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LOCALIZATION OF THE PSEUDOSPECTRA OF MATRICES THROUGH THE NUMERICAL RANGE

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Abstract. Numerical range and pseudospectra of a matrix play an important role in different areas and have several applications in different domains. The numerical range gives an estimate to the location of the pseudospectra, the present work proposes some properties of pseudospectra of a matrix and provides a connection between numerical range and pseudospectra of a matrix. Some upper bounds for pseudospectra of a matrix are given. Also we use the pseudospectra to solve a problem concerning the matrices which are nearly commuting.

1. Introduction

The study of pseudospectra of matrices has become a significant part of numerical linear algebra and related areas. Pseudospectra were introduced by Landau in 1975, who used the term ϵ-spectrum [8]. Four years later, J. M. Varah published a paper entitled "On the separation of two matrices," in which he use the notation $S_\epsilon(A)$ to ϵ-spectrum, and defined the ϵ-pseudospectrum using the smallest singular value $\sigma_{min}(A-\lambda I)$, see [15]. During the 1990s pseudospectra became an independent subject, for more details, see [3, 9, 10, 11, 12, 13, 14]. Numerous contributions related to pseudospectra were made by various people, including J. S. Baggett [1], A. Böttcher [2], T. A. Driscoll [3], M. Embree [14], N. Higham and F. Tisseur [6], S. C. Reddy [9], L. Reichel [10].

A *normal* matrix is one that satisfies $AA^* = A^*A$ where A^* is the conjugate transpose. It is known, see [14], that the ϵ-pseudospectrum of a normal matrix A is equal to the union of the closed ϵ-balls about the eigenvalues of A. In this paper we characterize the pseudospectra of a matrix using Taylor expansion. Some bounds of pseudospectra are given, too. We introduce a new concept

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pseudoprojection, different uses of pseudospectra are presented. Also some formulas which link the pseudospectra of a matrix with the numerical range are given.

2. Pseudospectra of a matrix

Four definitions of pseudospectra are given in [13], [14]. The first is related to perturbation, the second deals with the resolvent, the third is given with a normalized ϵ-pseudo-eigenvectors and the fourth definition involves the singular value decomposition.

Definition 1. Let $A \in \mathbb{C}^{n\times n}$ and $\epsilon \geq 0$ be arbitrary. The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$z \in \Lambda(A+E) \tag{1}$$

for some $E \in \mathbb{C}^{n\times n}$ with $\|E\| \leq \epsilon$. $\Lambda(A+E)$ denotes the spectrum of the matrix $(A+E)$.

The 0-pseudospectrum of A is just the spectrum of A i.e., $\Lambda_0(A) = \Lambda(A)$.

Definition 2. The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$\|(zI - A)^{-1}\| \geq \epsilon^{-1}. \tag{2}$$

I is the identity matrix and $(zI - A)^{-1}$ is the resolvent of A at z.

Definition 3. The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$\|(zI - A)v\| \leq \epsilon \tag{3}$$

for some $v \in \mathbb{C}^n$ with $\|v\| = 1$. z is an *ϵ-pseudo-eigenvalue* of A, and v is a corresponding *ϵ-pseudo-eigenvector.*

Definition 4. (Assuming that the norm is $\|.\|_2$.) The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$\sigma_{\min}(zI - A) \leq \epsilon. \tag{4}$$

$\sigma_{\min}(zI - A)$ denotes the smallest singular value of the matrix $(zI - A)$.

Theorem 1 ([13])**.** *The four definitions above are equivalent.*

Here we give some properties of the pseudospectra of a matrix.

Proposition 1. *Let $A \in \mathbb{C}^{n\times n}$, then*

(1) $\Lambda_{\epsilon_1}(A) \subseteq \Lambda_{\epsilon_2}(A),\ 0 \leq \epsilon_1 \leq \epsilon_2$.
(2) $\Lambda_\epsilon(A+F) \subseteq \Lambda_{\epsilon+\|F\|}(A),\ F \in \mathbb{C}^{n\times n}$.
(3) $\Lambda_{\epsilon_1}(A) + \Lambda_{\epsilon_2}(A) \subseteq \Lambda_{2\epsilon_1+2\epsilon_2}(2A)$.

In the third property a sum of sets has the usual meaning

$$\Lambda_{\epsilon_1}(A) + \Lambda_{\epsilon_2}(A) = \{z : z = z_1 + z_2,\ z_1 \in \Lambda_{\epsilon_1}(A), z_2 \in \Lambda_{\epsilon_2}(A)\}.$$

Proof. 1. Let $z \in \Lambda_{\epsilon_1}(A)$, there exists $E \in C^{n\times n}$ where $\|E\| \leq \epsilon_1$ such that $z \in \Lambda(A+E)$. Since $\epsilon_1 \leq \epsilon_2$, it follows $z \in \Lambda(A+E)$ where $\|E\| \leq \epsilon_2$. Hence $z \in \Lambda_{\epsilon_2}(A)$.

2. Let $z \in \Lambda_\epsilon(A+F)$, then $z \in \Lambda(A+F+E)$ where $\|E\| \leq \epsilon$. We have $\|F+E\| \leq \epsilon + \|F\|$, hence $z \in \Lambda_{\epsilon+\|F\|}(A)$.

3. Let $z \in \Lambda_{\epsilon_1}(A) + \Lambda_{\epsilon_2}(A)$, then $z = z_1 + z_2$ with $z_1 \in \Lambda_{\epsilon_1}(A)$ and $z_2 \in \Lambda_{\epsilon_2}(A)$. Assume that u_1 is the normalized ϵ-pseudo-eigenvector of A corresponding to z_1, so $(A+E_1)u_1 = z_1u_1$, $\|E_1\| \leq \epsilon_1$ and $(A+E_2)u_1 = z_2u_1 + w_2$, $\|E_2\| \leq \epsilon_2$, $w_2 \in C^n$. Thus $zu_1 = z_1u_1 + z_2u_1$, then

$$zu_1 = (2A + E_1 + E_2 - w_2u_1^*)u_1.$$

On the other hand, $\|E_1 + E_2 - w_2u_1^*\| \leq \epsilon_1 + \epsilon_2 + \|w_2\|$ with

$$\|w_2\| = \|(A+E_2)u_1 - z_2u_1\| \leq \|(z_2-A)u_1\| + \|E_2\| \leq \|(z_1-A)u_1\| + |z_1-z_2| + \epsilon_2.$$

Hence $\|w_2\| \leq \epsilon_1 + \epsilon_2 + |z_1 - z_2|$, therefore,

$$\|E_1 + E_2 - w_2u_1^*\| \leq 2\epsilon_1 + 2\epsilon_2 + |z_1 - z_2|.$$

Taking $z_1 = z_2$ it follows $z \in \Lambda_{2\epsilon_1+2\epsilon_2}(2A)$. □

Proposition 2. *The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that*

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{\|A^n\|}{|z^{n+1}|} \geq \epsilon^{-1} & \text{if } \|A\| < |z| \\ \displaystyle\sum_{n=0}^{\infty} \frac{|z^n|}{\|A^{n+1}\|} \geq \epsilon^{-1} & \text{if } |z| < \|A\|. \end{cases} \tag{5}$$

Proof. The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$\|(zI - A)^{-1}\| \geq \epsilon^{-1}.$$

On the other hand,

$$\|(zI-A)^{-1}\| = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{\|A^n\|}{|z^{n+1}|} & \text{if } \|A\| < |z| \\ \displaystyle\sum_{n=0}^{\infty} \frac{|z^n|}{\|A^{n+1}\|} & \text{if } |z| < \|A\|. \end{cases}$$

Hence, we obtain the desired result. □

In [11], it is shown that if V denotes any matrix of eigenvectors of A and $k(V) = \|V\|\|V^{-1}\|$ its *condition number* ($k(V) = \infty$ if A is not diagonalizable). Then we have

$$e^{t\alpha(A)} \leq \|e^{tA}\| \leq k(V)e^{t\alpha(A)}, t \geq 0, \tag{6}$$

where $\alpha(A) = \sup_{z\in\Lambda(A)} Re\, z$ is the *spectral abscissa* of A.

The analogous bound for matrix powers is

$$\rho(A)^n \leq \|A^n\| \leq k(V)\rho(A)^n, \tag{7}$$

where $\rho(A)$ is *the spectral radius* of A. The analogous bound for arbitrary functions $f(z)$ analytic in a neighborhood of the spectrum $\Lambda(A)$ is

$$\|f\|_{\Lambda(A)} \leq \|f(A)\| \leq k(V)\|f\|_{\Lambda(A)}, \tag{8}$$

where $\|f\|_{\Lambda(A)} = \sup_{z\in\Lambda(A)} |f(z)|$.

Theorem 2. *Let $A \in C^{n\times n}$ and $\epsilon \geq 0$ be arbitrary.*

(1) *If $z \in \Lambda_\epsilon(e^{tA}), t \geq 0$ and $\|e^{tA}\| < |z|$, resp. ($|z| < \|e^{tA}\|$) then*

$$k(V)\sum_{n=0}^{\infty} \frac{e^{tn\alpha(A)}}{|z^{n+1}|} \geq \epsilon^{-1}, \ \textit{resp.}\ (\sum_{n=0}^{\infty} \frac{|z^n|}{e^{t(n+1)\alpha(A)}} > \epsilon^{-1}).$$

(2) *If $z \in \Lambda_\epsilon(A)$ and $\|A\| < |z|$, resp. ($|z| < \|A\|$) then*

$$k(V)\sum_{n=0}^{\infty} \frac{\rho(A)^n}{|z^{n+1}|} \geq \epsilon^{-1}, \ \textit{resp.}\ (\sum_{n=0}^{\infty} \frac{|z^n|}{\rho(A)^{n+1}} > \epsilon^{-1}).$$

(3) *If $z \in \Lambda_\epsilon(f(A))$ and $\|f(A)\| < |z|$, resp. ($|z| < \|f(A)\|$) then*

$$k(V)\sum_{n=0}^{\infty} \frac{\|f^n\|_{\Lambda(A)}}{|z^{n+1}|} \geq \epsilon^{-1}, \ \textit{resp.}\ (\sum_{n=0}^{\infty} \frac{|z^n|}{\|f^{n+1}\|_{\Lambda(A)}} > \epsilon^{-1}).$$

Proof. By using (5)and (6), if $\|e^{tA}\| < |z|$, resp ($|z| < \|e^{tA}\|$), then

$$\epsilon^{-1} \leq \sum_{n=0}^{\infty} \frac{\|e^{tnA}\|}{|z^{n+1}|} \leq k(V)\sum_{n=0}^{\infty} \frac{e^{tn\alpha(A)}}{|z^{n+1}|}$$

resp.

$$(\epsilon^{-1} \leq \sum_{n=0}^{\infty} \frac{|z^n|}{\|e^{(n+1)tA}\|} \leq \sum_{n=0}^{\infty} \frac{|z^n|}{e^{t(n+1)\alpha(A)}}).$$

By using (5) and(7), if $\|A\| < |z|$ resp. ($|z| < \|A\|$) then

$$\epsilon^{-1} \leq \sum_{n=0}^{\infty} \frac{\|A^n\|}{|z^{n+1}|} \leq k(V)\sum_{n=0}^{\infty} \frac{\rho(A)^n}{|z^{n+1}|}$$

resp.

$$(\epsilon^{-1} \leq \sum_{n=0}^{\infty} \frac{|z^n|}{\|A^{n+1}\|} \leq \sum_{n=0}^{\infty} \frac{|z^n|}{\rho(A)^{n+1}}).$$

By using (5) and(8), if $\|f(A)\| < |z|$, resp. ($|z| < \|f(A)\|$), then

$$\epsilon^{-1} < \sum_{n=0}^{\infty} \frac{\|f^n(A)\|}{|z^{n+1}|} \leq k(V)\sum_{n=0}^{\infty} \frac{\|f^n\|_{\Lambda(A)}}{|z^{n+1}|}$$

resp.

$$(\epsilon^{-1} < \sum_{n=0}^{\infty} \frac{|z^n|}{\|f^{n+1}(A)\|} \leq \sum_{n=0}^{\infty} \frac{|z^n|}{\|f^{n+1}\|_{\Lambda(A)}}). \quad \square$$

Theorem 3. *Let $A \in C^{n\times n}$ and $\epsilon \geq 0$ be arbitrary. The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that*

$$\|u^*(zI - A)\| \leq \epsilon \tag{9}$$

for some $u \in \mathbb{C}^n$ with $\|u\| = 1$. u^* is the conjugate transpose of u.

Proof. Let $z \in \Lambda_\epsilon(A)$ and let v be its corresponding left eigenvector of the matrix $(A + E)$ with $\|E\| \leq \epsilon$. Thus $v^*(A + E) = zv^*$, then

$$\frac{v^*}{\|v\|}(zI - A) = \frac{v^*}{\|v\|}E.$$

Hence $\|u^*(zI - A)\| \leq \epsilon$ with $u = \dfrac{v^*}{\|v\|}$ and $\|u\| = 1$.

Now let $\|u^*(zI - A)\| \leq \epsilon$, then there exist η with $0 < \eta \leq \epsilon$ and $\phi \in C^n$ where $\|\phi\| = 1$, such that $u^*(zI - A) = \eta\phi^*$. Choosing $E = \eta u\phi^*$, it follows that $E \in C^{n\times n}$, $\|E\| = \|\eta u\phi^*\| \leq \eta\|u\|\|\phi^*\| \leq \eta \leq \epsilon$ and $u^*E = u^*(zI - A)$. Hence $z \in \Lambda_\epsilon(A)$. □

Proposition 3. *Let $A \in C^{n\times n}$ and $\epsilon \geq 0$ be arbitrary. Then there exist $\alpha \in C$ and $r_\epsilon > 0$, such that*

$$\Lambda_\epsilon(A) \subseteq \Lambda_{r_\epsilon}(\alpha I). \tag{10}$$

I is the identity matrix of dimension n.

Proof. Let $z_k \in \partial\Lambda_\epsilon(A)$, $k \in \{1, 2, \ldots, m\}$, where $\partial\Lambda_\epsilon(A)$ is the boundary of $\Lambda_\epsilon(A)$. Choosing α to be the barycenter of $\{(z_k, 1) \text{ with } k \in \{1, 2, \ldots, m\}\}$ and $r_\epsilon = \sup_{z_k\in\partial\Lambda_\epsilon(A)} |\alpha - z_k|$. Since αI is a normal matrix, then it is sufficient to prove that $\Lambda_\epsilon(A) \subseteq \overline{D}(\alpha, r_\epsilon)$ where $\overline{D}(\alpha, r_\epsilon)$ is the closed disk of radius r_ϵ and center α. If $z \in \Lambda_\epsilon(A)$, then $|\alpha - z| \leq r_\epsilon$. Hence $z \in \overline{D}(\alpha, r_\epsilon)$. □

Some properties of the pseudospectra of a matrix are given in [13], in particular, if

$$M = M_1 \oplus M_2 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

then $\Lambda_\epsilon(M) = \Lambda_\epsilon(M_1) \cup \Lambda_\epsilon(M_2)$.

Proposition 4. *Let $A \in C^{n\times n}$ and $\epsilon \geq 0$ be arbitrary. Then there exist a normal matrix B and $r_{\epsilon_k} > 0$ with $k \in \{1, 2, \cdots, l\}$, $l \leq n$, such that*

$$\Lambda_\epsilon(A) \subseteq \Lambda_{r_{\epsilon_k}}(B). \tag{11}$$

Proof. Let λ_i with $i \in \{1, 2, \cdots, n\}$ be the eigenvalues of A. Then there exist the matrices $A_k \in C^{n_k\times n_k}$ with $k \in \{1, 2, \cdots, l\}$, $l \leq n$ and $\sum_{k=1}^{l} n_k = n$ such that $A = \oplus_{k=1}^{l} A_k$. So $\Lambda(A) = \bigcup_{k=1}^{l} \Lambda(A_k)$, by (10), there exist $\beta_k \in C$ with $k \in \{1, 2, \ldots, l\}$ such that $\Lambda_\epsilon(A_k) \subseteq \Lambda_{r_{\epsilon_k}}(\beta_k I)$. Thus $\Lambda_\epsilon(A) \subseteq \bigcup_{k=1}^{l} \Lambda_{r_{\epsilon_k}}(\beta_k I)$, therefore, there exists a normal matrix B with eigenvalues β_k such that $\Lambda_\epsilon(A) \subseteq \Lambda_{r_{\epsilon_k}}(B)$. □

Let $\mathrm{Im}\,(C)$ denote any set of points in C. Let $\langle . \rangle$ be the application defined by

$$\langle \Lambda_\epsilon(A) \rangle = \overline{D}(\alpha, r_\epsilon).$$

α and r_ϵ are defined above.

Example. $\langle \Lambda_\epsilon(\gamma I) \rangle = \overline{D}(\gamma, \epsilon)$, $\gamma \in C$.

For any subset $S \subset \mathrm{Im}\,(C)$ and $\lambda \in R$, we define the set $\lambda S = \{\lambda s, s \in S\}$.

Definition 5. The application $f \colon \mathrm{Im}\,(C) \longrightarrow \mathrm{Im}\,(C)$ is a *pseudoprojection* if the following conditions hold

(1) $f(P_1 \cup P_2) = f(P_1) \cup f(P_2)$
(2) $f(\lambda P) = |\lambda| f(P)$ where $\lambda \in R$
(3) $f^2 = f$ where $f^2 = f \circ f$.

Theorem 4. *$\langle . \rangle$ is a pseudoprojection.*

Proof. (1) Given ϵ_1, ϵ_2 such that $0 < \epsilon_1 < \epsilon_2$, it follows

$$\begin{aligned}\langle \Lambda_{\epsilon_1}(A) \cup \Lambda_{\epsilon_2}(A) \rangle &= \langle \Lambda_{\epsilon_2}(A) \rangle = \overline{D}(\alpha_2, r_{\epsilon_2}) = \overline{D}(\alpha_1, r_{\epsilon_1}) \cup \overline{D}(\alpha_2, r_{\epsilon_2}) \\ &= \langle \Lambda_{\epsilon_1}(A) \rangle \cup \langle \Lambda_{\epsilon_2}(A) \rangle .\end{aligned}$$

(2) $\langle \lambda \Lambda_\epsilon(A) \rangle = \overline{D}(\alpha, \lambda r_\epsilon) = |\lambda| \overline{D}(\alpha, r_\epsilon) = |\lambda| \langle \Lambda_\epsilon(A) \rangle$, $\lambda \in R$.
(3) $\langle \langle \Lambda_\epsilon(A) \rangle \rangle = \left\langle \overline{D}(\alpha, r_\epsilon) \right\rangle = \langle \Lambda_{r_\epsilon}(\alpha I) \rangle = \overline{D}(\alpha, r_\epsilon) = \langle \Lambda_\epsilon(A) \rangle$. □

3. Numerical range

The numerical range $W(.)$ is a set of complex numbers associated with a given matrix $A \in C^{n \times n}$:

$$W(A) = \{x^* A x : x \in C^n, x^* x = 1\}. \tag{12}$$

Proposition 5 ([7]). *Let $A \in C^{n \times n}, B \in C^{n \times n}$ and $\gamma \in C$.*

$$\begin{aligned} W(A + B) &\subset W(A) + W(B), \\ W(\gamma A) &= \gamma W(A). \end{aligned}$$

Let $A \in C^{n \times n}$, consider the two matrices $H = \frac{1}{2}(A + A^*)$, $S = \frac{1}{2}(A - A^*)$. The matrix H is Hermitian and S is skew-Hermitian.

Proposition 6. *Let A, H and S be as described above, then*

$$W(A) = W(H) + W(S). \tag{13}$$

Proof. $A = H + S$, notice that $W(A) = \mathrm{Re}\, W(A) + i \mathrm{Im}\, W(A)$. We calculate:

$$\begin{aligned} x^* H x &= \frac{1}{2} x^*(A + A^*)x = \frac{1}{2}(x^* A x + x^* A^* x) \\ &= \frac{< x, Ax > + < x, A^* x >}{2} = \frac{< x, Ax > + < Ax, x >}{2} = \mathrm{Re}\, x^* A x. \end{aligned}$$

Thus, each $z \in W(H)$ is of the form $\operatorname{Re} z$ for some $z \in W(A)$ and vice versa. Hence $W(H) = \operatorname{Re} W(A)$.

$$\begin{aligned} x^*Sx &= \frac{1}{2}x^*(A - A^*)x = \frac{1}{2}(x^*Ax - x^*A^*x) = \frac{< x, Ax > - < x, A^*x >}{2} \\ &= \frac{< x, Ax > - < Ax, x >}{2} = i\operatorname{Im} x^*Ax. \end{aligned}$$

Thus, each $z \in W(S)$ is of the form $i\operatorname{Im} z$ for some $z \in W(A)$ and vice versa. Hence $W(S) = i\operatorname{Im} W(A)$, then the desired result is obtained. $\square$

Let the numerical positive abscissa of a matrix A be defined by

$$\omega^+(A) = \sup_{z \in W(A)} \operatorname{Re} z. \tag{14}$$

In the Hilbert space case, see [14], the numerical positive abscissa is given by the formula

$$\omega^+(A) = \sup \lambda \text{ where } \lambda \in \Lambda\left(\frac{A + A^*}{2}\right). \tag{15}$$

$\Lambda\left(\frac{A + A^*}{2}\right)$ denotes the spectrum of $\left(\frac{A + A^*}{2}\right)$.

Let the numerical negative abscissa of a matrix A be defined by

$$\omega^-(A) = \inf \lambda \text{ where } \lambda \in \Lambda\left(\frac{A + A^*}{2}\right). \tag{16}$$

Proposition 7. *For any matrix $A \in C^{n\times n}$,*

$$\omega^-(A) = \inf_{z \in W(A)} \operatorname{Re} z. \tag{17}$$

Proof.

$$\begin{aligned} \omega^-(A) &= \inf \lambda \, \text{where} \, \lambda \in \Lambda\left(\frac{A + A^*}{2}\right) \\ &= \inf_{\|x\|=1} \left\langle x, \frac{A + A^*}{2} x \right\rangle \\ &= \inf_{\|x\|=1} \frac{\langle x, Ax\rangle + \langle x, A^*x\rangle}{2} \\ &= \inf_{\|x\|=1} \frac{\langle x, Ax\rangle + \langle Ax, x\rangle}{2} \\ &= \inf_{\|x\|=1} \operatorname{Re} x^*Ax = \inf_{z \in W(A)} \operatorname{Re} z. \end{aligned}$$ $\square$

Theorem 5. *Let $A \in C^{n\times n}$ and $z \in C$. If $\omega^+(zI - A) = \omega_0$, then $\omega^-(A) = \operatorname{Re} z - \omega_0$.*

Proof. $\omega^+(zI - A) = \omega_0$, thus $\omega_0 = \sup \lambda$ where $\lambda \in \Lambda\left(\dfrac{zI - A + \overline{z}I - A^*}{2}\right)$. Then $(\operatorname{Re} z - \omega_0) = \inf \lambda$ where $\lambda \in \Lambda\left(\dfrac{A + A^*}{2}\right)$, hence the desired result is obtained. □

It is known [14] that, the pseudospectra cannot be much larger than the numerical range

$$\Lambda_\epsilon(A) \subseteq W(A) + \Delta_\epsilon. \tag{18}$$

We give another formula concerning the localization of the pseudospectra of a matrix using an upper bound independent from the matrix itself.

Lemma 1. *Let $A \in C^{n\times n}$, $u \in C^n$ and $\epsilon \geq 0$ be arbitrary. If $z \in \Lambda_\epsilon(A)$ then there exist $w \in C^n$, $b \in C^n$, where $\dfrac{b^* u}{\|u\|} = 1$ such that*

$$W\left(E - \frac{wb^*}{\|u\|}\right) = W(zI - A) \quad where \quad \|E\| \leq \epsilon. \tag{19}$$

Proof. Let $z \in \Lambda_\epsilon(A)$ then $(A + E)u = zu + w, \|E\| \leq \epsilon$ where $u, w \in C^n$. Thus $v^* E v = v^*(zI - A)v + \dfrac{v^* w}{\|u\|}$ where $v = \dfrac{u}{\|u\|}$. Hence

$$v^*\left(E - \frac{wb^*}{\|u\|}\right) v = v^*(zI - A)v$$

where $b^* v = 1$, then

$$W\left(E - \frac{wb^*}{\|u\|}\right) = W(zI - A).$$ □

Theorem 6. *Let A, u and ϵ be as described above. If $0 \in \Lambda_\epsilon(A)$ then*

$$\Lambda_\epsilon(A) \subseteq W\left(\frac{wb^*}{\|u\|}\right) + \Delta_{2\epsilon}. \tag{20}$$

Proof. Substituting $z = 0$ in (19), thus $W(A) = W\left(\dfrac{wb^*}{\|u\|} - E\right)$. Using (18), it follows $\Lambda_\epsilon(A) \subseteq W\left(\dfrac{wb^*}{\|u\|} - E\right) + \Delta_\epsilon$, then $\Lambda_\epsilon(A) \subseteq W(\dfrac{wb^*}{\|u\|}) + W(-E) + \Delta_\epsilon$. Assume that $z \in W(-E)$, then $z = x^*(-E)x$ where $\|x\| = 1$, thus $|z| \leq \epsilon$, hence $z \in \Delta_\epsilon$. Therefore, $\Lambda_\epsilon(A) \subseteq W\left(\dfrac{wb^*}{\|u\|}\right) + \Delta_{2\epsilon}$. □

Theorem 7. *Let $A \in C^{n\times n}$, $u \in C^n$ and $\epsilon \geq 0$ be arbitrary. If $z \in \Lambda_\epsilon(A)$ then there exist $\delta \in C$, $w \in C^n$ and $v \in C^n$, $\|v\| = 1$ such that*

$$W\left(\delta - (\bar{z} - A^*)(z - A)\right) = W\left((\bar{z} - A^*)\frac{wv^*}{\|u\|} + \frac{vw^*}{\|u\|}(z - A)\right). \tag{21}$$

Proof. Let $z \in \Lambda_\epsilon(A)$, then $(A+E)u = zu + w$, $\|E\| \le \epsilon$ implies $Eu = (z-A)u + w$. Assume without loss of generality that $E \in R^{n\times n}$. On other hand,

$$u^*(A^* + E) = \bar{z}u^* + w^*$$

then

$$u^*E = u^*(\bar{z} - A^*) + w^*,$$

thus

$$u^*E^2u = u^*(\bar{z} - A^*)(z-A)u + w^*w + u^*(\bar{z} - A^*)w + w^*(z-A)u.$$

Putting $\delta_1 = u^*E^2u - w^*w$, it follows that

$$\delta = v^*(\bar{z} - A^*)(z-A)v + v^*(\bar{z} - A^*)\frac{w}{\|u\|} + \frac{w^*}{\|u\|}(z-A)v,$$

where $v = \dfrac{u}{\|u\|}$, $\delta = \dfrac{\delta_1}{\|u\|^2}$. Hence

$$W\left(\delta - (\bar{z} - A^*)(z-A)\right) = W\left((\bar{z} - A^*)\frac{wv^*}{\|u\|} + \frac{vw^*}{\|u\|}(z-A)\right). \qquad \square$$

4. Almost commuting matrices

Let $A \in C^{n\times n}$, $B \in C^{n\times n}$, consider the commutator $[A,B] = AB - BA$. The pair of square matrices A and B is said to be nearly commute or almost commuting if $\|[A,B]\|$ is small, see [4], [5]. Consider the question: which matrices X is almost commuting with a given matrix A? In the following theorem, we propose an answer to this problem, a useful use of pseudospectra is given.

Definition 6. Given $\epsilon > 0$, let $A \in C^{n\times n}$, $B \in C^{n\times n}$. A and B are almost commuting if and only if

$$\|[A,B]\| \le \epsilon. \tag{22}$$

Theorem 8. *Given $\epsilon > 0$, let $A \in C^{n\times n}$, for some vectors $u \in C^n$, $v \in C^n$ with $\|u\| = \|v\| = 1$, we have A and $\frac{1}{2}(vu^*)$ are almost commuting.*

Proof. Let $z \in \Lambda_\epsilon(A)$, then $[A, vu^*] = Avu^* - vu^*A = (A-zI)vu^* - vu^*(A-zI)$. Hence $\|[A, vu^*]\| = \|(A-zI)vu^* - vu^*(A-zI)\| \le 2\epsilon$. $\square$

In general, almost commuting is not homogeneous: if $\|\,[A,B]\,\| \le \epsilon$ then, for every scalars p and q, $\|\,[pA, qB]\,\| \le |pq|\epsilon$. So, if $|pq|$ is not small, then the property of almost commuting is lost. Also this property is not linear, indeed, consider two linear combinations with scalar coefficients

$$[(pA+qB), (p'A + q'B)] = (pq' - qp')\,[A,B]\,.$$

If $(pq' - qp')$ is small, then, $[(pA+qB), (p'A+q'B)] < \epsilon$ implies that $[A,B]$ is not small and vice versa.

Corollary 1. *Given $\epsilon > 0$, let $A \in C^{n\times n}$ and $u \in C^n$. If $z \in \Lambda_\epsilon(A)$ then there exist $w \in C^n$, $v \in C^n$, $\|v\| = 1$ and $b \in C^n$ where $\dfrac{b^*u}{\|u\|} = 1$ such that*

$$W\left(\frac{1}{2}\left[A, \frac{vu^*}{\|u\|}\right] - \frac{wb^*}{\|u\|}\right) = W(zI - A). \tag{23}$$

Proof. Taking $E = \dfrac{1}{2}\left[A, \dfrac{vu^*}{\|u\|}\right]$, it follows that $\|E\| \leq \epsilon$. The result is obtained by substituting E in (19). □

Conclusion

The pseudospectra of a matrix are a set in the complex plane to which its pseudo-eigenvalues can be used to learn something else about the matrix and it can also give information that the spectrum alone cannot give. Pseudospectra are closely related to the numerical range, the behavior of pseudospectra determines the numerical range.

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