example 3.** Below we show how to transform the first triads of  $L^A, L^B, L^C$  into the first  $q$ -ads of  $\tilde{L}^A, \tilde{L}^B$  and  $\tilde{L}^C$ , respectively.

$$\begin{array}{lcl} L^A: a_{1,0}^3, b_{1,j-1}^3, c_{1,j-1}^3, a_{1,j}^1 & \xrightarrow{(d_0)} & \tilde{L}^A: a_{1,0}^3, \overbrace{d_0^1, d_0^2, \dots, d_0^{q-3}}^{(d_0)}, b_{1,j-1}^3, c_{1,j-1}^3, a_{1,j}^1 \\ L^B: b_{1,0}^3, a_{1,j-1}^3, \overline{bc}_{1,j}, \overline{bc}_{1,j} & \longrightarrow & \tilde{L}^B: b_{1,0}^3, d_0^1, d_0^2, \dots, d_0^{q-3}, a_{1,j-1}^3, \overline{bc}_{1,j}, \overline{bc}_{1,j} \\ L^C: c_{1,0}^3, a_{1,j-1}^3, \overline{bc}_{1,j}, \overline{bc}_{1,j} & & \tilde{L}^C: c_{1,0}^3, \overbrace{d_0^1, d_0^2, \dots, d_0^{q-3}}^{(d_0)}, a_{1,j-1}^3, \overline{bc}_{1,j}, \overline{bc}_{1,j} \end{array}$$

the first triads  the first  $q$ -ads

Second, we deal with the dummy agents. For each  $r \in [q-3]$  we construct the preference list  $\tilde{L}^{D^r}$  of dummy agent  $D^r$  based on  $L^C$  as follows. We set the item  $d_0^r$  as the most preferred item for  $D^r$ . Then, to get the  $\tau$ th  $q$ -ad for agent  $D^r$  for each  $\tau \in [22m]$ , we insert  $\langle d_{\tau-1} \rangle^{-r}$  at the beginning and  $d_\tau^r$  at the end of the  $\tau$ th triad of  $L^C$ . To obtain the closing block for  $D^r$  we append  $\langle d_{22m} \rangle^{-r}$  and also the item  $c_{1,0}^3$  to the end of its preference list.

**Example 4.** Below we show how to construct the first  $q$ -ads of the dummy agents, using the notation  $\langle d_0^{-r} \rangle$  for some  $r \in [q-3]$ . Note that the purpose of this construction is to ensure that each dummy agent  $D^r$  is assigned the item set  $\{d_\tau^r : 0 \leq \tau \leq |I|/3 - 1\}$ .



$$\begin{aligned}
\tilde{L}^{D_1}: & d_0^1, \overbrace{d_0^2, d_0^3, d_0^4, \dots, d_0^{q-3}}^{(d_0^{-1})}, a_{i,j-1}^3, \overline{bc}_{i,j}, \overline{bc}_{i,j}, d_1^1 \\
\tilde{L}^{D_2}: & d_0^2, \overbrace{d_0^1, d_0^3, d_0^4, \dots, d_0^{q-3}}^{(d_0^{-2})}, a_{i,j-1}^3, \overline{bc}_{i,j}, \overline{bc}_{i,j}, d_1^2 \\
\tilde{L}^{D_3}: & d_0^3, \overbrace{d_0^1, d_0^2, d_0^4, \dots, d_0^{q-3}}^{(d_0^{-3})}, a_{i,j-1}^3, \overline{bc}_{i,j}, \overline{bc}_{i,j}, d_1^3 \\
& \vdots \qquad \qquad \qquad \vdots \\
\tilde{L}^{D_{q-3}}: & d_0^{q-3}, \overbrace{d_0^1, d_0^2, d_0^3, \dots, d_0^{q-2}}^{(d_0^{-(q-3)})}, a_{i,j-1}^3, \overline{bc}_{i,j}, \overline{bc}_{i,j}, d_1^{q-3} \\
& \qquad \qquad \qquad \text{the first } q\text{-ads of dummy agents}
\end{aligned}$$

Next, we detail all the blocks in the new preferences defined as above. For each block  $B$  of the instance constructed in the proof of [Theorem 1](#) we let  $\tilde{B}$  denote the block we obtain by modifying  $B$  as described above. Each block comprises  $q$  lists, one for each agent, so index  $r$  in the definitions below takes on every value in  $[q-3]$ .

*Modified initial block  $\tilde{I}_0$ :*

$$\begin{aligned}
A: & a_{1,0}^3 \\
B: & b_{1,0}^3 \\
C: & c_{1,0}^3 \\
D^r: & d_0^r
\end{aligned}$$

Recall again that any complete NEF allocation assigns each agent its most preferred item. Hence, we immediately obtain the following.

**Proposition 5.** *Let  $\pi$  be a complete NEF allocation for  $(N, I, L)$ . Then  $\pi(a_{1,0}^3) = A$ ,  $\pi(b_{1,0}^3) = B$ ,  $\pi(c_{1,0}^3) = C$ , and  $\pi(d_0^r) = D^r$  for each  $r \in [q-3]$ .*

Let us now proceed with the details of modified literal blocks.

*Modified literal block  $\tilde{X}_{i,j}$  for some  $i \in [n]$  and  $j \in [\mu_i]$ :*

Let  $\tau \in [22m]$  be the number of  $q$ -ads preceding  $\tilde{X}_{i,j}$ . Recall that triads within an agent's preferences are to be read row by row, and within each row, from left to right. We first give the sublists of  $\tilde{L}^A$ ,  $\tilde{L}^B$ , and  $\tilde{L}^C$  within  $\tilde{X}_{i,j}$ .

$$\begin{aligned}
A: & \langle d_\tau \rangle, b_{i,j-1}^3, c_{i,j-1}^3, a_{i,j}^1, \langle d_{\tau+1} \rangle, b_{i,j}^1, \overline{ca}_{i,j}, \overline{ca}_{i,j}, \langle d_{\tau+2} \rangle, c_{i,j}^1, \beta_{i,j}, a_{i,j}^2, \\
& \langle d_{\tau+3} \rangle, c_{i,j}^2, \overline{ab}_{i,j}, \overline{ab}_{i,j}, \langle d_{\tau+4} \rangle, b_{i,j}^2, \gamma_{i,j}, a_{i,j}^3, \langle d_{\tau+5} \rangle, \overline{bc}_{i,j}, \overline{bc}_{i,j}, \alpha_{i,j} \\
B: & \langle d_\tau \rangle, a_{i,j-1}^3, \overline{bc}_{i,j}, \overline{bc}_{i,j}, \langle d_{\tau+1} \rangle, c_{i,j-1}^3, \alpha_{i,j}, b_{i,j}^1, \langle d_{\tau+2} \rangle, a_{i,j}^1, c_{i,j}^1, b_{i,j}^2, \\
& \langle d_{\tau+3} \rangle, c_{i,j}^2, \overline{ab}_{i,j}, \overline{ab}_{i,j}, \langle d_{\tau+4} \rangle, a_{i,j}^2, \gamma_{i,j}, b_{i,j}^3, \langle d_{\tau+5} \rangle, \overline{ca}_{i,j}, \overline{ca}_{i,j}, \beta_{i,j}, \overline{bc}_{i,j} \\
C: & \langle d_\tau \rangle, a_{i,j-1}^3, \overline{bc}_{i,j}, \overline{bc}_{i,j}, \langle d_{\tau+1} \rangle, b_{i,j-1}^3, \alpha_{i,j}, c_{i,j}^1, \langle d_{\tau+2} \rangle, b_{i,j}^1, \overline{ca}_{i,j}, \overline{ca}_{i,j}, \\
& \langle d_{\tau+3} \rangle, a_{i,j}^1, \beta_{i,j}, c_{i,j}^2, \langle d_{\tau+4} \rangle, a_{i,j}^2, b_{i,j}^2, c_{i,j}^3, \langle d_{\tau+5} \rangle, \overline{ab}_{i,j}, \overline{ab}_{i,j}, \alpha_{i,j}, \overline{bc}_{i,j}
\end{aligned}$$

Now we proceed with the sublists of  $\tilde{L}^{D^r}$  within  $\tilde{X}_{i,j}$  for some  $r \in [q-3]$ .

$$\begin{aligned}
D^r: & \langle d_\tau \rangle^{-r}, a_{i,j-1}^3, \overline{bc}_{i,j}, \overline{bc}_{i,j}, d_{\tau+1}^r, \langle d_{\tau+1} \rangle^{-r}, b_{i,j-1}^3, \alpha_{i,j}, c_{i,j}^1, d_{\tau+2}^r, \\
& \langle d_{\tau+2} \rangle^{-r}, b_{i,j}^1, \overline{ca}_{i,j}, \overline{ca}_{i,j}, d_{\tau+3}^r, \langle d_{\tau+3} \rangle^{-r}, a_{i,j}^1, \beta_{i,j}, c_{i,j}^2, d_{\tau+4}^r, \\
& \langle d_{\tau+4} \rangle^{-r}, a_{i,j}^2, b_{i,j}^2, c_{i,j}^3, d_{\tau+5}^r, \langle d_{\tau+5} \rangle^{-r}, \overline{ab}_{i,j}, \overline{ab}_{i,j}, \alpha_{i,j}, \overline{bc}_{i,j}, d_{\tau+6}^r
\end{aligned}$$

As we will later see in [Lemma 6](#), the arguments we applied to the original literal block  $X_{i,j}$  will remain applicable for  $\tilde{X}_{i,j}$  as well. The key to this statement will be the observation that any complete

NEF allocation must assign all dummies the items intended for them.

*Modified equivalence block  $\tilde{E}_{i,2j}$  for some  $i \in [n]$  and  $j \in [\mu_i/2]$ :* As before, agent  $A$  will have an empty sublist contained in  $\tilde{E}_{i,2j}$ , while every other agent will have two  $q$ -ads of their preference lists contained in  $\tilde{E}_{i,2j}$ . Let again  $\tau \in [22m]$  denote the number of  $q$ -ads preceding  $\tilde{E}_{i,2j}$  in each of the preference lists  $\tilde{L}^X$  where  $X \in \tilde{N} \setminus \{A\}$ .

$$\begin{aligned}
A: & - \\
B: & \langle d_\tau \rangle, \overline{ca}_{i,2j-1}, \alpha_{i,2j}, \beta_{i,2j-1}, \langle d_{\tau+1} \rangle, \alpha_{i,2j-1}, \overline{ca}_{i,2j}, \beta_{i,2j} \\
C: & \langle d_\tau \rangle, \overline{ab}_{i,2j}, \alpha_{i,2j+1}, \gamma_{i,2j}, \langle d_{\tau+1} \rangle, \alpha_{i,2j}, \overline{ab}_{i,2j+1}, \gamma_{i,2j+1} \\
D^r: & \langle d_\tau \rangle^{-r}, \overline{ab}_{i,2j}, \alpha_{i,2j+1}, \gamma_{i,2j}, d_{\tau+1}^r, \langle d_{\tau+1} \rangle^{-r}, \overline{ab}_{i,2j}, \alpha_{i,2j+1}, \gamma_{i,2j+1}, d_{\tau+2}^r
\end{aligned}$$

*Modified validity block  $\tilde{V}_k$  for some  $k \in [m]$ :* Set integers  $\tau$  and  $\rho$  such that the number of  $q$ -ads preceding  $\tilde{V}_k$  in agent  $A$ 's preference list is  $\tau$ , and the number of  $q$ -ads preceding  $\tilde{V}_k$  in every other agent's preference list is  $\rho$ . The choice items  $\ell_u$ ,  $\ell_v$ ,  $\ell_z$ ,  $\overline{\ell}_u$ ,  $\overline{\ell}_v$ , and  $\overline{\ell}_z$  are defined as for the validity block  $V_k$ . That is, let the clause  $c_k$  contain the  $j_u$ th occurrence of variable  $x_u$  in  $\varphi$ ; if  $x_u$  appears as a positive literal in  $c_k$ , then we set  $\ell_u = bc_{u,j_u}$  and  $\overline{\ell}_u = \overline{bc}_{u,j_u}$ , and similarly, if  $x_u$  appears as a negative literal in  $c_k$ , then we set  $\ell_u = \overline{bc}_{u,j_u}$  and  $\overline{\ell}_u = bc_{u,j_u}$ . Using analogous definitions for the items  $\ell_v$ ,  $\ell_z$ ,  $\overline{\ell}_v$ , and  $\overline{\ell}_z$ , the modified validity block  $\tilde{V}_k$  can be written as follows.

$$\begin{aligned}
A: & \langle d_\tau \rangle, \alpha_{u,j_u}, v_k, \ell_u, \langle d_{\tau+1} \rangle, \ell_v, \ell_z, v'_k, \\
& \langle d_{\tau+2} \rangle, \alpha_{v,j_v}, \alpha_{z,j_z}, \overline{\ell}_u, \langle d_{\tau+3} \rangle, \overline{\ell}_v, \overline{\ell}_z, v''_k \\
B: & \langle d_\rho \rangle, v_k, v'_k, v''_k \\
C: & \langle d_\rho \rangle, v_k, v'_k, v''_k \\
D^r: & \langle d_\rho \rangle^{-r}, v_k, v'_k, v''_k, d_{\rho+1}^r
\end{aligned}$$

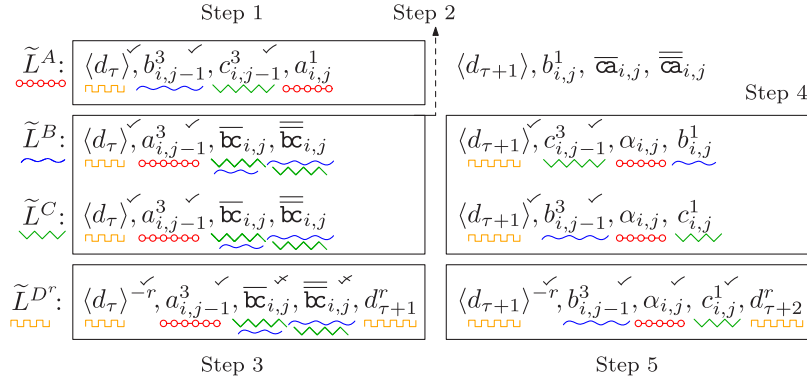
*Modified closing block  $\tilde{Z}$ :*

$$\begin{aligned}
A: & b_{n,\mu_n}^3, c_{n,\mu_n}^3, \langle d_{22m} \rangle \\
B: & a_{n,\mu_n}^3, c_{n,\mu_n}^3, \langle d_{22m} \rangle \\
C: & a_{n,\mu_n}^3, b_{n,\mu_n}^3, \langle d_{22m} \rangle \\
D^r: & a_{n,\mu_n}^3, b_{n,\mu_n}^3, \langle d_{22m} \rangle^{-r}, c_{1,0}^3
\end{aligned}$$

It is not hard to verify that the above modified preferences are well-formed, i.e., each preference list is a linear ordering of the set of items.

**Correctness.** Proving the correctness of our construction can be done along the same lines as in the proof of [Theorem 1](#). We start by introducing a smoothness notion for allocations involving  $q$  agents. Let a  $q$ -adic prefix be a prefix of some preference list of length  $qt+1$  for some integer  $t$  with  $qt+1 \leq |I|$ . An allocation  $\pi$  is  $q$ -smooth on the  $q$ -adic prefix  $\tilde{L}^X(1 : qt+1)$  for some agent  $X \in \tilde{N}$  and  $t \in [|I|/3 - 1]$ , if  $\pi$  assigns to each agent exactly one item from each  $q$ -ad contained in  $\tilde{L}^X(1 : qt+1)$ , and additionally, assigns to  $X$  its most preferred item. We also say that  $\pi$  is  $q$ -smooth until a given block  $B$  in  $(\tilde{N}, \tilde{I}, \tilde{L})$ , if for each agent  $X \in \tilde{N}$  it is smooth on the prefix of  $\tilde{L}^X$  ending right before the block  $B$ . We can quickly state an analog of [Proposition 3](#); since the proof uses exactly the same arguments as the proof of [Proposition 3](#), we omit it.

**Proposition 6.** *Let  $\pi$  be a complete NEF allocation for  $(\tilde{N}, \tilde{I}, \tilde{L})$  that is smooth on a  $q$ -adic prefix  $P$  of  $\tilde{L}^X$  for some agent  $X \in \tilde{N}$ . Then  $\pi$  allocates at least one item to  $X$  from the  $q$ -ad following  $P$  in  $\tilde{L}^X$ . More generally,  $\pi$  allocates at least  $t$  items to  $X$  from the  $t$  consecutive  $q$ -ads following  $P$  in  $\tilde{L}^X$  for each  $t$  satisfying  $|P| + qt \leq |I|$ .*



**Fig. 7.** Illustration of the proof of Lemma 6 for some literal block  $\tilde{X}_{i,j}$ , describing the first five steps of our arguments (Steps 1–5). The figure depicts the first two  $q$ -ads for each agent, with the  $q$ -ads considered in each step grouped together. We retain the notation from Figs. 3–6 with the addition that we underline items allocated to (specific) dummy agents with a meander-style (orange) line.

We proceed by showing an analog of Lemma 1 that also deals with dummies.

**Lemma 6.** Let  $\pi$  be a complete NEF allocation for  $(\tilde{N}, \tilde{I}, \tilde{L})$ , and let  $i \in [n]$  and  $j \in [\mu_i]$ . Let  $\tau$  denote the number of  $q$ -ads preceding  $\tilde{X}_{i,j}$  in the preference lists. If  $\pi$  is  $q$ -smooth until the literal block  $\tilde{X}_{i,j}$ , then statements (i) and (ii) of Lemma 1 hold, and additionally:

- (iii) For each  $\tau+1 \leq t \leq \tau+6$  and  $r \in [q-3]$  we have  $\pi(d_t^r) = D^r$ .
- (iv)  $\pi$  remains  $q$ -smooth until the block following  $\tilde{X}_{i,j}$ .

**Proof.** The proof is a direct analog of the proof of Lemma 1. Again we use induction on indices  $i$  and  $j$ . So fix some  $i \in [n]$  and  $j \in [\mu_i]$ , and let  $\tau$  be the number of  $q$ -ads preceding  $\tilde{X}_{i,j}$ . The precise induction statement we use the following: we assume that claim (iii) holds for  $\tau$ , and that claim (iv), as well as claims (i) and (ii) from Lemma 1 hold for the last literal block  $\tilde{X}_{i',j'}$  preceding  $\tilde{X}_{i,j}$ , unless  $i = j = 1$ .

We claim  $\pi(a_{i,j-1}^3) = A$ ,  $\pi(b_{i,j-1}^3) = B$ ,  $\pi(c_{i,j-1}^3) = C$  and  $\pi(d_t^r) = D^r$ ; recall that  $\tau$  is the number of  $q$ -ads preceding  $\tilde{X}_{i,j}$ . If  $i = j = 1$  and hence  $\tau = 0$ , then this follows from Proposition 5. Otherwise, recall that  $\tilde{X}_{i',j'}$  is the last literal block preceding  $\tilde{X}_{i,j}$ . Observe also that since  $\pi$  is  $q$ -smooth until  $\tilde{X}_{i,j}$ , it is also  $q$ -smooth until  $\tilde{X}_{i',j'}$ . Hence, the claim follows from our induction hypothesis for  $\tilde{X}_{i',j'}$ , using that the number of  $q$ -ads preceding  $\tilde{X}_{i',j'}$  is  $\tau - 6$ ; recall that if  $i > 1$ , then  $a_{i,0}^3 = a_{i-1,\mu_{i-1}}^3$ ,  $b_{i,0}^3 = b_{i-1,\mu_{i-1}}^3$ , and  $c_{i,0}^3 = c_{i-1,\mu_{i-1}}^3$ .

We prove the induction statements  $q$ -ad by  $q$ -ad using essentially the same arguments as in the proof of Lemma 1. Nevertheless, we clearly have to take into account the presence of dummy agents and dummy items, and show that our arguments can be applied in the modified instance as well. At each step, when we consider a given set of  $q$ -ads, we rely on the  $q$ -smoothness of  $\pi$  on the  $q$ -adic prefixes preceding these given triads. Using Proposition 6 and the structure of  $\tilde{X}_{i,j}$  we will then prove that  $\pi$  remains  $q$ -smooth also on the  $q$ -adic prefixes ending with the given  $q$ -ads. We provide an illustration in Figs. 7–10, which describe the chain of our reasoning broken down into Steps 1–16. To avoid repetition, we will omit those parts of the proof that require no arguments other than those already presented in the proof of Lemma 1.

Since  $\pi$  is  $q$ -smooth until  $\tilde{X}_{i,j}$ , by Proposition 6 we know that each agent has to obtain at least one item from its  $q$  most preferred items in  $\tilde{X}_{i,j}$  to ensure necessary envy-freeness. Consider the first  $q$ -ads for agents  $A$ ,  $B$ , and  $C$  within  $\tilde{X}_{i,j}$ , each starting with the item series  $\langle d_\tau \rangle$ ; for an illustration see Fig. 7. By our inductive

assumption, we know that all of these items are allocated to dummy agents; namely  $\pi(d_t^r) = D^r$  for each  $r$ . Thus, we can argue about the remaining three items within the first  $q$ -ads of  $\tilde{L}^A$ ,  $\tilde{L}^B$ , and  $\tilde{L}^C$  exactly as we did for the corresponding triads in the proof of Lemma 1 to obtain  $\pi(a_{i,j}^1) = A$  and  $\{\pi(\bar{b}_{i,j}), \pi(\bar{c}_{i,j})\} = \{B, C\}$ , as shown in Steps 1 and 2 of Fig. 7.

Considering now the first  $q$ -ad of the preference list of the dummy agent  $D^r$  within  $\tilde{X}_{i,j}$ , this means that  $\pi$  assigns exactly one item from this  $q$ -ad to each agent other than  $D^r$ . Therefore, by Proposition 6,  $\pi$  must assign the single remaining item, namely  $d_{\tau+1}^r$ , to  $D^r$  (Step 3 of Fig. 7).

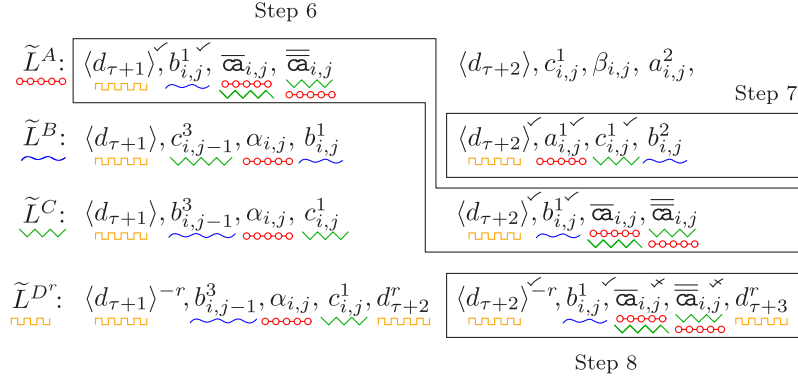
Considering the second  $q$ -ads for agents  $B$  and  $C$  as shown in Step 4, we can see that  $\tilde{L}^B(1 : q(\tau+2))$  and  $\tilde{L}^C(1 : q(\tau+2))$  both end with the item  $\alpha_{i,j}$ . As  $\pi$  is  $q$ -smooth until  $\tilde{X}_{i,j}$ , and by the above properties of  $\pi$  the prefix  $\tilde{L}^B(1 : q(\tau+2) - 1)$  contains  $\tau+2$  items allocated by  $\pi$  to each agent except for agent  $A$ , who gets  $\tau+1$  items. Similarly, the prefix  $\tilde{L}^C(1 : q(\tau+2) - 1)$  contains  $\tau+2$  items allocated by  $\pi$  to each agent except for agent  $A$ , who gets  $\tau+1$  items. Thus,  $\alpha_{i,j}$  can be allocated neither to  $C$  or a dummy agent (since then  $\tilde{L}^B(1 : q(\tau+2))$  would violate envy-freeness by Proposition 1), nor to  $B$  (since then  $\tilde{L}^C(1 : q(\tau+2))$  would violate envy-freeness by Proposition 1). This proves  $\pi(\alpha_{i,j}) = A$ , which in turn leads to  $\pi(d_{\tau+2}^r) = D^r$  for each  $r \in [q-3]$ , as shown in Step 5 in Fig. 7.

Proceeding in this fashion with Steps 6–8 in Fig. 8, we can argue that  $\{\pi(\bar{a}_{i,j}), \pi(\bar{b}_{i,j})\} = \{C, A\}$ ,  $\pi(b_{i,j}^2) = B$ , and also that  $\pi(d_{\tau+3}^r) = D^r$  for each  $r \in [q-3]$ . Next, we again use the reasoning of the previous paragraph to show that  $\pi(\beta_{i,j}) = B$ ,  $\pi(a_{i,j}^2) = A$  and  $\pi(c_{i,j}^2) = C$ ; see Step 9 in Fig. 9.

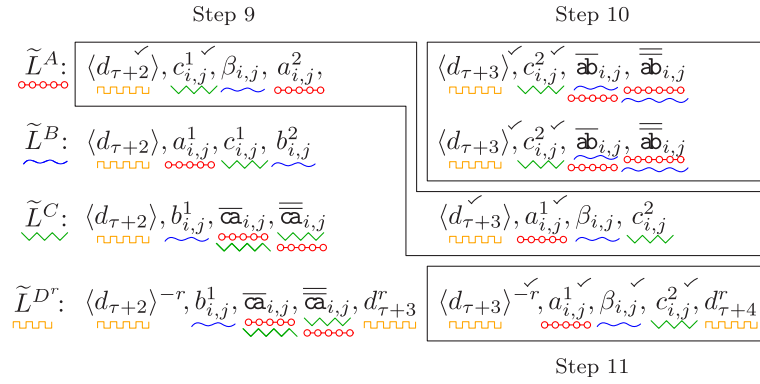
Considering the fourth  $q$ -ads for agents  $A$ ,  $B$  and some dummy agent  $D^r$ , Steps 10 and 11 in Fig. 9 show that we can obtain  $\{\pi(\bar{a}_{i,j}), \pi(\bar{b}_{i,j})\} = \{A, B\}$  and  $\pi(d_{\tau+4}^r) = D^r$  for each  $r \in [q-3]$ . Considering the fifth  $q$ -ads within  $\tilde{X}_{i,j}$  as shown in Steps 12–14 of Fig. 10 we obtain  $\pi(c_{i,j}^3) = C$ , and then by the reasoning we applied in Steps 4 and 9 before we also get  $\pi(\gamma_{i,j}) = C$ , which yields  $\pi(b_{i,j}^3) = B$  and  $\pi(c_{i,j}^3) = C$ , and finally  $\pi(d_{\tau+5}^r) = D^r$  for each  $r \in [q-3]$ .

Looking at the last  $q$ -ads for agents  $A$ ,  $B$ , and  $C$ , we can apply the same arguments as in the proof of Lemma 1 to show that  $\tilde{X}_{i,j}$  either has type 1 or type 2 (Step 15 in Fig. 10). This proves that claims (i) and (ii) of Lemma 1 hold for  $\tilde{X}_{i,j}$ . We finish our proof by observing that  $\pi(d_{\tau+6}^r) = D^r$  for each  $r \in [q-3]$  (Step 16 in Fig. 10), which proves claims (iii) and (iv).  $\square$

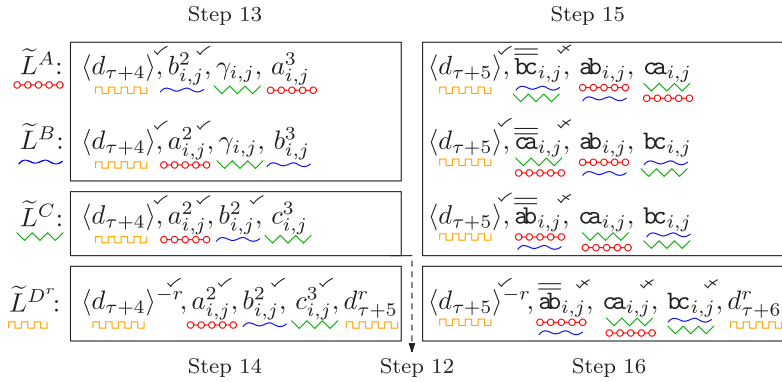
**Lemma 7.** Let  $\pi$  be a complete NEF allocation for  $(\tilde{N}, \tilde{I}, \tilde{L})$ , and let  $i \in [n]$  and  $j \in [\mu_i/2]$ . Let also  $\tau$  denote the number of  $q$ -ads



**Fig. 8.** Illustration of the proof of Lemma 6 for some literal block  $\tilde{X}_{i,j}$ , describing Steps 6–8 of our arguments. The figure depicts the second and third  $q$ -ads for each agent.



**Fig. 9.** Illustration of the proof of Lemma 6 for some literal block  $\tilde{X}_{i,j}$ , describing Steps 9–11 of our arguments. The figure depicts the third and fourth  $q$ -ads for each agent.



**Fig. 10.** Illustration of the proof of Lemma 6 for some literal block  $\tilde{X}_{i,j}$ , describing Steps 12–16 of our arguments. The figure depicts the fifth and sixth  $q$ -ads for each agent.

in  $\tilde{L}^B$  and  $\tilde{L}^C$  preceding the equivalence block  $\tilde{E}_{i,2j}$ . If  $\pi$  is  $q$ -smooth until  $\tilde{E}_{i,2j}$ , then:

- (i)  $\pi(d_{\tau+1}^r) = \pi(d_{\tau+2}^r) = D^r$  for each  $r \in [q-3]$ .
- (ii) The literal blocks  $\tilde{X}_{i,2j-1}$ ,  $\tilde{X}_{i,2j}$ , and  $\tilde{X}_{i,2j+1}$  all have the same type (where indices are taken modulo  $\mu_i$  so that  $\tilde{X}_{i,2\mu_i+1} = \tilde{X}_{i,1}$ ).
- (iii)  $\pi$  remains  $q$ -smooth until the block following  $\tilde{E}_{i,2j}$ .

**Proof.** The proof is by induction on  $i$  and  $j$ , and is a direct analog of the proof of Lemma 2. By our assumption on the  $q$ -smoothness

of  $\pi$  we can apply Lemma 6 for any literal block  $\tilde{X}_{i,j'}$ , as they all precede  $\tilde{E}_{i,2j}$  (see Fig. 2). Thus, we get  $\{\pi(\check{a}_{i,j'}), \pi(\check{\bar{a}}_{i,j'})\} = \{A, B\}$  and  $\pi(\gamma_{i,j'}) = C$ , and similarly,  $\{\pi(\check{a}_{i,j'}), \pi(\check{\bar{a}}_{i,j'})\} = \{A, C\}$  and  $\pi(\beta_{i,j'}) = B$  for each  $j' \in [\mu_i]$ . We also know that  $\pi(d_{\tau}^r) = D^r$  for each  $r \in [q-3]$ : for  $i = j = 1$  this follows from Lemma 6 for the last literal block, while for  $i + j > 2$  this follows by our induction hypothesis that claim (i) holds for the dummy items in the equivalence block preceding  $\tilde{E}_{i,2j}$ . Hence, the single item that  $\pi$  can assign to  $D^r$  from its first  $q$ -ad within the block  $\tilde{E}_{i,2j}$  is  $d_{\tau+1}^r$ , so by Proposition 6 we get  $\pi(d_{\tau}^r) = D^r$ . Repeating this argument

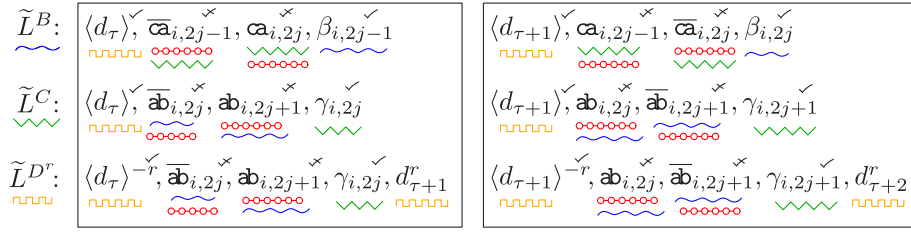


Fig. 11. Illustration of the proof of Lemma 7 for some equivalence block  $\tilde{E}_{i,2j}$ .

again for the second  $q$ -ad for  $D^r$  we obtain  $\pi(d_{\tau+2}^r) = D^r$  as well, proving claim (i). See Fig. 11 for an illustration.

Considering the first  $q$ -ad of  $\tilde{L}^B$  within the block, the  $q$ -smoothness of  $\pi$  implies that among all items in the prefix  $\tilde{L}^B(1 : (\tau+1)q)$ , ending with  $\bar{a}_{i,2j}$ ,  $\pi$  assigns exactly  $\tau+1$  items to  $B$ . If the literal blocks  $\tilde{X}_{i,2j}$  and  $\tilde{X}_{i,2j-1}$  are not of the same type, then  $\pi$  would allocate both items  $\bar{a}_{i,2j-1}$  and  $\bar{a}_{i,2j}$  to the same agent ( $A$  or  $C$ ), meaning that  $\pi$  would assign  $\tau+2$  items to either  $A$  or to  $C$  from the prefix  $\tilde{L}^B(1 : (\tau+1)q)$ , more than it assigns to  $B$ , contradicting the envy-freeness of  $\pi$  by Proposition 1. Similarly, considering the first  $q$ -ad of  $\tilde{L}^C$  within  $\tilde{E}_{i,2j}$  the same arguments yield that  $\tilde{X}_{i,2j}$  and  $\tilde{X}_{i,2j+1}$  must be of the same type (when taking indices modulo  $\mu_i$ ), proving claim (ii).

Observe that the above facts also imply that  $\pi$  allocates an item from every  $q$ -ad of  $\tilde{E}_{i,2j}$  to each agent, showing that  $\pi$  remains  $q$ -smooth until the block following  $\tilde{E}_{i,2j}$ . This finishes the proof of the lemma.  $\square$

Next we prove an analog of Lemma 3 saying that a complete NEF allocation is  $q$ -smooth on most parts of the constructed instance, and it is *almost*  $q$ -smooth on the remaining parts. Formally, an allocation  $\pi$  is  $q$ -half-smooth on  $\tilde{L}^X(1 : 2qt + 1)$ , or equivalently, on the first  $2t$   $q$ -ads for some agent  $X \in \tilde{N}$  and integer  $t$  (with  $2(t+1)q \leq |\tilde{I}|$ ), if  $\pi$  assigns to each agent exactly two items from the two  $q$ -ads in  $\tilde{L}^X(2q(t'-1) + 1 : 2qt' + 1)$  for each  $t' \in [t]$ , and additionally, assigns to  $X$  its most preferred item.

**Lemma 8.** Suppose  $\pi$  is a complete NEF allocation for  $(\tilde{N}, \tilde{I}, \tilde{L})$ . Then  $\pi$  is  $q$ -smooth on  $\tilde{L}^A$  until the first validity block, and it is  $q$ -smooth on  $\tilde{L}^X$  for every other agent  $X \in \tilde{N} \setminus \{A\}$  until the closing block. Furthermore,  $\pi$  is  $q$ -half-smooth on  $\tilde{L}^A$  until the closing block, and satisfies  $\pi(d_t^r) = D^r$  for each  $r \in [q-3]$  and  $t \in [22m] = \lfloor |\tilde{I}|/q - 1 \rfloor$ .

**Proof.** Since  $\pi$  is  $q$ -smooth until the first literal block due to Proposition 5, Lemmas 6 and 7 imply that  $\pi$  is  $q$ -smooth until the first validity block  $\tilde{V}_1$ . Then, using again Lemmas 6 and 7 we obtain that  $\pi(d_t^r) = D^r$  holds for every index  $t$  such that  $d_t^r$  is contained in a  $q$ -ad of  $\tilde{L}^{D^r}$  preceding the first validity block.

We will now show that, for each  $k \in [m]$ ,  $\pi$  is  $q$ -half-smooth on  $\tilde{L}^A$  until the block following  $\tilde{V}_k$ , it is  $q$ -smooth on  $\tilde{L}^X$  for every other agent  $X \in \tilde{N} \setminus \{A\}$  until the block following  $\tilde{V}_k$ , and moreover, for any  $r \in [q-3]$ ,  $\pi(d_t^r) = D^r$  for all dummy items  $d_t^r$  appearing in  $\tilde{L}^{D^r}$  either within  $\tilde{V}_k$  or earlier. Observe that by the previous paragraph, this claim for  $k = m$  suffices to prove the lemma.

We show our claim using induction on  $k$ ; see Fig. 12 for an illustration. Let  $\tau$  and  $\rho$  denote the number of  $q$ -ads in  $\tilde{L}^A$  and in  $\tilde{L}^B$ , respectively, that precede  $\tilde{V}_k$ . Note that if  $k = m$ , then  $\tau+3 = \rho$ , since the last  $q$ -ads of  $\tilde{V}_k$  must end at the same position for every agent. Otherwise, i.e., if  $k \in [m-1]$ , then  $\tau+3 < \rho$ . Therefore, for any  $r \in [q-3]$  all items in  $\{d_t^r, d_{\tau+1}^r, d_{\tau+2}^r, d_{\tau+3}^r, d_\rho^r\}$  have already appeared in the  $q$ -adic prefix of  $D^r$  that ends right before  $\tilde{V}_k$ . By the first paragraph of this proof (for  $k = 1$ ) and by

our inductive hypothesis (for  $k > 1$ ) we know that  $\pi$  allocates each of these items to  $D^r$ .

Consider the  $q$ -ads of  $\tilde{L}^B$  and  $\tilde{L}^C$  and the first two  $q$ -ads of  $\tilde{L}^A$  within  $\tilde{V}_k$ ; see Step 1 on Fig. 12. By our assumptions on the  $q$ -smoothness of  $\pi$  until  $\tilde{V}_1$ , Lemma 6 implies that  $\pi$  assigns each choice item within  $\tilde{V}_k$  either to  $B$  or to  $C$ . Note that  $\pi$  is  $q$ -smooth on  $\tilde{L}^B$  and  $\tilde{L}^C$  until  $\tilde{V}_k$ : for  $k = 1$  this follows from our first paragraph, for  $k > 1$  from our inductive hypothesis. Therefore, Proposition 6 implies that  $\pi$  must allocate both  $B$  and  $C$  at least one item from  $\{v_k, v'_k, v''_k\}$ . Similarly, we know that  $\pi$  is  $q$ -half-smooth on  $\tilde{L}^A$  until  $\tilde{V}_k$ , and hence (using an analog of Proposition 6 for  $q$ -half-smoothness, based on the same reasoning) we obtain that  $\pi$  must allocate at least one item from  $\{v_k, v'_k\}$  to  $A$ . This means  $\{\pi(v_k), \pi(v'_k), \pi(v''_k)\} = \{A, B, C\}$ .

Taking into account the unique  $q$ -ad of  $\tilde{L}^{D^r}$  within  $\tilde{V}_k$  for some  $r \in [q-3]$ , it follows that  $\pi(d_{\rho+1}^r) = D^r$ ; see Step 2 on Fig. 12. Consequently,  $\pi$  remains  $q$ -smooth on  $\tilde{L}^X$  until the block following  $\tilde{V}_k$  for each agent  $X$  other than  $A$ .

Since  $\pi$  can assign at most one item from  $\{v_k, v'_k, v''_k\}$  to agent  $A$ , we get that  $\pi$  must assign exactly two items from the first two  $q$ -ads of  $\tilde{L}^A$  within  $\tilde{V}_k$  to each agent. Arguing the same way for the third and fourth  $q$ -ads of  $\tilde{L}^A$ , shown in Step 3 on Fig. 12, we obtain that  $\pi$  remains  $q$ -half-smooth on  $\tilde{L}^A$  until the block following  $\tilde{V}_k$ , finishing our proof.  $\square$

Using Lemmas 7 and 8 we immediately get the following corollary; its proof is the same as the proof of Lemma 4, so we omit it.

**Lemma 9.** Let  $\pi$  be a complete NEF allocation for  $(\tilde{N}, \tilde{I}, \tilde{L})$ , and let  $i \in [n]$ . Then all literal blocks in  $Y_i$  have the same type; we call this the type of  $Y_i$ .

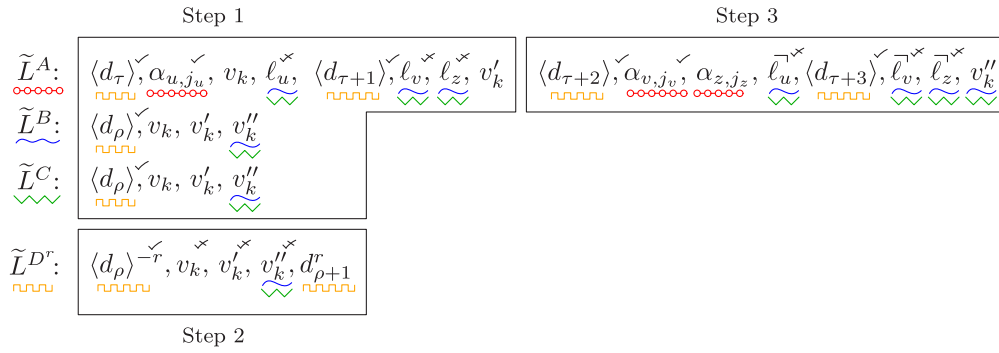
It is straightforward to verify that using Lemmas 6–9 the same arguments we applied to prove the correctness of the reduction in Section 3 also imply Lemma 10 below, and consequently also Theorem 2; we leave the details to the reader.

**Lemma 10.** There exists a valid allocation for the input formula  $\varphi$  if and only if the constructed instance  $(\tilde{N}, \tilde{I}, \tilde{L})$  admits a complete NEF allocation.

## 5. Conclusion

We examined a fundamental fair division setting for indivisible items under ordinal preferences and focused on necessary envy-freeness. In the literature on cardinal valuations, a natural relaxation of envy-freeness is the ‘up to one item’ (EF1) relaxation in which envy is allowed as long as it goes away after ignoring some item (Caragiannis et al., 2019). Similar to EF1, one can define a necessary version of EF1 called NEF1 where the requirements of NEF are met if for any envy comparison between agents, we ignore some item. Such a notion is easily satisfied by





**Fig. 12.** Illustration of the proof of Lemma 8 depicting some validity block  $\tilde{V}_k$ .

allocating most valuable items among agents in a round robin manner (Aziz, 2020b).

We resolved an outstanding open problem and proved that checking whether a necessary envy-free allocation exists is NP-complete when the number of agents is at least three. It will be interesting to identify conditions under which the problem is polynomial-time solvable. For example, does it help if the preferences are single-peaked? Another interesting direction is to explore the probability for an instance to admit a necessary envy-free allocation under some well-studied probabilistic models of generating instances.

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## Data availability

No data was used for the research described in the article.

## References

- Aziz, H., 2016. A generalization of the AL method for fair allocation of indivisible objects. *Econ. Theory Bull.* 4, 307–324.
- Aziz, H., 2020a. Developments in multi-agent fair allocation. In: *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI 2020)*. AAAI Press, pp. 13563–13568.
- Aziz, H., 2020b. Simultaneously achieving ex-ante and ex-post fairness. In: *Proceedings of the 16th Conference on Web and Internet Economics (WINE 2020)*. Springer, pp. 341–355.
- Aziz, H., Gaspers, S., Mackenzie, S., Walsh, T., 2015. Fair assignment of indivisible objects under ordinal preferences. *Artificial Intelligence* 227, 71–92.
- Aziz, H., Schlotter, I., Walsh, T., 2016. Control of fair division. In: *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI 2016)*. AAAI Press, pp. 67–73.
- Bouveret, S., Chevaleyre, Y., Maudet, N., 2016. Fair allocation of indivisible goods. In: Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A.D. (Eds.), *Handbook of Computational Social Choice*. Cambridge University Press, pp. 284–310.
- Bouveret, S., Endriss, U., Lang, J., 2010. Fair division under ordinal preferences: Computing envy-free allocations of indivisible goods. In: *Proceedings of the 19th European Conference on Artificial Intelligence (ECAI 2010)*. In: *Frontiers in Artificial Intelligence and Applications*, vol. 215, IOS Press, pp. 387–392.
- Brams, S.J., Edelman, P.H., Fishburn, P.C., 2001. Paradoxes of fair division. *J. Philos.* 98 (6), 300–314.
- Brams, S.J., Kilgour, D.M., Klamler, C., 2014. Two-person fair division of indivisible items: An efficient, envy-free algorithm. *Notices Amer. Math. Soc.* 61 (2), 130–141.
- Brams, S.J., Taylor, A.D., 1996. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press.
- Bredereck, R., Figiel, A., Kaczmarczyk, A., Knop, D., Niedermeier, R., 2021. High-multiplicity fair allocation made more practical. In: *Proceedings of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2021)*. Association for Computing Machinery, pp. 260–268.
- Bredereck, R., Kaczmarczyk, A., Knop, D., Niedermeier, R., 2019. High-multiplicity fair allocation: Lenstra empowered by N-fold integer programming. In: *Proceedings of the 2019 ACM Conference on Economics and Computation (EC 2019)*. Association for Computing Machinery, pp. 505–523.
- Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A.D., Shah, N., Wang, J., 2019. The unreasonable fairness of maximum Nash welfare. *ACM Trans. Econ. Comput.* 7 (3), 12.
- Goldman, J., Procaccia, A.D., 2014. Spliddit: Unleashing fair division algorithms. *ACM SIGecom Exch.* 13 (2), 41–46.
- Moulin, H., 2003. *Fair Division and Collective Welfare*. MIT Press.
- Schaefer, T.J., 1978. The complexity of satisfiability problems. In: *Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC 1978)*. Association for Computing Machinery, pp. 216–226.