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ON CLASSES OF FUNCTIONS WITH VARYING ARGUMENT OF COEFFICIENTS

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Abstract. In this paper we give the coefficients estimates, distortion properties, radii of starlikeness and convexity for classes of functions with varying argument of coefficients defined by convolution and subordination.

1. Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U}=\mathcal{U}(1)$, where

$$\mathcal{U}(r)=\{z\in\mathbb{C}:|z|<r\}.$$

and let $\mathcal{A}(p,k)$ $(p,k\in\mathbb{N}=\{1,2,3,\dots\},\ p<k)$ denote the class of functions $f\in\mathcal{A}$ of the form

(1)
$$f(z)=a_pz^p+\sum_{n=k}^{\infty}a_nz^n\quad(z\in\mathcal{U};\ a_p>0).$$

For multivalent function $f\in\mathcal{A}(p,k)$ the normalization

(2)
$$\left.\frac{f(z)}{z^{p-1}}\right|_{z=0}=0\text{ and }\left.\frac{f'(z)}{z^{p-1}}\right|_{z=0}=p.$$

is classical. One can obtain interesting results by applying normalization of the form

(3)
$$\left.\frac{f(z)}{z^{p-1}}\right|_{z=0}=0\text{ and }\left.\frac{f'(z)}{z^{p-1}}\right|_{z=\rho}=p.$$

where ρ is a fixed point of the unit disk $\mathcal{U}$. In particular, for $p=1$ we obtain Montel's normalization (cf. [7]).

We denote by $\mathcal{A}_\rho(p,k)$ the classes of functions $f\in\mathcal{A}(p,k)$ with the normalization (3). It will be called the class of functions with two fixed points.

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Also, by $\mathcal{T}(p,k;\eta)$ $(\eta \in \mathbb{R})$ we denote the class of functions $f \in \mathcal{A}(p,k)$ of the form (1) for which all of non-vanishing coefficients satisfy the condition

$$\arg(a_n) = \pi + (p-n)\eta \quad (n = k, k+1, \ldots). \tag{4}$$

For $\eta = 0$ we obtain the class $\mathcal{T}(p,k;0)$ of functions with negative coefficients. Moreover, we define

$$\mathcal{T}(p,k) := \bigcup_{\eta\in\mathbb{R}} \mathcal{T}(p,k;\eta). \tag{5}$$

The classes $\mathcal{T}(p,k)$ and $\mathcal{T}(p,k;\eta)$ are called the classes of functions with varying argument of coefficients. The class $\mathcal{T}(1,2)$ was introduced by Silverman [8] (see also [10]).

Let $\alpha \in \langle 0,p)$, $r \in (0,1\rangle$. A function $f \in \mathcal{A}(p,k)$ is said to be *convex of order* α in $\mathcal{U}(r)$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathcal{U}(r)).$$

A function $f \in \mathcal{A}(p,k)$ is said to be *starlike of order* α *in* $\mathcal{U}(r)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{U}(r)). \tag{6}$$

We denote by $\mathcal{S}^c(\alpha)$ the class of all functions $f \in \mathcal{A}(p,p+1)$ which are convex of order α in $\mathcal{U}$ and by $\mathcal{S}^*_p(\alpha)$ we denote the class of all functions $f \in \mathcal{A}(p,p+1)$ which are starlike of order α in $\mathcal{U}$. We also set

$$\mathcal{S}^c = \mathcal{S}^c_1(0) \text{ and } \mathcal{S}^* = \mathcal{S}^*_1(0).$$

It is easy to show that for a function f from the class $\mathcal{T}(p,k)$ the condition (6) is equivalent to the following

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \alpha \quad (z \in \mathcal{U}(r)). \tag{7}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{A}(p,k)$. We define *the radius of starlikeness of order* α and *the radius of convexity of order* α for the class $\mathcal{B}$ by

$$R^*_\alpha(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left(\sup\{r \in (0,1] : f \text{ is starlike of order } \alpha \text{ in } \mathcal{U}(r)\}\right),$$

$$R^c_\alpha(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left(\sup\{r \in (0,1] : f \text{ is convex of order } \alpha \text{ in } \mathcal{U}(r)\}\right),$$

respectively.

We say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function $\omega \in \mathcal{A}$ $(|\omega(z)| \le |z|,\ z \in \mathcal{U})$ such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

For functions $f, g \in \mathcal{A}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote the *Hadamard product* (or *convolution*) of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathcal{U}).$$

Let A, B be real parameters, $-1 \leq A < B \leq 1$, and let $\varphi, \phi \in \mathcal{A}_0(p, k)$.

By $\mathcal{W}(p, k; \phi, \varphi; A, B)$ we denote the class of functions $f \in \mathcal{A}(p, k)$ such that

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} \prec \frac{1 + Az}{1 + Bz}. \tag{8}$$

In particular, we have

$$\mathcal{S}^* = \mathcal{W}\left(1, 2; \frac{z}{(1-z)^2}, \frac{z}{1-z}; 1, -1\right),$$
$$\mathcal{S}^c = \mathcal{W}\left(1, 2; \frac{z(z+1)}{(1-z)^3}, \frac{z}{(1-z)^2}; 1, -1\right).$$

Now, we define the classes of functions with varying argument of coefficients related to the class $\mathcal{W}(p, k; \phi, \varphi; A, B)$. Let us denote

$$\mathcal{TW}(p, k; \phi, \varphi; A, B) := \mathcal{T}(p, k) \cap \mathcal{W}(p, k; \phi, \varphi; A, B),$$
$$\mathcal{TW}(p, k; \phi, \varphi; A, B; \eta) := \mathcal{T}(p, k; \eta) \cap \mathcal{W}(p, k; \phi, \varphi; A, B),$$
$$\mathcal{TW}_\rho(p, k; \phi, \varphi; A, B; \eta) := \mathcal{A}_\rho(p, k) \cap \mathcal{TW}(p, k; \phi, \varphi; A, B; \eta),$$
$$\mathcal{TW}_\rho(p, k; \phi, \varphi; A, B) := \mathcal{A}_\rho(p, k) \cap \mathcal{TW}(p, k; \phi, \varphi; A, B).$$

For the presented investigations we assume that φ, ϕ are the functions of the form

$$\varphi(z) = z^p + \sum_{n=k}^{\infty} \alpha_n z^n, \;\; \phi(z) = z^p + \sum_{n=k}^{\infty} \beta_n z^n \quad (z \in \mathcal{U}), \tag{9}$$

where

$$0 \leq \alpha_n < \beta_n \quad (n = k, k+1, \ldots).$$

Moreover, let us put

$$d_n := (1 + B)\beta_n - (1 + A)\alpha_n \quad (n = k, k+1, \ldots). \tag{10}$$

The families $\mathcal{W}_\rho(p, k; \phi, \varphi; A, B; \eta)$ and $\mathcal{W}_\rho(p, k; \phi, \varphi; A, B)$ unify various new and well-known classes of analytic functions.

The object of the present paper is to investigate the coefficients estimates, distortion properties and the radii of starlikeness and convexity for the class of functions with varying argument of coefficients. Some remarks depicting consequences of the main results are also mentioned.

2. Coefficients estimates

We first mention a sufficient condition for the function to belong to the class $\mathcal{W}(p,k;\phi,\varphi;A,B)$.

Theorem 1. *Let $\{d_n\}$ be defined by* (10), *and let $-1 \le A < B \le 1$. If a function f of the form* (1) *satisfies the condition*

$$\sum_{n=k}^{\infty} d_n |a_n| \le (B-A)\, a_p, \tag{11}$$

then f belongs to the class $\mathcal{W}(p,k;\phi,\varphi;A,B)$.

Proof. A function f of the form (1) belongs to the class $\mathcal{W}(p,k;\phi,\varphi;A,B)$ if and only if there exists a function ω, $|\omega(z)| \le |z| \quad (z \in \mathcal{U})$, such that

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} = \frac{1+A\omega(z)}{1+B\omega(z)} \quad (z \in \mathcal{U}),$$

or equivalently

$$\left| \frac{(\phi * f)(z) - (\varphi * f)(z)}{B(\phi * f)(z) - A(\varphi * f)(z)} \right| < 1 \quad (z \in \mathcal{U}). \tag{12}$$

Thus, it is sufficient to prove that

$$\left| \frac{(\phi * f)(z) - (\varphi * f)(z)}{z^p} \right| - \left| \frac{B(\phi * f)(z) - A(\varphi * f)(z)}{z^p} \right| < 0 \quad (z \in \mathcal{U}).$$

Indeed, letting $|z| = r$ $(0 \le r < 1)$, we have

$$\begin{aligned}
&\left| \frac{(\phi * f)(z) - (\varphi * f)(z)}{z^p} \right| - \left| \frac{B(\phi * f)(z) - A(\varphi * f)(z)}{z^p} \right| \\
&= \left| \sum_{n=k}^{\infty} (\beta_n - \alpha_n)\, a_n z^{n-p} \right| - \left| (B-A)\, a_p - \sum_{n=k}^{\infty} (B\beta_n - A\alpha_n)\, a_n z^{n-p} \right| \\
&\le \sum_{n=k}^{\infty} (\beta_n - \alpha_n)\, |a_n|\, r^{n-p} - (B-A)\, a_p + \sum_{n=k}^{\infty} (B\beta_n - A\alpha_n)\, |a_n|\, r^{n-p} \\
&\le \sum_{n=k}^{\infty} d_n |a_n|\, r^{n-p} - (B-A)\, a_p < 0,
\end{aligned}$$

whence $f \in \mathcal{W}(p,k;\phi,\varphi;A,B)$. □

Our next theorem shows that the condition (11) is necessary as well for functions of the form (1), with (4) to belong to the class $\mathcal{TW}(p,k;\phi,\varphi;A,B;\eta)$.

Theorem 2. *Let f be a function of the form* (1), *satisfying the argument property* (4). *Then f belongs to the class $\mathcal{TW}(p,k;\phi,\varphi;A,B;\eta)$ if and only if the condition* (11) *holds true.*

Proof. In view of Theorem 1 we need only show that each function f from the class $\mathcal{TW}_k^p(\phi,\varphi;A,B;\eta)$ satisfies the coefficient inequality (11). Let $f \in \mathcal{TW}(p,k;\phi,\varphi;A,B;\eta)$. Then by (12) and (1), we have

$$\left|\frac{\sum\limits_{n=k}^{\infty}(\beta_n-\alpha_n)\,a_n z^{n-p}}{(B-A)\,a_p-\sum\limits_{n=k}^{\infty}(B\beta_n-A\alpha_n)\,a_n z^{n-p}}\right|<1 \quad (z\in\mathcal{U}).$$

Therefore putting $z=re^{i\eta}$ $(0\le r<1)$, and applying (4) we obtain

$$\frac{\sum\limits_{n=k}^{\infty}(\beta_n-\alpha_n)\,a_n r^{n-p}}{(B-A)\,a_p-\sum\limits_{n=k}^{\infty}(B\beta_n-A\alpha_n)\,a_n r^{n-p}}<1 \tag{13}$$

It is clear, that the denominator of the left hand said can not vanish for $r\in\langle 0,1)$. Moreover, it is positive for $r=0$, and in consequence for $r\in\langle 0,1)$. Thus, by (13) we have

$$\sum_{n=k}^{\infty}\left[(1+B)\,\beta_n-(1+A)\,\alpha_n\right]|a_n|\,r^{n-p}<(B-A)\,a_p,$$

which, upon letting $r\to 1^-$, readily yields the assertion (11). □

By applying Theorem 2 we can deduce following result.

Theorem 3. *Let f be a function of the form* (1)*, satisfying the argument property* (4)*. A function f of the form* (1) *belongs to the class $\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B;\eta)$ if and only if it satisfies* (3) *and*

$$\sum_{n=k}^{\infty}\left(pd_n-(B-A)\,n\,|\rho|^{n-p}\right)|a_n|\le p(B-A), \tag{14}$$

where $\{d_n\}$ is defined by (10)*.*

Proof. For a function f of the form (1) with the normalization (3), we have

$$a_p=1+\sum_{n=k}^{\infty}\frac{n}{p}\,|a_n|\,|\rho|^{n-p}\,. \tag{15}$$

Applying the equality (15)) to (11), we obtain the assertions (14). □

Since the condition (14) is independent of η, Theorem 3 yields the following theorem.

Theorem 4. *Let f be a function of the form* (1)*, satisfying the argument property* (4)*. Then f belongs to the class $\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B)$ if and only if the condition* (14) *holds true.*

By applying Theorem 3 we obtain the following lemma.

Lemma 1. *Let $\{d_n\}$ be defined by* (10), $\rho \in \mathcal{U}$, *and let us assume, that there exists an integer* n_0 $(n_0 \in \mathbb{N}_k = \{k, k+1, \ldots\})$ *such that*

$$pd_{n_0} - (B-A)\, n_0\, |\rho|^{n_0-p} \leq 0. \tag{16}$$

Then the function

$$f_{n_0}(z) = \left(1 + a\frac{n_0}{p}\rho^{n_0-p}\right) z^p - ae^{i(p-n_0)\eta} z^{n_0}$$

belongs to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;A,B;\eta)$ *for all positive real numbers* a. *Moreover, for all* n $(n \in \mathbb{N}_k)$ *such that*

$$pd_n - (B-A)\, n\, |\rho|^{n-p} > 0, \tag{17}$$

the functions

$$f_n(z) = \left(1 + a\frac{n_0}{p}\rho^{n_0-p} + b\frac{n}{p}z^{n-p}\right) z^p - ae^{i(p-n_0)\eta} z^{n_0} - be^{i(p-n)\eta} z^n,$$

where

$$b = \frac{p\,(B-A) + \left((B-A)\, n_0\, |\rho|^{n_0-p} - pd_{n_0}\right) a}{pd_n - (B-A)\, n\, |\rho|^{n-p}},$$

belong to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;A,B;\eta)$.

By Lemma 1 and Theorem 3, we have following corollary.

Corollary 1. *Let a function* f *of the form* (1) *belongs to the class* $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;A,B;\eta)$ *and let* $\{d_n\}$ *be defined by* (10). *Then all of the coefficients* a_n *for which*

$$pd_n - (B-A)\, n\, |\rho|^{n-p} = 0$$

are unbounded.

Moreover, if there exists an integer n_0 $(n_0 \in \mathbb{N}_k = \{k, k+1, \ldots\})$ *such that*

$$pd_{n_0} - (B-A)\, n_0\, |\rho|^{n_0-p} < 0,$$

then all of the coefficients of the function f *are unbounded. In the remaining cases*

$$|a_n| \leqq \frac{p\,(B-A)}{pd_n - (B-A)\, n\, |\rho|^{n-p}}. \tag{18}$$

The result is sharp, the functions f_n *of the form*

$$f_{n,\eta}(z) = \frac{pd_n z^p - p\,(B-A)\, e^{i(p-n)\eta} z^n}{pd_n - (B-A)\, n\, |\rho|^{n-p}} \quad (z \in \mathcal{U};\ n = k, k+1, \ldots) \tag{19}$$

are the extremal functions.

Remark 1. The coefficients estimates for the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;A,B)$ are the same as for the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;A,B;\eta)$.

By putting $\rho = 0$ in Theorems 3 and 4, and Corollary 1, we have the corollaries listed below.

Corollary 2. *Let f be a function of the form* (1)*, satisfying the argument property* (4)*. Then f belongs to the class $\mathcal{TW}_0(p,k;\phi,\varphi;A,B;\eta)$ if and only if*

$$\sum_{n=k}^{\infty} d_n |a_n| \le B-A, \tag{20}$$

where $\{d_n\}$ is defined by (10)*.*

Corollary 3. *Let f be a function of the form* (1)*, satisfying the argument property* (4)*. Then f belongs to the class $\mathcal{TW}_0(p,k;\phi,\varphi;A,B)$ if and only if the condition* (20) *holds true.*

Corollary 4. *If a function f of the form* (1) *belongs to the class $\mathcal{TW}_0(p,k;\phi,\varphi;A,B;\eta)$, then*

$$|a_n| \le \frac{B-A}{d_n} \quad (n=k,k+1,\ldots), \tag{21}$$

where d_n is defined by (10)*. The result is sharp. The functions $f_{n,\eta}$ of the form*

$$f_{n,\eta}(z) = z^p - \frac{B-A}{d_n} e^{i(p-n)\eta} z^n \quad (z\in\mathcal{U};\ n=k,k+1,\ldots) \tag{22}$$

are the extremal functions.

Corollary 5. *If a function f of the form* (1) *belongs to the class $\mathcal{TW}_0(p,k;\phi,\varphi;A,B)$, then the coefficients estimates* (21) *holds true. The result is sharp. The functions $f_{n,\eta}$ $(\eta\in\mathbb{R})$ of the form* (22) *are the extremal functions.*

3. Distortion theorems

From Theorem 2 we have the following lemma.

Lemma 2. *Let a function f of the form* (1) *belong to the class $\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B;\eta)$. If the sequence $\{d_n\}$ defined by* (10) *satisfies the inequality*

$$0 < d_k - (B-A)\frac{k}{p}|\rho|^{k-p} \le d_n - (B-A)\frac{n}{p}|\rho|^{n-p} \quad (n=k,k+1,\ldots), \tag{23}$$

then

$$\sum_{n=k}^{\infty} |a_n| \le \frac{B-A}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}}.$$

Moreover, if

$$0 < \frac{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}}{k} \le \frac{d_n - (B-A)\frac{n}{p}|\rho|^{n-p}}{n} \quad (n=k,k+1,\ldots), \tag{24}$$

then

$$\sum_{n=k}^{\infty} n\,|a_n| \leq \frac{k\,(B-A)}{d_k-(B-A)\,\frac{k}{p}\,|\rho|^{k-p}}.$$

Remark 2. The second part of Lemma 2 we can rewritten in terms of σ-neighborhood N_σ defined by

$$N_\sigma=\left\{f(z)=a_pz^p+\sum_{n=k}^{\infty}a_nz^n\in\mathcal{T}\,(p,k;\eta):\quad \sum_{n=k}^{\infty}n\,|a_n|\leq\sigma\right\}$$

in the following form:
if the sequence $\{d_n\}$ defined by (10) satisfies (24), then

$$\mathcal{TW}_\rho\,(p,k;\phi,\varphi;A,B;\eta)\subset N_\sigma.$$

where

$$\delta=\frac{k\,(B-A)}{d_k-(B-A)\,\frac{k}{p}\,|\rho|^{k-p}}.$$

Theorem 5. *Let a function f belong to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;A,B;\eta)$ and let $|z|=r<1$. If the sequence $\{d_n\}$ defined by* (10) *satisfies* (23), *then*

$$pa_pr^p-\frac{B-A}{d_k-(B-A)\,\frac{k}{p}\,|\rho|^{k-p}}r^k\leq|f(z)|\leq\frac{d_kr^p+(B-A)\,r^k}{d_k-(B-A)\,\frac{k}{p}\,|\rho|^{k-p}},\tag{25}$$

Moreover, if (24) *holds, then*

$$\phi'(r)\leq|f'(z)|\leq\frac{d_kr^p+k\,(B-A)\,r^{k-1}}{d_k-(B-A)\,\frac{k}{p}\,|\rho|^{k-p}}.\tag{26}$$

where

$$\phi(r):=\begin{cases} r^p & (r\leqq\rho)\\ \frac{d_kr^p-(B-A)r^k}{d_k-(B-A)\frac{k}{p}|\rho|^{k-p}} & (r>\rho)\,. \end{cases}\tag{27}$$

The result is sharp, with the extremal function $f_{k,\eta}$ of the form (19) *and $f(z)=z$.*

Proof. Suppose that the function f of the form (1) belongs to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;A,B;\eta)$. By Lemma 2 we have

$$\begin{aligned}|f'(z)|&=\left|pa_pz^{p-1}+\sum_{n=k}^{\infty}na_nz^{n-1}\right|\leq r^{p-1}\left(pa_p+\sum_{n=k}^{\infty}n\,|a_n|\,r^{n-p}\right)\\&\leq r^{p-1}\left(p+\sum_{n=k}^{\infty}n\,|a_n|\,|\rho|^{n-p}+\sum_{n=k}^{\infty}n\,|a_n|\,r^{n-p}\right)\\&\leq r^{p-1}\left(p+\left(|\rho|^{k-p}+r^{k-p}\right)\sum_{n=k}^{\infty}n\,|a_n|\right)\leq\frac{d_kr^{p-1}+k(B-A)r^k}{d_k-(B-A)\frac{k}{p}|\rho|^{k-p}},\end{aligned}$$

and

$$|f'(z)| \geq r^{p-1}\left(pa_p - \sum_{n=k}^{\infty} n\,|a_n|\, r^{n-p}\right) = r^{p-1}\left(p + \sum_{n=k}^{\infty}(|\rho|^{n-p} - r^{n-p}) n\,|a_n|\right). \tag{28}$$

If $r \leq \rho$, then we obtain $|f'(z)| \geq r^{p-1}$. If $r > \rho$, then the sequence $\{(\rho^{n-p} - r^{n-p})\}$ is decreasing and negative. Thus, by (26), we obtain

$$|f'(z)| \geq r^{p-1}\left(p - (r^{k-p} - |\rho|^{k-p})\sum_{n=2}^{\infty} a_n\right) \geq \frac{pd_k r^p - k\,(B-A)\, r^k}{d_k - (B-A)\,\frac{k}{p}\,|\rho|^{k-p}},$$

and we have the assertion (26). Making use of Lemma 2, in conjunction with (15), we readily obtain the assertion (25) of Theorem 5. □

Theorem 5 implies the following results.

Corollary 6. *Let a function f belong to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;A,B)$. the sequence $\{d_n\}$ defined by* (10) *satisfies* (23)*, then the assertion* (25) *holds true. Moreover, if we assume* (24)*, then the assertion* (26) *holds true. The result is sharp, with the extremal functions $f_{k,\eta}\quad(\eta \in \mathbb{R})$ of the form* (19).

Corollary 7. *Let a function f belong to the class $\mathcal{TW}_0\,(p,k;\phi,\varphi;A,B;\eta)$ and let the sequence $\{d_n\}$ be defined by* (10). *If*

$$d_k \leq d_n \quad (n = k, k+1, \ldots)\,, \tag{29}$$

then

$$r^p - \frac{B-A}{d_k} r^k \leq |f(z)| \leq r^p + \frac{B-A}{d_k} r^k \quad (|z| = r < 1)\,. \tag{30}$$

Moreover, if

$$nd_k \leq kd_n \quad (n = k, k+1, \ldots)\,, \tag{31}$$

then

$$pr^{p-1} - \frac{k\,(B-A)}{d_k} r^{k-1} \leq |f'(z)| \leq pr^{p-1} + \frac{k\,(B-A)}{d_k} r^{k-1} \quad (|z| = r < 1)\,. \tag{32}$$

The result is sharp, with the extremal function $f_{k,\eta}$of the form (22).

Corollary 8. *Let a function f belong to the class $\mathcal{TW}_0\,(p,k;\phi,\varphi;A,B)$. If the sequence $\{d_n\}$ defined by* (10) *satisfies* (29)*, the assertion* (30) *holds true. Moreover, if we assume* (31)*, then the assertion* (32) *holds true. The result is sharp, with the extremal functions $f_{k,\eta}$ $(\eta \in \mathbb{R})$ of the form* (22).

4. The radii of convexity and starlikeness

Theorem 6. *The radius of starlikeness of order α for the class $\mathcal{TW}(p,k;\phi,\varphi;A,B;\eta)$ is given by*

$$R_{\alpha}^{*}\left(\mathcal{TW}(p,k;\phi,\varphi;A,B;\eta)\right)=\inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{(n-\alpha)\,(B-A)}\right)^{\frac{1}{n-p}}, \tag{33}$$

where d_n is defined by (10). The functions $f_{n,\eta}$ of the form

$$f_{n,\eta}(z)=a_p\left(z^p-\frac{B-A}{d_n}e^{i(p-n)\eta}z^n\right)\qquad (z\in\mathcal{U};\ n=k,k{+}1,\ldots;a_p>0) \tag{34}$$

are the extremal functions.

Proof. A function $f\in\mathcal{T}(p,k;\eta)$ of the form (1) is starlike of order α in the disk $\mathcal{U}(r)$, $0<r\leq 1$, if and only if it satisfies the condition (7). Since

$$\left|\frac{zf'(z)}{f(z)}-p\right|=\left|\frac{\sum_{n=k}^{\infty}(n-p)a_nz^n}{a_pz^p+\sum_{n=k}^{\infty}a_nz^n}\right|\leq\frac{\sum_{n=k}^{\infty}(n-p)\left|a_n\right|\left|z\right|^{n-p}}{a_p-\sum_{n=k}^{\infty}\left|a_n\right|\left|z\right|^{n-p}},$$

putting $|z|=r$ the condition (7) is true if

$$\sum_{n=k}^{\infty}\frac{n-\alpha}{p-\alpha}\left|a_n\right|r^{n-p}\leq a_p. \tag{35}$$

By Theorem 2, we have

$$\sum_{n=k}^{\infty}\frac{d_n}{B-A}\left|a_n\right|\leq a_p, \tag{36}$$

Thus, the condition (35) is true if

$$\frac{n-\alpha}{p-\alpha}r^{n-p}\leq\frac{d_n}{B-A}\qquad (n=k,k+1,\ldots),$$

that is, if

$$r\leq\left(\frac{(p-\alpha)\,d_n}{(n-\alpha)\,(B-A)}\right)^{\frac{1}{n-p}}\qquad (n=k,k+1,\ldots). \tag{37}$$

It follows that each function $f\in\mathcal{TW}(p,k;\phi,\varphi;A,B;\eta)$ is starlike of order α in the disk $\mathcal{U}(r)$, where

$$r=\inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{(n-\alpha)\,(B-A)}\right)^{\frac{1}{n-p}}.$$

The functions $f_{n,\eta}$ of the form (34) realize equality in (36), and the radius r can not be larger. Thus we have (33). □

The following result may be proved in much the same way as Theorem 6.

Theorem 7. *The radius of convexity of order α for the class $\mathcal{TW}(p,k;\phi,\varphi;A,B;\eta)$ is given by*

$$R_\alpha^c(\mathcal{TW}(p,k;\phi,\varphi;A,B;\eta)) = \inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{n(n-\alpha)(B-A)}\right)^{\frac{1}{n-p}},$$

where d_n is defined by (10)*. The functions $f_{n,\eta}$ of the form* (34) *are the extremal functions.*

It is clear that for

$$a_p = \frac{d_n}{d_n-(B-A)\frac{n}{p}|\rho|^{n-p}} > 0$$

the extremal functions $f_{n,\eta}$ of the form (34) belong to the class $\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B;\eta)$. Moreover, we have

$$\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B;\eta) \subset \mathcal{TW}(p,k;\phi,\varphi;A,B;\eta).$$

Thus, by Theorems 6 and 7 we have the following results.

Corollary 9. *Let the sequence $\left\{d_n-(B-A)\frac{n}{p}|\rho|^{n-p}\right\}$, where d_n is defined by* (10)*, be positive. The radius of starlikeness of order α for the class $\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B;\eta)$ is given by*

$$R_\alpha^*(\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B;\eta)) = \inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{(n-\alpha)(B-A)}\right)^{\frac{1}{n-p}}.$$

The functions $f_{n,\eta}$ of the form (19) *are the extremal functions.*

Corollary 10. *Let the sequence $\left\{d_n-(B-A)\frac{n}{p}|\rho|^{n-p}\right\}$, where d_n is defined by* (10)*, be positive. The radius of convexity of order α for the class $\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B;\eta)$ is given by*

$$R_\alpha^c(\mathcal{TW}_\rho(p,k;\phi,\varphi;A,B;\eta)) = \inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{n(n-\alpha)(B-A)}\right)^{\frac{1}{n-p}},$$

where d_n is defined by (10)*.*

5. Remarks

Choosing the functions ϕ and φ in the condition (8) we can define new classes of functions. In particular, the class

$$\mathcal{W}_n(p,k;\varphi;A,B) := \mathcal{W}\left(p,k;z\varphi'(z),\sum_{k=0}^{n-1}\varphi\left(x^k z\right);A,B\right) \quad (x^n=1)$$

contains functions $f\in\mathcal{A}(p,k)$, such that

$$\frac{z(\varphi*f)'(z)}{\sum_{k=0}^{n-1}(\varphi*f)(x^k z)} \prec \frac{1+Az}{1+Bz}.$$

It is related to the well-known class of starlike functions with n-symmetric points. Moreover, putting $n = 1$ we obtain the class $\mathcal{W}(p,k;\varphi;A,B) = \mathcal{W}_1(p,k;\varphi;A,B)$ defined by the following condition

$$\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} \prec \frac{1+Az}{1+Bz}.$$

The class is related to the class of p-valent starlike functions.

Let λ be a convex parameter. A function $f \in \mathcal{A}(p,k)$ belongs to the class

$$\mathcal{V}_\lambda(p,k;\varphi;A,B) := \mathcal{W}\left(p,k;\lambda\frac{\varphi(z)}{z} + (1-\lambda)\varphi'(z), z; A, B\right)$$

if it satisfies the condition

$$\lambda\frac{(\varphi * f)(z)}{z} + (1-\lambda)(\varphi * f)'(z) \prec \frac{1+Az}{1+Bz}.$$

Moreover, a function $f \in \mathcal{A}(p,k)$ belongs to the class

$$\mathcal{U}_\lambda(p,k;\varphi;A,B) := \mathcal{W}\left(p,k;\lambda\frac{\varphi(z)}{z} + (1-\lambda)\varphi'(z); A, B\right)$$

if it satisfies the condition

$$\frac{z(\varphi * f)'(z) + (1-\lambda)z^2(\varphi * f)''(z)}{\lambda(\varphi * f)(z) + (1-\lambda)z(\varphi * f)'(z)} \prec \frac{1+Az}{1+Bz}.$$

The classes $\mathcal{W}_n(p,k;\varphi;A,B)$, $\mathcal{U}_\lambda(p,k;\varphi;A,B)$ and $\mathcal{V}_\lambda(p,k;\varphi;A,B)$ generalize well-known important classes, which were investigated in earlier works, see for example [1]-[6], and [9]. Most of these classes were defined by using linear operators and special functions.

If we apply the results presented in this paper to the classes discussed above, we can obtain a lot of partial results. Some of these results where obtained in earlier works.

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