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FIXED POINTS THEOREMS FOR MONOTONE SET-VALUED MAPS IN PSEUDO-ORDERED SETS

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Abstract. In this paper, we first establish the existence of the greatest and the least fixed points for monotone set-valued maps defined on non-empty pseudo-ordered sets. Furthermore, we prove that the set of all fixed points of two classes of monotone set-valued maps defined on a non-empty complete trellis is also a non-empty complete trellis. As a consequence we obtain a generalization of the Skala's result [4, Theorem 37].

1. Introduction and preliminaries

Let X be a non-empty set and let $\trianglerighteq$ be a binary relation defined on its. If the binary relation $\trianglerighteq$ is reflexive and antisymmetric, we say that $(X, \trianglerighteq)$ is a pseudo-ordered set or a psoset. We will usually omit the pair notation and call X a pseudo-ordered set also. Every subset A of X is a pseudo-ordered set with the induced pseudo-ordered from X and will be called a pseudo-ordered set. Let $x, y \in X$. If $x \neq y$ and $x \trianglerighteq y$, then we shall write $x \triangleright y$.

Let A be a non-empty subset of a psoset $(X, \trianglerighteq)$. An element u is said to be an upper bound of A (respectively v a lower bound of A) if $x \trianglerighteq u$ for every $x \in A$ (respectively $v \trianglerighteq x$ for every $x \in A$). An element s of X is called a greatest element or the maximum of A and denoted by $s = \max_{\trianglerighteq}(A)$ if s is an upper bound of A and $s \in A$. An element ℓ is the least or the minimum element of A and denoted by $\ell = \min_{\trianglerighteq}(A))$ if ℓ is a lower bound of A and $\ell \in A$. When the least upper bound (l.u.b.) s of A exists, we shall denoted its by $s = \sup_{\trianglerighteq}(A)$. Dually if the greatest lower bound (g.l.b.) of A exists, we shall denoted its by $\ell = \inf_{\trianglerighteq}(A)$.

A psoset $(X, \trianglerighteq)$ is said to be a trellis if every pair of elements of $(X, \trianglerighteq)$ has a greatest lower bound (g.l.b) and a least upper bound (l.u.b). A psoset $(X, \trianglerighteq)$ is said to be a complete trellis if every non-empty subset of X has a g.l.b and a l.u.b. More details for those notions can be found in H. L. Skala (see [5, 4]).

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Let $(X, \trianglerighteq)$ be a non-empty pseudo-ordered set and $f\colon X \to X$ a map. We shall say that f is monotone if for every $x, y \in X$, with $x \trianglerighteq y$, then we have $f(x) \trianglerighteq f(y)$.

An element x of X is said to be a fixed point of a map $f\colon X \to X$ if $f(x) = x$. The set of all fixed points of f is denoted by $\mathrm{Fix}(f)$.

Let X be a non-empty set and 2^X be the set of all non-empty subsets of X. A set-valued map on X is any map $T\colon X \to 2^X$. An element x of X is called a fixed point of T if $x \in T(x)$. We denote by $\mathrm{Fix}(T)$ the set of all fixed points of T.

In this paper, we shall use the following definition of monotonicity for set-valued maps.

Definition 1.1. Let $(X, \trianglerighteq)$ be a non-empty pseudo-ordered set. A set-valued map $T\colon X \to 2^X$ is said to be monotone if for any $x, y \in X$ with $x \triangleright y$, then for every $a \in T(x)$ and $b \in T(y)$, we have $a \trianglerighteq b$.

In this work, we shall need the following notion of inverse relation.

Definition 1.2. Let X be a non-empty set and let $\trianglerighteq$ be a relation on its. The inverse relation $\trianglelefteq$ of $\trianglerighteq$ is defined for every $x, y \in X$ by:

$$(x \trianglelefteq y) \Leftrightarrow (y \trianglerighteq x).$$

In this paper, we shall need the two following technical lemmas which their proofs will be given in the Appendix.

Lemma 1.3. *Let $\trianglerighteq$ be a pseudo-order relation defined on a non-empty set X and let $\trianglelefteq$ be its inverse relation. Then, $\trianglelefteq$ is a pseudo-order relation on X.*

Lemma 1.4. *Let $\trianglerighteq$ be a pseudo-order relation defined on a non-empty set X, let $\trianglelefteq$ be its inverse relation and let A be a non-empty subset of X. Then, we have*

(i) *if* $\sup_{\trianglerighteq}(A)$ *exists, so* $\inf_{\trianglelefteq}(A)$ *exists too and* $\sup_{\trianglerighteq}(A) = \inf_{\trianglelefteq}(A)$;
(ii) *if* $\inf_{\trianglerighteq}(A)$ *exists, hence* $\sup_{\trianglelefteq}(A)$ *exists also and* $\inf_{\trianglerighteq}(A) = \sup_{\trianglelefteq}(A)$;
(iii) *if* $\min_{\trianglerighteq}(A)$ *exists, then* $\max_{\trianglelefteq}(A)$ *exists too and* $\min_{\trianglerighteq}(A) = \max_{\trianglelefteq}(A)$;
(iv) *if* $\max_{\trianglerighteq}(A)$ *exists, so* $\min_{\trianglelefteq}(A)$ *exists too and* $\max_{\trianglerighteq}(A) = \min_{\trianglelefteq}(A)$.
(v) *if $T\colon X \to 2^X$ is a monotone set-valued map for $\trianglerighteq$, then T is also a set-valued map for $\trianglelefteq$.*

In 1971, H. Skala introduced the notions of pseudo-ordered sets and trellises and gave some fixed points theorems in this setting (see Theorems 36 and 37 in [4]). Later on, S. Parameshwara Bhatta and all [3, 1] studied the fixed point property in pseudo-ordered sets. In this work, we first establish the existence of the greatest and the least fixed points for monotone set-valued maps defined on non-empty pseudo-ordered sets. Furthermore, we prove that the set of all fixed points of of two classes of monotone set-valued maps defined on a non-empty complete trellis is also a non-empty complete trellis. As a consequence, we reobtain the Skala's result [4, Theorem 37].

2. Least and greatest fixed points for monotone set-valued maps in pseudo-ordered sets

In this section, we shall establish the existence of the least and the greatest fixed points for monotone set-valued maps defined on non-empty pseudo-ordered sets. First, we shall prove our key result in this paper.

Theorem 2.1. *Let* $(X, \trianglerighteq)$ *be a non-empty pseudo-ordered set with a least element* ℓ. *Assume that every non-empty subset of* X *has a supremum in* $(X, \trianglerighteq)$. *Then, the set of all fixed points* $\mathrm{Fix}(T)$ *of every monotone set-valued map* $T\colon X \to 2^X$ *is non-empty and has a least element.*

Proof. Let $(X, \trianglerighteq)$ be a non-empty pseudo-ordered set with a least element ℓ and let $T\colon X \to 2^X$ be a monotone set-valued map.

First step. We have $\mathrm{Fix}(T) \neq \emptyset$. Indeed, let $\mathcal{F}$ be the family of all subsets A of X satisfying the following three conditions:

(i) $\ell \in A$;
(ii) $T(A) \subset A$;
(iii) for every non-empty subset B of A, we have $\sup_{\trianglerighteq}(B) \in A$.

Since $X \in \mathcal{F}$, so $\mathcal{F} \neq \emptyset$. Set $S = \bigcap_{A \in \mathcal{F}} A$.

Claim 1. The subset S is the least non-empty element of $\mathcal{F}$ for the inclusion relation. Indeed, as $\ell \in A$ for every $A \in \mathcal{F}$, so $\ell \in S$. Since $S = \bigcap_{A \in \mathcal{F}} A$, then

$$T(S) = T(\bigcap_{A \in \mathcal{F}} A) \subset \bigcap_{A \in \mathcal{F}} T(A) \subset \bigcap_{A \in \mathcal{F}} A.$$

Thus, we get $T(S) \subset S$. Now, let $D \subset S$ such that $D \neq \emptyset$. Then, $D \subset A$ for every $A \in \mathcal{F}$. So, $\sup_{\trianglerighteq}(D) \in A$ for every $A \in \mathcal{F}$. Hence, we obtain $\sup_{\trianglerighteq}(D) \in S$. Therefore, S is the least non-empty element of $\mathcal{F}$ for the inclusion relation. Then, we set $m = \sup_{\trianglerighteq}(S)$.

Claim 2. We have $m \in \mathrm{Fix}(T)$. Indeed, since $m \in S$ and $T(S) \subset S$, then for every $a \in T(m)$, we have $a \trianglerighteq m$. By absurd assume that $m \notin T(m)$. So, we get $a \triangleright m$, for every $a \in T(m)$. Next, we shall associate for every $a \in T(m)$ a subset B_a defined by

$$B_a = \{x \in S : x \trianglerighteq a\}.$$

As $\ell = \min_{\trianglerighteq}(X)$, so $\ell \in B_a$. We shall show that $B_a \in \mathcal{F}$. Let $x \in B_a$ and $y \in T(x)$. So $x \in S$. As $m = \sup_{\trianglerighteq}(S)$, then $x \trianglerighteq m$. We claim that $x \neq m$. Indeed, if $x = m$, so $m \trianglerighteq a$. Hence we get $a = m$. That is not possible. Then, $x \triangleright m$. Hence, from the monotonicity of T, we get $y \trianglerighteq a$, for every $y \in T(x)$. So, $T(x) \subset B_a$, for every $x \in B_a$. Thus, we have $T(B_a) \subset B_a$. Now, let $C \subset B_a$ and $C \neq \emptyset$. So, $C \subset S$. Then, $t = \sup_{\trianglerighteq}(C) \in S$. On the other hand By definition of B_a we deduce that a is an upper bound of C. Hence, we obtain $t \trianglerighteq a$. Thus $\sup_{\trianglerighteq}(C) \in B_a$. Therefore, $B_a \in \mathcal{F}$ for every $a \in T(m)$. As S is the least non-empty element of $\mathcal{F}$ for the inclusion relation, so we get $S \subset B_a$ for every $a \in T(m)$. On the other hand, we know that $B_a \subset S$ for every $a \in T(m)$. Therefore, we obtain $S = B_a$, for every $a \in T(m)$. Then, as $m \in B_a$, so $m \trianglerighteq a$ for every $a \in T(m)$. Thus, we get $m = a$, for every $a \in T(m)$. So, $T(m) = \{m\}$. That is a contradiction with our assumption that $m \notin T(m)$. Therefore, $m \in T(m)$.

Second step. The subset $\mathrm{Fix}(T)$ has a least element. Indeed from the first step above, we know that $\mathrm{Fix}(T) \neq \emptyset$. Next, we consider the following subset B of X defined by

$$B = \{x \in X : x \trianglerighteq z \text{ for every } z \in \mathrm{Fix}(T)\}.$$

As $\ell = \min_{\trianglerighteq}(X)$, so $\ell \in B$. Hence, we get $\ell = \min_{\trianglerighteq}(B)$. By absurd assume that $\mathrm{Fix}(T)$ has not a least element. So, for every $x \in B$, we have $x \triangleright z$ for every $z \in \mathrm{Fix}(T)$. Next, we shall show that $T(B) \subset B$. Indeed, if $x \in B$, $y \in T(x)$ and $z \in \mathrm{Fix}(T)$, then by the monotonicity of T we get $y \trianglerighteq z$ for every $y \in T(x)$ and $z \in \mathrm{Fix}(T)$. Hence, we get $T(x) \subset B$, for every $x \in B$. Thus, $T(B) \subset B$. Now, let C be a non-empty subset of B and let $c = \sup_{\trianglerighteq}(C)$. By definition of B, we know that every element z of $\mathrm{Fix}(T)$ is an upper bound of C. So, we get $c \trianglerighteq z$ for every $z \in \mathrm{Fix}(T)$. Thus, we have $c \in B$. Hence, from claim 1, we get $B \in \mathcal{F}$. Since S is the least element of $\mathcal{F}$, so $S \subset B$. On the other hand, by Claim 1, we know that the supremum m of S is a fixed point of T. Hence, $m \in B$. Thus, m is the least fixed of T. That is a contradiction with our assumption. Therefore, $\mathrm{Fix}(T)$ has a least element. □

As a consequence of Theorem 2.1, we get the following result.

Corollary 2.2. *Let $(X, \leq)$ be a non-empty partially ordered ordered set with a least element ℓ. Assume that every non-empty subset of X has a supremum in $(X, \leq)$. Then, the set of all fixed points $\mathrm{Fix}(T)$ of every monotone set-valued map $T \colon X \to 2^X$ is non-empty and has a least element.*

Next, by combining Lemmas 1.3 and 1.4 and Theorem 2.1 we obtain the the existence of the greatest fixed point for monotone set-valued maps defined on non-empty pseudo-ordered sets.

Theorem 2.3. *Let $(X, \trianglerighteq)$ be a non-empty pseudo-ordered set with a greatest element g. Assume that every non-empty subset of X has an infimum in $(X, \trianglerighteq)$. Then, the set of all fixed points of every monotone set-valued map $T \colon X \to 2^X$ is non-empty and has a greatest element.*

Proof. Let $(X, \trianglerighteq)$ be a non-empty pseudo-ordered set with a greatest element g such that every non-empty subset of X has an infimum in $(X, \trianglerighteq)$. Let $T \colon X \to 2^X$ be a monotone set-valued map for the pseudo-order relation $\trianglerighteq$ and let $\trianglelefteq$ be its inverse relation. Then from Lemma 1.2, we know that $\trianglelefteq$ is a pseudo-order relation on X. On the other hand by Lemma 1.3, $\min_{\trianglelefteq}(X)$ exists and we have $\min_{\trianglelefteq}(X) = g$. As by our hypothesis $T \colon X \to 2^X$ is a monotone set-valued map for $\trianglerighteq$, so from Lemma 1.3 the set-valued map T is also a monotone set-valued map for $\trianglelefteq$. Thus, all hypothesis of Theorem 2.1 are satisfied. Therefore, The set $\mathrm{Fix}(T)$ of all fixed points of T is non-empty and has a least element in $(X, \trianglelefteq)$, m, say. Then from Lemma 1.3, we get $m = \min_{\trianglelefteq}(\mathrm{Fix}(T)) = \max_{\trianglerighteq}(\mathrm{Fix}(T))$. □

Combining Theorems 2.1 and 2.3, we obtain the existence of the least and the greatest fixed points of monotone set-valued maps defined on non-empty complete trellises.

Corollary 2.4. *Let $(X, \trianglerighteq)$ be a non-empty complete trellis. Then, the set of all fixed points* $\mathrm{Fix}(T)$ *of every monotone set-valued map $T\colon X \to 2^X$ is non-empty and has a least and a greatest element.*

For complete lattice, we obtain the following result.

Corollary 2.5. *Let $(X, \leq)$ be a non-empty complete lattice. Then, the set of all fixed points* $\mathrm{Fix}(T)$ *of every monotone set-valued map $T\colon X \to 2^X$ is non-empty and has a least and a greatest element.*

3. Fixed points for monotone set-valued maps in complete trellises

In this section, we shall establish that the set of all fixed points of two classes of monotone set-valued maps defined on a non-empty compete trellis is also a non-empty compete trellis. First, we shall prove the following result.

Theorem 3.1. *Let $(X, \trianglerighteq)$ be a non-empty complete trellis and let $T\colon X \to 2^X$ be a monotone set-valued map such that for every $x \in X$ there is $y \in T(x)$ such that $x \trianglerighteq y$. Then, the set of all fixed points* $\mathrm{Fix}(T)$ *of T is a non-empty complete trellis.*

Proof. Let $(X, \trianglerighteq)$ be a non-empty complete trellis and $T\colon X \to 2^X$ be a monotone set-valued map such that for every $x \in X$, there is $y \in T(x)$, such that $x \trianglerighteq y$. Then by Corollary 2.4, we know that $\mathrm{Fix}(T)$ is non-empty and has a least and a greatest element. Let A be a non-empty subset of $\mathrm{Fix}(T)$.

Claim 1. The infimum of A in $\mathrm{Fix}(T)$ belongs to $\mathrm{Fix}(T)$. Indeed, consider the following subset D of X defined by

$$D = \{x \in X : x \trianglerighteq z \text{ for every } z \in A\}.$$

From Corollary 2.4, we know that the set-valued map T has a least fixed point. So, $D \neq \emptyset$. Let $d = \sup_{\trianglerighteq}(D)$. We shall prove that $d \in T(d)$. Indeed assume on the contrary that $d \notin T(d)$. Since every element z of A is an upper bound of D, so we get $d \trianglerighteq z$ for every $z \in A$. As $d \notin T(d)$, then $d \triangleright z$ for every $z \in A$. We claim that $T(d) \subset D$. Indeed, let $x \in T(d)$. So, by the monotonicity of T we get $x \trianglerighteq z$ for every $z \in A$. Thus, we have $T(d) \subset D$. Hence, we obtain $x \trianglerighteq d$ for every $x \in T(d)$. On the other hand, by our hypothesis we know that there is an element $t \in T(d)$ such that $d \trianglerighteq t$. Hence, from the antisymmetry of the relation $\trianglerighteq$ we deduce that $t = d$ and $d \in T(d)$. That is a contradiction. Hence, $d \in \mathrm{Fix}(T)$. Now, let B be the following subset of $\mathrm{Fix}(T)$ defined by

$$B = \{x \in \mathrm{Fix}(T) : x \trianglerighteq z \text{ for every } z \in A\}.$$

From Corollary 2.4, we know that the set-valued map T has a least fixed point. So, $B \neq \emptyset$. Let $m = \sup_{\trianglerighteq}(B)$. As $B \subset D$, then we get $m \trianglerighteq d$. On the other hand, we know that $d \in B$. Hence, we get $d \trianglerighteq m$. So, from the antisymmetry of the relation $\trianglerighteq$ we deduce that $m = d$. Then, $m \in \mathrm{Fix}(T)$. Therefore, the infimum of A in $\mathrm{Fix}(T)$ belongs to $\mathrm{Fix}(T)$.

Claim 2. The supremum of A in $\mathrm{Fix}(T)$ belongs to $\mathrm{Fix}(T)$. Indeed, let E be the following subset of X defined by

$$E = \{x \in X : z \trianglerighteq x \text{ for every } z \in A\}.$$

From Corollary 2.4, we know that T has a greatest fixed point. Then $\mathrm{Fix}(T) \neq \emptyset$. As $(X, \trianglerighteq)$ is a nonempty complete trellis, so let $g = \max(X)$. Hence, $g \in E$. Thus, $E \neq \emptyset$ and $g = \max(E)$. Now, we claim that $E \cap \mathrm{Fix}(T) \neq \emptyset$. Assume in the contrary that $E \cap \mathrm{Fix}(T) = \emptyset$. Then, $T(E) \subset E$. Indeed, let $x \in E$, $y \in T(x)$ and let $z \in A$. As $z \triangleright x$ and T is monotone, so for every $z \in A$, we get $z \trianglerighteq y$. Thus, we have $T(x) \subset E$ for every $x \in E$. Hence $T(E) \subset E$. On the other hand, as by our definition $T(x) \neq \emptyset$ for every $x \in X$. From the axiom of choice, there exists a map $\Phi \colon 2^X \to X$ such for every nonempty subset A of X we have $\Phi(A) \in A$. Then, for every $x \in X$ we define a new map $f \colon X \to X$ by setting: $f(x) = \Phi(T(x))$. We claim that f is a monotone map from $(X, \trianglerighteq)$ to $(X, \trianglerighteq)$. Indeed, let $x, y \in X$ with $x \triangleright y$. Since $f(x) \in T(x)$, $f(y) \in T(y)$ and T is monotone, then we get $f(x) \trianglerighteq f(y)$. Hence, f is a monotone map. Let F be a nonempty subset of E, $f = \inf(F)$ and $x \in F$. As for every $z \in A$ we have $z \trianglerighteq x$, then z is a lower bound of F. Hence, we get $z \trianglerighteq f$. Thus, every nonempty subset of E has an infimum in E and $(E, \trianglerighteq)$ has a greatest element. Therefore, all hypothesis of Theorem 3.3 in [6] are satisfied for the monotone map $f \colon E \to E$. Hence, $\mathrm{Fix}(f) \neq \emptyset$. Since $\mathrm{Fix}(f) \subset E \cap \mathrm{Fix}(T)$, so we get $E \cap \mathrm{Fix}(T) \neq \emptyset$. That is a contradiction. Therefore, $E \cap \mathrm{Fix}(T) \neq \emptyset$. Then, the set of all supremums of A in $(\mathrm{Fix}(T), \trianglerighteq)$: $G = E \cap \mathrm{Fix}(T)$ is nonempty. Let $\ell = \inf_{\trianglerighteq}(G)$. Then we get $\ell \in E$. We claim that $\ell \in \mathrm{Fix}(T)$. On the contrary assume that $\ell \notin \mathrm{Fix}(T)$. Now, let $x \in G$ and $t \in T(e)$ be given. As $\ell \triangleright x$, $x \in \mathrm{Fix}(T)$ and T is monotone, so we get $t \trianglerighteq x$. Thus, t is a lower bound of G. As $\ell = \inf_{\trianglerighteq}(G)$, then we deduce that we have $t \trianglerighteq \ell$, for every $t \in T(\ell)$. On the other hand, we know that by our hypothesis there is an element $g \in T(\ell)$ such that $\ell \trianglerighteq g$. So, from the antisymmetry of the relation $\trianglerighteq$ we deduce that $\ell = g$. Then, $\ell \in \mathrm{Fix}(T)$. Therefore, the infimum of A in $\mathrm{Fix}(T)$ belongs to $\mathrm{Fix}(T)$. □

As a consequence of Theorem 3.1, we reobtain the Skala's result [4, Theorem 37].

Corollary 3.2. *Let $(X, \trianglerighteq)$ be a non-empty complete trellis and let $f \colon X \to X$ be a monotone map such that for every $x \in X$, $x \trianglerighteq f(x)$. Then, the set of all fixed points* $\mathrm{Fix}(f)$ *of f is a non-empty complete trellis.*

Using Lemmas 1.3 and 1.4 and Theorem 3.1, we get the following dual result.

Theorem 3.3. *Let* $(X, \trianglerighteq)$ *be a non-empty complete trellis and let* $T\colon X \to 2^X$ *be a monotone set-valued map such that for every* $x \in X$ *there is* $y \in T(x)$ *satisfying* $y \trianglerighteq x$. *Then, the set of all fixed points* $\mathrm{Fix}(T)$ *of* T *is a non-empty complete trellis.*

As a corollary of Theorem 3.3, we obtain the following result for monotone map. That is a dual result of Theorem 37 in [4].

Corollary 3.4. *Let* $(X, \trianglerighteq)$ *be a non-empty complete trellis and let* $f\colon X \to X$ *be a monotone map such that for every* $x \in X$, $f(x) \trianglerighteq x$. *Then, the set of all fixed points* $\mathrm{Fix}(f)$ *of* f *is a non-empty complete trellis.*

4. Appendix

In this section, we shall give the proofs of Lemmas 1.3 and 1.4.

Proof of Lemma 1.3. Let $\trianglerighteq$ be a pseudo-order defined on a non-empty set X and let $\trianglelefteq$ be its inverse relation.

(i) The relation $\trianglelefteq$ is reflexive. Let $x \in X$. Then, $x \trianglerighteq x$. So, $x \trianglelefteq x$. Hence, $\trianglelefteq$ is reflexive.

(ii) The relation $\trianglelefteq$ is antisymmetric. Let $x, y \in X$ such that $x \trianglelefteq y$ and $y \trianglelefteq x$. So, we get $y \trianglerighteq x$ and $x \trianglerighteq y$. Since $\trianglerighteq$ is antisymmetric, then we obtain $x = y$. Thus, the relation $\trianglelefteq$ is antisymmetric. □

Proof of Lemma 1.4. Let $\trianglerighteq$ be a pseudo-order defined on a non-empty set X, let $\trianglelefteq$ be its inverse relation and let A be a non-empty subset of X.

(i) Assume that $\sup_{\trianglerighteq}(A)$ exists. Set $s = \sup_{\trianglerighteq}(A)$. Now, let $x \in A$. Then, $x \trianglerighteq s$. So, we get $s \trianglelefteq x$ for every $x \in A$. Thus, s is a $\trianglelefteq$–lower bound of A. Let ℓ be another $\trianglelefteq$–lower bound of A. So, we have $\ell \trianglelefteq x$ for every $x \in A$. Hence, $x \trianglerighteq \ell$. Then, ℓ is a $\trianglerighteq$–upper bound of A. As $s = \sup_{\trianglerighteq}(A)$, so $s \trianglerighteq \ell$. Hence, we get $\ell \trianglelefteq s$. Thus, s is the greatest $\trianglelefteq$–lower bound of A. Then, $s = \inf_{\trianglelefteq}(A)$.

(ii) Assume that $\inf_{\trianglerighteq}(A)$ exists. Set $\ell = \inf_{\trianglerighteq}(A)$. Now, let $x \in A$. Then, $\ell \trianglerighteq x$. So, we get $x \trianglelefteq \ell$ for every $x \in A$. Thus, ℓ is a $\trianglelefteq$–upper bound of A. Let m be another $\trianglelefteq$–upper bound of A. So, we have $x \trianglelefteq m$ for every $x \in A$. Hence, $m \trianglerighteq x$. Then, m is a $\trianglerighteq$–lower bound of A. As $\ell = \inf_{\trianglerighteq}(A)$, so $m \trianglerighteq \ell$. Thus, we have $\ell \trianglelefteq m$. Thus, ℓ is the least $\trianglelefteq$–upper bound of A. Then, $\ell = \sup_{\trianglelefteq}(A)$.

(iii) Let $m = \min_{\trianglerighteq}(A)$. Then, $m = \inf_{\trianglerighteq}(A)$ and $m \in A$. From (ii) above, we get $m = \sup_{\trianglelefteq}(A)$. As $m \in A$, hence we deduce that $m = \max_{\trianglelefteq}(A)$.

(iv) Let $s = \max_{\trianglerighteq}(A)$. So, $s = \sup_{\trianglerighteq}(A)$ and $m \in A$. From (ii) above, we get $s = \inf_{\trianglelefteq}(A)$. As $s \in A$, hence we obtain $s = \min_{\trianglelefteq}(A)$.

(v) Let let $T\colon X \to 2^X$ be a monotone set-valued map in $(X, \trianglerighteq)$. Let $x, y \in X$ such that $x \vartriangleleft y$. So, we have $y \vartriangleright x$. As T is $\trianglerighteq$–monotone, so we for every $a \in T(x)$ and $b \in T(y)$, we get $b \trianglerighteq a$. Hence, we deduce that for every $a \in T(x)$ and $b \in T(y)$, we have $a \trianglelefteq b$. Thus, T is $\trianglelefteq$–monotone. □

References

[1] S. P. Bhatta. Weak chain-completeness and fixed point property for pseudo-ordered sets. *Czechoslovak Math. J.*, 55(130)(2):365–369, 2005.
[2] P. Crawley and R. Dilworth. *Algebraic theory of lattices*. Prentice-Hall, 1973.
[3] S. Parameshwara Bhatta and H. Shashirekha. A characterization of completeness for trellises. *Algebra Universalis*, 44(3-4):305–308, 2000.
[4] H. Skala. *Trellis theory*. American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 121.
[5] H. L. Skala. Trellis theory. *Algebra Universalis*, 1:218–233, 1971/72.
[6] A. Stouti and A. Maaden. Fixed points and common fixed points theorems in pseudo-ordered sets. *Proyecciones*, 32(4):409–418, 2013.

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