

ON THE CATEGORY OF WEAK BIALGEBRAS

GABRIELLA BÖHM, JOSÉ GÓMEZ-TORRECILLAS, AND ESPERANZA LÓPEZ-CENTELLA

ABSTRACT. Weak (Hopf) bialgebras are described as (Hopf) bimonoids in appropriate duoidal (also known as 2-monoidal) categories. This interpretation is used to define a category \mathbf{wba} of weak bialgebras over a given field. As an application, the “free vector space” functor from the category of small categories with finitely many objects to \mathbf{wba} is shown to possess a right adjoint, given by taking (certain) group-like elements. This adjunction is proven to restrict to the full subcategories of groupoids and of weak Hopf algebras, respectively. As a corollary, we obtain equivalences between the category of small categories with finitely many objects and the category of pointed cosemisimple weak bialgebras; and between the category of small groupoids with finitely many objects and the category of pointed cosemisimple weak Hopf algebras.

INTRODUCTION.

Some generalizations of (Hopf) bialgebras — which have been studied intensively on their own right — were shown to be instances of (Hopf) bimonoids in appropriately constructed braided (or even symmetric) monoidal categories. This was done, for example, in [10] for Turaev’s group (Hopf) bialgebras [26] and in [11] for Makhlof and Silvestrov’s hom (Hopf) bialgebras [16]. Such a description allows for a unified treatment of all these structures, it conceptually explains the origin of some results obtained earlier by other means and it also makes available the general theory of (Hopf) bimonoids in braided monoidal categories.

Weak (Hopf) bialgebras in [8], however, do not seem to be (Hopf) bimonoids in any braided monoidal category. Our first aim in this paper is to describe them rather as (Hopf) bimonoids in so-called *duoidal categories*.

Duoidal categories were introduced by Aguiar and Mahajan in [3] under the original name ‘2-monoidal category’. These are categories with two, possibly different, monoidal structures. They are required to be compatible in the sense that the functors and natural transformations defining the first monoidal structure, are comonoidal with respect to the second monoidal structure. Equivalently, the functors and natural transformations defining the second monoidal structure, are monoidal with respect to the first monoidal structure. Whenever both monoidal structures coincide, we re-obtain the notion of braided monoidal category. More details will be recalled in Section 1. A *bimonoid* in a duoidal category is a monoid with respect to the first monoidal structure and a comonoid with respect to the second monoidal structure. The compatibility axioms are formulated in terms of the coherence morphisms between the monoidal structures, see [3] (and a short review in Section 1). In the spirit of [4], a bimonoid is said to be a *Hopf monoid* provided that it induces a right Hopf comonad in the sense of [9], see Section 1.

An inspiring example in [3, Example 6.43] says that small categories can be described as bimonoids in an appropriately chosen duoidal category: in the category of spans over a given set (the set of objects). This construction is re-visited in Section 3. By this motivation we aim to find an appropriate duoidal

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category whose bimonoids are ‘quantum categories’; that is, weak bialgebras. Recall that weak bialgebras are examples of Takeuchi’s \times_R -bialgebras [25], equivalently, of Lu’s bialgebroids [14]; such that the base algebra R carries a separable Frobenius structure [22, 20]. Bialgebroids whose base algebra R is central, were described in [3, Example 6.44] as bimonoids in the duoidal category of R -bimodules. It was also discussed there that arbitrary bialgebroids are beyond this framework because the candidate — Takeuchi’s \times_R -operation — does not define a monoidal product in general.

In Section 4 we study the category of bimodules over $R \otimes R^{op}$ for a separable Frobenius algebra R . Observing that in this case Takeuchi’s \times_R -product becomes isomorphic to some (twisted) bimodule tensor product over $R \otimes R^{op}$, we equip this category with a duoidal structure. Moreover, we show that its bimonoids are precisely the weak bialgebras whose base algebra is isomorphic to R .

This interpretation of weak bialgebras as bimonoids allows us to define a category **wba** of weak bialgebras (by applying a more general construction in Section 2). Morphisms, from a weak bialgebra H with separable Frobenius base algebra R , to a weak bialgebra H' with separable Frobenius base algebra R' , are pairs of coalgebra maps $R \rightarrow R'$ and $H \rightarrow H'$ with additional properties that ensure that they induce a morphism of monoidal comonads — in the sense of [24] — from the monoidal comonad induced by H on the category of $R \otimes R^{op}$ -bimodules to the monoidal comonad induced by H' on the category of $R' \otimes R'^{op}$ -bimodules.

The vector space spanned by any small category with finitely many objects carries a weak bialgebra structure [5, 18]. This turns out to yield the object map of a functor k from the category **cat** of small categories with finitely many objects to **wba**, see Section 5. In Sections 6 and 7 we show that it possesses a right adjoint: For the interval category **2** and any weak bialgebra H , we consider the set $g(H) := \mathbf{wba}(k(\mathbf{2}), H)$. In general, it is isomorphic to a subset of the set of so-called ‘group-like elements’; that is, of coalgebra maps from the base field to H (not to be mixed with the weakly group-like elements in [8] and [27]). In favorable situations — for example, if H is cocommutative or H is a weak Hopf algebra — $g(H)$ is proven to be isomorphic to the set of group-like elements. For any weak bialgebra H , $g(H)$ is interpreted as the morphism set of a category and it is shown to obey $\mathbf{wba}(k(A), H) \cong \mathbf{cat}(A, g(H))$, for any small category A with finitely many objects. The unit of this adjunction is a natural isomorphism. The component of the counit at some weak bialgebra H is an isomorphism if and only if H is pointed cosemisimple (as a coalgebra). So we obtain an equivalence between **cat** and the full subcategory in **wba** of all pointed cosemisimple weak bialgebras.

Defining Hopf monoids in duoidal categories as bimonoids that induce right Hopf comonads in the sense of [9], the Hopf monoids in the duoidal category of spans turn out to be precisely the small groupoids. In the duoidal category of bimodules over $R \otimes R^{op}$, for a separable Frobenius algebra R , Hopf monoids turn out to be precisely the weak Hopf algebras with base algebra isomorphic to R . In Section 8 we show that the adjunction between **cat** and **wba** restricts to an adjunction between the category **grp** of small groupoids with finitely many objects, and the full subcategory **wha** in **wba** of all weak Hopf algebras. Consequently, the equivalence between **cat** and the full subcategory in **wba** of all pointed cosemisimple weak bialgebras restricts to an equivalence between **grp** and the full subcategory in **wha** of all pointed cosemisimple weak Hopf algebras. This extends the well-known relation between groups and pointed cosemisimple Hopf algebras, see for example [1].

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1. PRELIMINARIES.

Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, with underlying category \mathcal{C} , monoidal product \otimes and unit I . For any objects A, B, C of \mathcal{C} , we will denote by α the associator natural isomorphism

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C),$$

and by λ and ρ the unit natural isomorphisms

$$\lambda_A : I \otimes A \rightarrow A, \quad \rho_A : A \otimes I \rightarrow A.$$

The composition of morphisms will be denoted by juxtaposition.

1.1. Duoidal categories. A *duoidal category* (introduced in [3] under the name of 2-monoidal category) is a five tuple $(\mathcal{C}, \circ, I, \bullet, J)$, where (\mathcal{C}, \circ, I) and $(\mathcal{C}, \bullet, J)$ are monoidal categories, along with a transformation (called *the interchange law*)

$$(1.1) \quad \gamma_{A,B,C,D} : (A \bullet B) \circ (C \bullet D) \rightarrow (A \circ C) \bullet (B \circ D)$$

which is natural in A, B, C and D , and three morphisms

$$(1.2) \quad \mu_J : J \circ J \rightarrow J, \quad \Delta_I : I \rightarrow I \bullet I, \quad \tau : I \rightarrow J$$

such that the axioms below are satisfied.

Compatibility of units. The units I and J are compatible in the sense that (J, μ_J, τ) is a monoid in (\mathcal{C}, \circ, I) and (I, Δ_I, τ) is a comonoid in $(\mathcal{C}, \bullet, J)$. Equivalently, the following diagrams commute:

$$\begin{array}{ccc}
 (1.3) \quad \begin{array}{ccc} I & \xrightarrow{\Delta_I} & I \bullet I \\ \Delta_I \downarrow & & \downarrow \Delta_I \bullet I \\ I \bullet I & \xrightarrow{I \bullet \Delta_I} & I \bullet (I \bullet I) \end{array} & (1.4) \quad \begin{array}{ccccc} J \bullet I & \xleftarrow{\tau \bullet I} & I \bullet I & \xrightarrow{I \bullet \tau} & I \bullet J \\ & \searrow \lambda_I^\bullet & \downarrow \Delta_I & \swarrow \rho_I^\bullet & \\ & & I & & \end{array} \\
 (1.5) \quad \begin{array}{ccc} (J \circ J) \circ J & \xrightarrow{\mu_J \circ J} & J \circ J \\ \alpha_{J,J,J}^\circ \downarrow & & \downarrow \mu_J \\ J \circ (J \circ J) & & \\ J \circ \mu_J \downarrow & & \\ J \circ J & \xrightarrow{\mu_J} & J \end{array} & (1.6) \quad \begin{array}{ccccc} I \circ J & \xrightarrow{\tau \circ J} & J \circ J & \xleftarrow{J \circ \tau} & J \circ I \\ & \searrow \lambda_J^\circ & \downarrow \mu_J & \swarrow \rho_J^\circ & \\ & & J & & \end{array}
 \end{array}$$

Associativity. The following diagrams commute, for any objects A, B, C, D, E, F :

$$\begin{array}{ccc}
((A \bullet B) \circ (C \bullet D)) \circ (E \bullet F) & \xrightarrow{\alpha^\circ} & (A \bullet B) \circ ((C \bullet D) \circ (E \bullet F)) \\
\downarrow \gamma \circ (E \bullet F) & & \downarrow (A \bullet B) \circ \gamma \\
((A \circ C) \bullet (B \circ D)) \circ (E \bullet F) & & (A \bullet B) \circ ((C \circ E) \bullet (D \circ F)) \\
\downarrow \gamma & & \downarrow \gamma \\
((A \circ C) \circ E) \bullet ((B \circ D) \circ F) & \xrightarrow{\alpha^\circ \bullet \alpha^\circ} & (A \circ (C \circ E)) \bullet (B \circ (D \circ F))
\end{array}
\quad (1.7)$$

$$\begin{array}{ccc}
((A \bullet B) \bullet C) \circ ((D \bullet E) \bullet F) & \xrightarrow{\alpha^\bullet \circ \alpha^\bullet} & (A \bullet (B \bullet C)) \circ (D \bullet (E \bullet F)) \\
\downarrow \gamma & & \downarrow \gamma \\
((A \bullet B) \circ (D \bullet E)) \bullet (C \circ F) & & (A \circ D) \bullet ((B \bullet C) \circ (E \bullet F)) \\
\downarrow \gamma \bullet (C \circ F) & & \downarrow (A \circ D) \bullet \gamma \\
((A \circ D) \bullet (B \circ E)) \bullet (C \circ F) & \xrightarrow{\alpha^\bullet} & (A \circ D) \bullet ((B \circ E) \bullet (C \circ F))
\end{array}
\quad (1.8)$$

Unitality. The following diagrams commute, for any objects A, B :

$$\begin{array}{ccc}
I \circ (A \bullet B) & \xrightarrow{\Delta_I \circ (A \bullet B)} & (I \bullet I) \circ (A \bullet B) \\
\downarrow \lambda_{A \bullet B}^\circ & & \downarrow \gamma \\
A \bullet B & \xleftarrow{\lambda_A^\circ \bullet \lambda_B^\circ} & (I \circ A) \bullet (I \circ B)
\end{array}
\quad (1.9)$$

$$\begin{array}{ccc}
(A \bullet B) \circ I & \xrightarrow{(A \bullet B) \circ \Delta_I} & (A \bullet B) \circ (I \bullet I) \\
\downarrow \rho_{A \bullet B}^\circ & & \downarrow \gamma \\
A \bullet B & \xleftarrow{\rho_A^\circ \bullet \rho_B^\circ} & (A \circ I) \bullet (B \circ I)
\end{array}
\quad (1.10)$$

$$\begin{array}{ccc}
J \bullet (A \circ B) & \xleftarrow{\mu_J^\bullet \bullet (A \circ B)} & (J \circ J) \bullet (A \circ B) \\
\downarrow \lambda_{A \circ B}^\bullet & & \uparrow \gamma \\
A \circ B & \xleftarrow{\lambda_A^\bullet \circ \lambda_B^\bullet} & (J \bullet A) \circ (J \bullet B)
\end{array}
\quad (1.11)$$

$$\begin{array}{ccc}
(A \circ B) \bullet J & \xleftarrow{(A \circ B) \bullet \mu_J} & (A \circ B) \bullet (J \circ J) \\
\downarrow \rho_{A \circ B}^\bullet & & \uparrow \gamma \\
A \circ B & \xleftarrow{\rho_A^\bullet \circ \rho_B^\bullet} & (A \bullet J) \circ (B \bullet J)
\end{array}
\quad (1.12)$$

A *bimonoid* in \mathcal{C} is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$ where (H, μ, η) is a monoid in (\mathcal{C}, \circ, I) , (H, Δ, ϵ) is a comonoid in $(\mathcal{C}, \bullet, J)$ and both structures are compatible in the sense that the following four diagrams commute.

$$\begin{array}{ccc}
(H \bullet H) \circ (H \bullet H) & \xrightarrow{\gamma} & (H \circ H) \bullet (H \circ H) \\
\uparrow \Delta \circ \Delta & & \downarrow \mu \bullet \mu \\
H \circ H & \xrightarrow{\mu} H \xrightarrow{\Delta} H \bullet H
\end{array}
\quad (1.13)$$

$$\begin{array}{ccc}
H \circ H & \xrightarrow{\epsilon \circ \epsilon} & J \circ J \\
\downarrow \mu & & \downarrow \mu_J \\
H & \xrightarrow{\epsilon} & J
\end{array}
\quad (1.14)$$

$$\begin{array}{ccc}
I & \xrightarrow{\eta} & H \\
\downarrow \Delta_I & & \downarrow \Delta \\
I \bullet I & \xrightarrow{\eta \bullet \eta} & H \bullet H
\end{array}
\quad (1.15)$$

$$\begin{array}{ccc}
& H & \\
\eta \nearrow & & \searrow \epsilon \\
I & \xrightarrow{\tau} & J
\end{array}
\quad (1.16)$$

A *morphism of bimonoids* is a morphism of the underlying monoids and comonoids.

1.2. Monoidal comonads. Let $\mathbf{T} = (T, \delta, \varepsilon)$ be a comonad on a monoidal category $(\mathcal{C}, \otimes, I)$ with comultiplication $\delta : T \rightarrow T^2$ and counit $\varepsilon : T \rightarrow \mathcal{C}$. We call \mathbf{T} a *monoidal comonad* if the endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$ is monoidal and δ and ε are monoidal natural transformations. (We use the term ‘monoidal functor’ in the sense of [15, Section XI.2]. That is, we mean by it the existence of a morphism $T_0 : I \rightarrow TI$ and a natural transformation $T_2 : T(-) \otimes T(-) \rightarrow T((-) \otimes (-))$ obeying the evident associativity and unitality conditions. Some authors (for instance, the authors of [3]) use the name ‘lax monoidal’ functor

for the same notion. Also for the dual notion, both names ‘comonoidal functor’ (used in this paper) and ‘colax monoidal functor’ (used in [3]) do coexist.) As in [15, Section XI.2], monoidality of δ and ε means commutativity of the diagrams

$$\begin{array}{ccc}
 TA \otimes TB & \xrightarrow{T_2} & T(A \otimes B) \\
 \delta_A \otimes \delta_B \downarrow & & \downarrow \delta_{A \otimes B} \\
 T^2 A \otimes T^2 B & \xrightarrow{T_2} T(TA \otimes TB) \xrightarrow{TT_2} & T^2(A \otimes B)
 \end{array}
 \quad
 \begin{array}{ccc}
 I & \xrightarrow{T_0} & TI \\
 \parallel \downarrow & & \downarrow \delta_I \\
 I & \xrightarrow{T_0} TI \xrightarrow{TT_0} & T^2 I
 \end{array}$$

$$\begin{array}{ccc}
 TA \otimes TB & \xrightarrow{T_2} & T(A \otimes B) \\
 \epsilon_A \otimes \epsilon_B \downarrow & & \downarrow \epsilon_{A \otimes B} \\
 A \otimes B & \xlongequal{\quad} & A \otimes B
 \end{array}
 \quad
 \begin{array}{ccc}
 I & \xrightarrow{T_0} & TI \\
 \searrow & & \downarrow \epsilon_I \\
 & & I
 \end{array}$$

for any objects A, B . (Comonoidal monads, the dual concept, were introduced in [17] under the name of Hopf monads.) A bimonoid H in a duoidal category $(\mathcal{C}, \circ, I, \bullet, J)$ induces monoidal comonads $(-) \bullet H$ and $H \bullet (-)$ on (M, \circ, I) and comonoidal monads $(-) \circ H$ and $H \circ (-)$ on (M, \bullet, J) , see [4].

Let $\mathbf{T}' = (T', \delta', \varepsilon')$ be a second monoidal comonad on a monoidal category $(\mathcal{C}', \otimes', I')$. Recall from [24, Definition 3.1], that a morphism from \mathbf{T} to \mathbf{T}' is a pair $(F : \mathcal{C} \rightarrow \mathcal{C}', \Phi : FT \rightarrow T'F)$ where F is a comonoidal functor and Φ is a comonad morphism (in the sense of [21, §1]) rendering commutative also the diagrams

(1.17)

$$\begin{array}{ccc}
 F(TA \otimes TB) & \xrightarrow{FT_2} FT(A \otimes B) \xrightarrow{\Phi(A \otimes B)} & T'F(A \otimes B) \\
 F_2 \downarrow & & \downarrow T'F_2 \\
 FTA \otimes' FTB & \xrightarrow{\Phi A \otimes' \Phi B} T'FA \otimes' T'FB \xrightarrow{T'_2} & T'(FA \otimes' FB)
 \end{array}
 \quad
 \begin{array}{ccc}
 FI & \xrightarrow{F_0} & I' \\
 FT_0 \downarrow & & \downarrow T'_0 \\
 FTI & \xrightarrow{\Phi I} T'FI \xrightarrow{T'F_0} & T'I'
 \end{array}$$

for any objects A, B of \mathcal{C} .

A monoidal comonad \mathbf{T} is said to be a *right Hopf comonad* whenever for any objects A and B ,

$$TA \otimes TB \xrightarrow{\delta_A \otimes TB} T^2 A \otimes TB \xrightarrow{T_2} T(TA \otimes B)$$

is an isomorphism (natural in the objects A and B).

1.3. Separable Frobenius algebras. Let k be a field and R be an algebra over k . Recall that R is said to be a *Frobenius algebra* if there exists a Frobenius structure (ψ, e) consisting of a k -linear map $\psi : R \rightarrow k$ — called the *Frobenius functional* — and an element $e = \sum_i e_i \otimes f_i \in R \otimes R$ (the summation symbol will be omitted for brevity) — called the *Frobenius element* — such that

$$(1.18) \quad r = \psi(re_i)f_i = e_i\psi(f_ir) \quad \forall r \in R.$$

A Frobenius algebra R is necessarily finite dimensional with finite dual basis $e_i \otimes \psi(f_i -) \in R \otimes \text{Hom}(R, k)$.

Any Frobenius algebra R possesses a coalgebra structure with R -bilinear coassociative comultiplication $\delta : R \rightarrow R \otimes R$, $r \mapsto re = er$ and counit $\psi : R \rightarrow k$. Moreover, the category of right (respectively left) R -modules is isomorphic to the category of right (respectively left) R -comodules (see for example [2]). Thus, each right R -module M is endowed with a right R -comodule structure via the coaction $m \mapsto me_i \otimes f_i$. Conversely, any R -comodule $m \mapsto m_0 \otimes m_1$ is an R -module via the action $m \otimes r \mapsto m_0\psi(m_1r)$.

For any Frobenius algebra R there is a unique algebra automorphism $\theta : R \rightarrow R$, the so-called *Nakayama automorphism*, obeying

$$(1.19) \quad \psi(sr) = \psi(\theta(r)s) \quad \forall r, s \in R.$$

From (1.18) and (1.19) we get the explicit forms

$$(1.20) \quad \theta(r) = \psi(e_i r) f_i, \quad \theta^{-1}(r) = \psi(r f_i) e_i$$

of the Nakayama automorphism and its inverse, and the following identities for all $r \in R$.

$$(1.21) \quad r e_i \otimes f_i = e_i \otimes f_i r$$

$$(1.22) \quad e_i r \otimes f_i = e_i \otimes \theta(r) f_i$$

$$(1.23) \quad e_i \otimes \theta(f_i) = \theta^{-1}(e_i) \otimes f_i$$

$$(1.24) \quad \theta(e_i) \otimes f_i = f_i \otimes e_i = e_i \otimes \theta^{-1}(f_i)$$

An algebra R is *separable* if its multiplication is a split epimorphism of R -bimodules (see for example [13]). Equivalently, there exists an element $\sum_i a_i \otimes b_i \in R \otimes R$, called a separability structure for R , such that (omitting again the summation symbol)

$$(1.25) \quad r a_i \otimes b_i = a_i \otimes b_i r \quad \forall r \in R \quad \text{and} \quad a_i b_i = 1.$$

In this case the element $\sum_i a_i \otimes b_i$ is an idempotent in the enveloping algebra $R^e := R \otimes R^{op}$, that is,

$$(1.26) \quad a_i a_j \otimes b_j b_i = a_i \otimes b_i.$$

We say that R is a *separable Frobenius algebra* if there exists a Frobenius structure (ψ, e) for R such that $e = \sum_i e_i \otimes f_i \in R^e$ is also a separability structure (see [22]). In this case the canonical epimorphism $M \otimes N \rightarrow M \otimes_R N$ is split by

$$M \otimes_R N \mapsto M \otimes N, \quad m \otimes_R n \mapsto \sum_i m e_i \otimes f_i n,$$

for any right R -module M and any left R -module N . It is evident that the opposite algebra R^{op} and the tensor product $R \otimes S$ of separable Frobenius algebras R, S are also separable Frobenius algebras.

1.4. Weak bialgebras. Recall from [8] that a *weak bialgebra* $(H, \mu, \eta, \Delta, \epsilon)$ over a field k is an associative unital k -algebra (H, μ, η) and a coassociative counital k -coalgebra (H, Δ, ϵ) such that

$$(1.27) \quad \Delta(ab) = \Delta(a)\Delta(b),$$

$$(1.28) \quad \epsilon(ab_1)\epsilon(b_2c) = \epsilon(abc) = \epsilon(ab_2)\epsilon(b_1c)$$

$$(1.29) \quad (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = \Delta^2(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$$

for all $a, b, c \in H$. Here (and throughout) we use a simplified version of Heynemann-Sweedler's notation: $\Delta(a) = a_1 \otimes a_2$, implicit summation understood. The axioms expressing weak multiplicativity of the counit (that is, (1.28)) can be rephrased equivalently as

$$(1.30) \quad \epsilon(a_1)\epsilon(1_2c) = \epsilon(ac) = \epsilon(a_1)\epsilon(1_1c),$$

for all $a, c \in H$. Indeed, (1.28) \Rightarrow (1.30) is evident. Conversely, for any $b \in H$,

$$1_1 \otimes \Delta(1_2b) = 1_1 \otimes 1_2 b_1 \otimes 1_3 b_2 = 1_1 \otimes 1_2 1_1' b_1 \otimes 1_2' b_2 = 1_1 \otimes 1_2 b_1 \otimes b_2,$$

by the multiplicativity and the coassociativity of Δ , by (1.29), and by the multiplicativity of Δ again. Therefore,

$$\epsilon(ab_1)\epsilon(b_2c) = \epsilon(a_1)\epsilon(1_2 b_1)\epsilon(b_2c) = \epsilon(a_1)\epsilon((1_2 b)_1)\epsilon((1_2 b)_2c) = \epsilon(a_1)\epsilon(1_2 bc) = \epsilon(abc),$$

where the first and the last equalities follow by (1.30) and the penultimate one follows by counitality of Δ . One can argue symmetrically about the other axiom in (1.28).

Frequently, we will write H to denote the weak bialgebra, understanding that the structure is given. In a weak bialgebra H the counital (idempotent) maps $H \rightarrow H$ defined by the formulae

$$(1.31) \quad \begin{aligned} \sqcap^L(h) &= \varepsilon(1_1 h) 1_2, & \sqcap^R(h) &= 1_1 \varepsilon(h 1_2) \\ \overline{\sqcap}^L(h) &= \varepsilon(h 1_1) 1_2, & \overline{\sqcap}^R(h) &= 1_1 \varepsilon(1_2 h) \end{aligned}$$

play an important role. The following identities are immediate consequences of the above definitions.

$$(1.32) \quad \begin{aligned} \sqcap^L \sqcap^R &= \overline{\sqcap}^L, & \sqcap^R \sqcap^L &= \overline{\sqcap}^R, & \sqcap^R \overline{\sqcap}^L &= \sqcap^R, & \sqcap^L \overline{\sqcap}^R &= \sqcap^L, \\ \overline{\sqcap}^R \sqcap^R &= \sqcap^R, & \overline{\sqcap}^L \sqcap^L &= \sqcap^L, & \sqcap^R \overline{\sqcap}^R &= \overline{\sqcap}^R, & \sqcap^L \overline{\sqcap}^L &= \overline{\sqcap}^L. \end{aligned}$$

By ([8, Proposition 2.4]), the coinciding images of \sqcap^R and $\overline{\sqcap}^R$; and the coinciding images of \sqcap^L and $\overline{\sqcap}^L$ are unital subalgebras of H — they are called the ‘right’ and ‘left’ subalgebra, respectively — and they commute with each other. That is, for all $h, h' \in H$

$$(1.33) \quad \sqcap^R(h) \sqcap^L(h') = \sqcap^L(h') \sqcap^R(h).$$

Moreover,

$$(1.34) \quad \Delta(1) = \sqcap^R(1_1) \otimes 1_2 = 1_1 \otimes \sqcap^L(1_2) = \sqcap^R(1_2) \otimes \sqcap^L(1_1).$$

By ([8, Proposition 2.11], [22]), they are separable Frobenius (co)algebras with respective separability Frobenius structures

$$(\epsilon|_{\sqcap^R(H)}, 1_1 \otimes \sqcap^R(1_2)) \quad \text{and} \quad (\epsilon|_{\sqcap^L(H)}, \sqcap^L(1_1) \otimes 1_2).$$

Corestriction yields coalgebra maps $\sqcap^R : H \rightarrow \sqcap^R(H)$ and $\sqcap^L : H \rightarrow \sqcap^L(H)$; symmetrically, anti-coalgebra maps $\overline{\sqcap}^R : H \rightarrow \sqcap^R(H)$ and $\overline{\sqcap}^L : H \rightarrow \sqcap^L(H)$. Moreover, the maps \sqcap^R and $\overline{\sqcap}^L$; and also the maps $\overline{\sqcap}^R$ and \sqcap^L , induce mutually inverse anti-algebra and anti-coalgebra isomorphisms between the separable Frobenius (co)algebras $\sqcap^R(H)$ and $\sqcap^L(H)$, see for example [6, Proposition 1.18].

By ([8, Lemma 2.5]), \sqcap^R is a right $\sqcap^R(H)$ -module map and \sqcap^L is a left $\sqcap^L(H)$ -module map. Symmetrically, $\overline{\sqcap}^R$ is a left $\sqcap^R(H)$ -module map and $\overline{\sqcap}^L$ is a right $\sqcap^L(H)$ -module map. In formulae, the following identities hold true for all $h, h' \in H$.

$$(1.35) \quad \begin{aligned} \sqcap^R(h \sqcap^R(h')) &= \sqcap^R(h) \sqcap^R(h'), & \sqcap^L(\sqcap^L(h)h') &= \sqcap^L(h) \sqcap^L(h'), \\ \overline{\sqcap}^R(\overline{\sqcap}^R(h) h') &= \overline{\sqcap}^R(h) \overline{\sqcap}^R(h'), & \overline{\sqcap}^L(h \overline{\sqcap}^L(h')) &= \overline{\sqcap}^L(h) \overline{\sqcap}^L(h'). \end{aligned}$$

These maps also obey the so-called counital properties

$$(1.36) \quad h_1 \sqcap^R(h_2) = \sqcap^L(h_1)h_2 = \overline{\sqcap}^R(h_2)h_1 = h_2 \overline{\sqcap}^L(h_1) = h$$

and the identities

$$(1.37) \quad \begin{aligned} \sqcap^R(\sqcap^R(h)h') &= \sqcap^R(hh'), & \sqcap^L(h \sqcap^L(h')) &= \sqcap^L(hh'), \\ \overline{\sqcap}^R(h \overline{\sqcap}^R(h')) &= \overline{\sqcap}^R(hh'), & \overline{\sqcap}^L(h \overline{\sqcap}^L(h')) &= \overline{\sqcap}^L(hh'), \end{aligned}$$

for all $h, h' \in H$. In this work we will need a few more identities from [8]:

$$(1.38) \quad \begin{aligned} h_1 \otimes \sqcap^L(h_2) &= 1_1 h \otimes 1_2, & \Delta(\sqcap^L(h)) &= \sqcap^L(h) 1_1 \otimes 1_2, \\ h_1 \otimes \sqcap^R(h_2) &= h 1_1 \otimes \sqcap^R(1_2), & \Delta(\sqcap^R(h)) &= 1_1 \otimes \sqcap^R(h) 1_2, \end{aligned}$$

for all $h \in H$.

A weak bialgebra H is a *weak Hopf algebra* if there exists a k -linear map $S : H \rightarrow H$, called the *antipode*, satisfying the following axioms for all $h \in H$.

$$(1.39) \quad h_1 S(h_2) = \sqcap^L(h), \quad S(h_1) h_2 = \sqcap^R(h), \quad S(h_1) h_2 S(h_3) = S(h).$$

By [8, Theorem 2.10], the antipode is anti-multiplicative and anti-comultiplicative. That is, for all $h, h' \in H$,

$$(1.40) \quad S(hh') = S(h')S(h) \quad \text{and} \quad S(h)_1 \otimes S(h)_2 = S(h_2) \otimes S(h_1).$$

By [8, Lemma 2.9], the following identities hold true.

$$(1.41) \quad \begin{aligned} \sqcap^R S &= \sqcap^R \sqcap^L = S \sqcap^L, & \overline{\sqcap}^R S &= \sqcap^R = S \overline{\sqcap}^L, \\ \sqcap^L S &= \sqcap^L \sqcap^R = S \sqcap^R, & \overline{\sqcap}^L S &= \sqcap^L = S \overline{\sqcap}^R. \end{aligned}$$

For more on weak bialgebras and weak Hopf algebras, we refer to [8].

2. CATEGORIES OF BIMONOIDS.

Let \mathbf{duo} denote the category whose objects are duoidal categories and whose morphisms are functors which are comonoidal with respect to both monoidal structures. Consider a functor $M : \mathcal{S} \rightarrow \mathbf{duo}$ from an arbitrary category \mathcal{S} to \mathbf{duo} . In this section we associate a category (of some bimonoids) to M .

Lemma 2.1. *Let X and X' be objects of \mathcal{S} and let H and H' be bimonoids in MX and MX' , respectively. For a morphism $q : X \rightarrow X'$ in \mathcal{S} and a morphism $Q : (Mq)H \rightarrow H'$ in MX' , the following assertions are equivalent.*

(a) *The functor $Mq : MX \rightarrow MX'$ and the natural transformation*

$$(Mq)(- \bullet H) \xrightarrow{(Mq)_2^\bullet} (Mq)(-) \bullet' (Mq)H \xrightarrow{(Mq)(-) \bullet' Q} (Mq)(-) \bullet' H'$$

constitute a morphism of monoidal comonads $((MX, \circ), (-) \bullet H) \rightarrow ((MX', \circ'), (-) \bullet' H')$.

(b) *The following diagrams commute, for any objects A, B of MX .*

$$(2.1) \quad \begin{array}{ccc} (Mq)H & \xrightarrow{Q} & H' \\ (Mq)\Delta \downarrow & & \downarrow \Delta' \\ (Mq)(H \bullet H) & & \\ (Mq)_2^\bullet \downarrow & & \\ (Mq)H \bullet' (Mq)H & \xrightarrow{Q \bullet' Q} & H' \bullet' H' \end{array}$$

$$(2.2) \quad \begin{array}{ccc} (Mq)H & \xrightarrow{Q} & H' \\ (Mq)\epsilon \downarrow & & \downarrow \epsilon' \\ (Mq)J & & \\ (Mq)_0^\bullet \downarrow & & \\ J' & \xlongequal{\quad} & J' \end{array}$$

$$(2.3) \quad \begin{array}{ccc} (Mq)((A \bullet H) \circ (B \bullet H)) & \xrightarrow{(Mq)\gamma} & (Mq)((A \circ B) \bullet (H \circ H)) \\ (Mq)_2^\circ \downarrow & & \downarrow (Mq)_2^\bullet \\ (Mq)(A \bullet H) \circ' (Mq)(B \bullet H) & & (Mq)(A \circ B) \bullet' (Mq)(H \circ H) \\ (Mq)_2^\bullet \circ' (Mq)_2^\bullet \downarrow & & \downarrow (Mq)_2^\circ \bullet' (Mq)(H \circ H) \\ ((Mq)A \bullet' (Mq)H) \circ' ((Mq)B \bullet' (Mq)H) & & ((Mq)A \circ' (Mq)B) \bullet' (Mq)(H \circ H) \\ \gamma' \downarrow & & \downarrow ((Mq)A \circ' (Mq)B) \bullet' (Mq)\mu \\ ((Mq)A \circ' (Mq)B) \bullet' ((Mq)H \circ' (Mq)H) & & ((Mq)A \circ' (Mq)B) \bullet' (Mq)H \\ ((Mq)A \circ' (Mq)B) \bullet' (Q \circ' Q) \downarrow & & \downarrow ((Mq)A \circ' (Mq)B) \bullet' Q \\ ((Mq)A \circ' (Mq)B) \bullet' (H' \circ' H') & \xrightarrow{((Mq)A \circ' (Mq)B) \bullet' \mu'} & ((Mq)A \circ' (Mq)B) \bullet' H' \end{array}$$

$$\begin{array}{ccc}
(Mq)I & \xrightarrow{(Mq)_0^\circ} & I' \\
(Mq)\Delta \downarrow & & \downarrow \Delta' \\
(Mq)(I \bullet I) & & \\
(Mq)_2^\bullet \downarrow & & \\
(Mq)I \bullet' (Mq)I & & I' \bullet' I' \\
(Mq)I \bullet' (Mq)\eta \downarrow & & \downarrow I' \bullet' \eta' \\
(Mq)I \bullet' (Mq)H \xrightarrow{(Mq)I \bullet' Q} (Mq)I \bullet' H' \xrightarrow{(Mq)_0^\circ \bullet' H'} I' \bullet' H' & &
\end{array}
\tag{2.4}$$

Proof. Here and throughout, for brevity, we omit explicitly denoting the associator isomorphisms.

The diagrams (2.3) and (2.4) are identical to the diagrams in (1.17) for the functors $F = Mq$, $T = (-) \bullet H$ and $T' = (-) \bullet' H'$. So we only need to show that the pair in part (a) is a morphism of comonads if and only if (2.1) and (2.2) commute.

Assume first that the functor Mq and the natural transformation $((Mq)(-) \bullet' Q)(Mq)_2^\bullet$ constitute a morphism of comonads. This means commutativity of the diagrams

$$\begin{array}{ccc}
(Mq)(-) \bullet H & \xrightarrow{(Mq)(-\bullet\Delta)} & (Mq)(-) \bullet H \bullet H \\
(Mq)_2^\bullet \downarrow & & \downarrow (Mq)_2^\bullet \\
(Mq)(-) \bullet' (Mq)H & & (Mq)(-) \bullet H \bullet' (Mq)H \\
(Mq)(-) \bullet' Q \downarrow & & \downarrow (Mq)(-) \bullet' Q \\
(Mq)(-) \bullet' H' & \xrightarrow{(Mq)(-) \bullet' \Delta'} & (Mq)(-) \bullet' H' \bullet' H' \\
& & \downarrow (Mq)_2^\bullet \bullet' H' \\
& & (Mq)(-) \bullet' (Mq)H \bullet' H' \\
& & \downarrow (Mq)(-) \bullet' Q \bullet' H' \\
& & (Mq)(-) \bullet' H' \bullet' H'
\end{array}
\quad
\begin{array}{ccc}
(Mq)(-) \bullet H & \xrightarrow{(Mq)(-\bullet\epsilon)} & (Mq)(-) \bullet J \\
(Mq)_2^\bullet \downarrow & & \downarrow (Mq)_2^\bullet \\
(Mq)(-) \bullet' (Mq)H & & (Mq)(-) \bullet' (Mq)H \\
(Mq)(-) \bullet' Q \downarrow & & \downarrow (Mq)(-) \bullet' Q \\
(Mq)(-) \bullet' H' & \xrightarrow{(Mq)(-) \bullet' \epsilon'} & (Mq)(-) \bullet' H' \bullet' J' \\
& & \downarrow \rho_{Mq}^\bullet \\
& & Mq
\end{array}$$

Evaluate the equal pairs of paths around these diagrams at the monoidal unit J . Precompose the resulting morphisms with $((Mq)\lambda_H^\bullet)^{-1}$ in both cases and postcompose them with $\lambda_{H' \bullet' H'}^\bullet ((Mq)_0^\bullet \bullet' H' \bullet' H')$ in the case of the first diagram and postcompose them with $(Mq)_0^\bullet$ in the case of the second diagram. Using naturality of the morphisms λ^\bullet , $\lambda^{\bullet'}$, $\rho^{\bullet'}$ and $(Mq)_2^\bullet$, \bullet -comonoidality of Mq , the identity $\rho_J^\bullet = \lambda_J^\bullet$ (holding true in every monoidal category) and functoriality of the monoidal product \bullet' , they yield (2.1) and (2.2), respectively. The converse implication follows by noting first that $(Mq, (Mq)_2^\bullet)$ is a comonad morphism from the comonad $(-) \bullet H$ to $(-) \bullet' (Mq)H$ by coassociativity, counitality and naturality of $(Mq)_2^\bullet$. Second, since $Q : (Mq)H \rightarrow H'$ is a morphism of comonoids, $(MX', (-) \bullet' Q)$ is a comonad morphism from the comonad $(-) \bullet' (Mq)H$ to $(-) \bullet' H'$. Then also their composite is a comonad morphism. \square

Remark 2.2. If the functor $Mq : MX \rightarrow MX'$ is double comonoidal in the sense of [3, Definition 6.55], then the last two diagrams in part (b) of Lemma 2.1 turn out to be equivalent to the following diagrams.

$$\begin{array}{ccc}
 (Mq)(H \circ H) & \xrightarrow{(Mq)_2^\circ} & (Mq)H \circ' (Mq)H \xrightarrow{Q \circ' Q} H' \circ' H' \\
 (Mq)\mu \downarrow & & \downarrow \mu' \\
 (Mq)H & \xrightarrow{Q} & H'
 \end{array}
 \qquad
 \begin{array}{ccc}
 (Mq)I & \xrightarrow{(Mq)_0^\circ} & I' \\
 (Mq)\eta \downarrow & & \downarrow \eta' \\
 (Mq)H & \xrightarrow{Q} & H'
 \end{array}$$

However, in our most important example in Section 4, the functors $Mq : MX \rightarrow MX'$ are not double comonoidal. So we need to cope with the more general situation in Lemma 2.1.

Definition 2.3. Let M be a functor from an arbitrary category \mathcal{S} to the category **duo** of duoidal categories. The associated category $\mathbf{bmd}(M)$ is defined to have objects which are pairs, consisting of an object X of \mathcal{S} and a bimonoid H in MX . Morphisms are pairs consisting of a morphism $q : X \rightarrow X'$ in \mathcal{S} and a morphism $Q : (Mq)H \rightarrow H'$ in MX' , obeying the equivalent conditions in Lemma 2.1.

The composite of any morphisms of monoidal comonads is a morphism of monoidal comonads again, see [24, Definition 3.3]. Hence the composition of morphisms in $\mathbf{bmd}(M)$ is well-defined by their description in part (a) of Lemma 2.1.

If \mathcal{S} is the singleton category (having only one object and its identity morphism), then the functors $M : \mathcal{S} \rightarrow \mathbf{duo}$ are in bijection with the objects of **duo**; that is, with the duoidal categories M . As kindly pointed out by the referee, in this case $\mathbf{bmd}(M)$ is the usual category of bimonoids in the duoidal category M : Its objects are the bimonoids and the morphisms are the morphisms in M which are both morphisms of monoids (w.r.t. \circ) and morphisms of comonoids (w.r.t. \bullet); see Remark 2.2.

Remark 2.4. Note that Definition 2.3 is one choice of several symmetric possibilities. With this choice, we obtain the adjunction in Section 5 and Section 7. An analogous definition could be based on the monoidal comonad $((MX, \circ), H \bullet (-))$. If applied to the functor $\mathbf{span} : \mathbf{set} \rightarrow \mathbf{duo}$ in Section 3, it would lead to the same category of small categories. If applied to the functor $\mathbf{bim}(-^e) : \mathbf{sfr} \rightarrow \mathbf{duo}$ in Section 4, however, it would result in a different notion of morphism between weak bialgebras (related to that in Section 4 by interchanging the roles of the ‘left’ and ‘right’ subalgebras). This symmetric variant of the category of weak bialgebras admits a symmetric adjunction with the category of small categories, see also Remark 6.9.

As a further symmetry, one can change the notion of morphism between duoidal categories to functors which are monoidal with respect to both monoidal structures. Then two symmetric variants of morphisms between bimonoids can be defined in terms of the induced comonoidal monads $((MX, \bullet), H \circ (-))$ and $((MX, \bullet), (-) \circ H)$. (Note that while weak bialgebra is a self-dual structure, its morphisms in Section 4 are not. A category of weak bialgebras whose morphisms are dual to those in Section 4 can be obtained by this dual construction. The possibility of finding a contravariant adjunction to the category of small categories has not been investigated in this case.)

3. EXAMPLE: **cat** AS A CATEGORY OF BIMONOIDS.

3.1. The category $\mathbf{span}(X)$ [3, Example 6.17]. For any set X , a *span* over X is a triple (A, s, t) where A is a set and $s, t : A \rightarrow X$ is a pair of maps, called the *source* and *target* maps, respectively. A morphism between the spans (A, s, t) and (A', s', t') over X is a map $f : A \rightarrow A'$ such that the diagrams

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 s \searrow & & \swarrow s' \\
 & X &
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 t \searrow & & \swarrow t' \\
 & X &
 \end{array}$$

commute. For brevity, we write A instead (A, s, t) , understanding that s and t are given. We denote by $\text{span}(X)$ the category of spans over X . For any spans A and B , define the sets

$$\begin{aligned} A \circ B &:= \{(a, b) \in A \times B : s(a) = t(b)\} \\ A \bullet B &:= \{(a, b) \in A \times B : s(a) = s(b) \text{ and } t(a) = t(b)\}. \end{aligned}$$

We turn $A \circ B$ and $A \bullet B$ into spans over X by defining, for $(a, b) \in A \circ B$,

$$s(a, b) := s(b) \quad \text{and} \quad t(a, b) := t(a),$$

and for $(a, b) \in A \bullet B$,

$$s(a, b) := s(a) = s(b) \quad \text{and} \quad t(a, b) := t(a) = t(b).$$

Each one of these operations is functorial and endows the category $\text{span}(X)$ with a monoidal structure, with the obvious associators. The unit object I of $(\text{span}(X), \circ)$ is the discrete span $(X, \text{id}, \text{id})$ and the unit object J of $(\text{span}(X), \bullet)$ is the complete span $(X \times X, p_1, p_2)$ with $p_1(x, y) = x$ and $p_2(x, y) = y$. Furthermore, $(\text{span}(X), \circ, I, \bullet, J)$ is a duoidal category with the structure below. Let A, B, C, D be spans over X . The interchange law

$$\gamma_{A,B,C,D} : (A \bullet B) \circ (C \bullet D) \rightarrow (A \circ C) \bullet (B \circ D)$$

simply sends (a, b, c, d) to (a, c, b, d) . The structure map $\Delta_I : I \rightarrow I \bullet I$ is the identity and $\mu_J : J \circ J \rightarrow J$ and $\tau : I \rightarrow J$ are uniquely determined since the object J is terminal in the category $\text{span}(X)$.

A bimonoid in the duoidal category $\text{span}(X)$ is, equivalently, a small category (see [3, Example 6.43]).

Consider the following functor span from the category set of (small) sets to duo . It sends a set X to the duoidal category $\text{span}(X)$ above. Regarding its action on a map of sets $q : X \rightarrow X'$, note that q induces a morphism $\text{span}(q)$ in duo from $\text{span}(X)$ to $\text{span}(X')$: The functor $\text{span}(q)$ takes an object $t : X \leftarrow A \rightarrow X : s$ to $qt : X' \leftarrow A \rightarrow X' : qs$ and it acts on the morphisms as the identity map. It is easily seen to be comonoidal with respect to both monoidal structures \circ and \bullet , via

$$\begin{aligned} \text{span}(q)_2^\circ : A \circ B &\rightarrow A \circ' B, & (a, b) &\mapsto (a, b) \\ \text{span}(q)_0^\circ : X &\rightarrow X', & x &\mapsto q(x) \\ \text{span}(q)_2^\bullet : A \bullet B &\rightarrow A \bullet' B, & (a, b) &\mapsto (a, b) \\ \text{span}(q)_0^\bullet : X \times X &\rightarrow X' \times X' & (x, y) &\mapsto (q(x), q(y)). \end{aligned}$$

The final aim of this section is to prove the following.

Theorem 3.2. *The category $\text{bmd}(\text{span})$ is isomorphic to the category of small categories.*

Proof. Since there is exactly one comonoid structure (the ‘diagonal’ one) on any object (A, s, t) of $(\text{span}(X), \bullet, J)$, it follows that objects in $\text{bmd}(\text{span})$ are pairs (X, A) of a set X and a monoid A in $\text{span}(X)$ or, equivalently, a small category A with object set X , see [3].

The morphisms in $\text{bmd}(\text{span})$ are pairs $(q : X \rightarrow X', Q : A \rightarrow A')$ of maps for which $qs = s'Q$, $qt = t'Q$ and which render commutative the four diagrams in Lemma 2.1. Evaluating these diagrams on elements of the appropriate set, we see that (2.1) commutes for any maps $(q : X \rightarrow X', Q : A \rightarrow A')$; (2.2) commutes if and only if Q is a morphism of spans, (2.3) commutes if and only if Q preserves composition; and (2.4) commutes if and only if Q preserves identity morphisms. Shortly, these diagrams commute if and only if there is a functor with object map q and morphism map Q . \square

Applying the above construction to the restriction of the functor span to the full subcategory of finite sets in set , we obtain the full subcategory cat of small categories with finitely many objects.

4. EXAMPLE: wba AS A CATEGORY OF BIMONOIDS.

Let sfr denote the category whose objects are separable Frobenius (co)algebras (over a given base field k), and whose morphisms are defined as follows. Given separable Frobenius algebras R and R' with respective Nakayama automorphisms θ and θ' , a morphism in sfr from R to R' is a coalgebra map $f : R \rightarrow R'$ such that $\theta'f = f\theta$. In what follows we construct a functor $\text{bim}(-^e)$ from sfr to duo .

In the monoidal category of $R^e := R \otimes R^{op}$ -bimodules, we will not explicitly denote the associator constraints. However, since we work simultaneously with various R^e -actions, the corresponding unit constraints will be written out for an easier reading.

4.1. The first monoidal structure. Let R be an object in \mathbf{sfr} with idempotent Frobenius element $e_i \otimes f_i$, Frobenius functional $\psi : R \rightarrow k$ and Nakayama automorphism $\theta : R \rightarrow R$. The category $\mathbf{bim}(R^e)$ of R^e -bimodules is monoidal via the monoidal product $\circ = \otimes_{R^e}$, and unit $I = R^e$ with the R^e -bimodule structure given by its multiplication as a k -algebra; that is, with the actions

$$(4.1) \quad (s \otimes r)(x \otimes y) = sx \otimes yr \quad \text{and} \quad (x \otimes y)(s \otimes r) = xs \otimes ry.$$

Given R^e -bimodules M and N , the unit constraints are

$$\begin{aligned} \lambda_M^\circ : I \circ M &\rightarrow M, & (x \otimes y) \circ m &\mapsto (x \otimes y)m \\ \rho_M^\circ : M \circ I &\rightarrow M, & m \circ (x \otimes y) &\mapsto m(x \otimes y). \end{aligned}$$

The product $M \circ N$ is an R^e -bimodule via the actions

$$(4.2) \quad (s \otimes r)(m \circ n) = (s \otimes r)m \circ n \quad \text{and} \quad (m \circ n)(s \otimes r) = m \circ n(s \otimes r).$$

The canonical R^e -bimodule epimorphism

$$\pi_{M,N}^\circ : M \otimes N \rightarrow M \circ N, \quad m \otimes n \mapsto m \circ n$$

is split by

$$\iota_{M,N}^\circ : M \circ N \rightarrow M \otimes N, \quad m \circ n \mapsto m(e_i \otimes f_j) \otimes (f_i \otimes e_j)n.$$

Thus, $M \circ N$ is isomorphic to the vector subspace (in fact R^e -subbimodule)

$$\iota_{M,N}^\circ(M \circ N) = M(e_i \otimes f_j) \otimes (f_i \otimes e_j)N$$

of $M \otimes N$. Alternatively,

$$M \circ N \cong \{ x \in M \otimes N : \iota_{M,N}^\circ \pi_{M,N}^\circ x = x \}.$$

Recall from [5, Lemma 2.2] that the monoids in this monoidal category $(\mathbf{bim}(R^e), \circ, I)$ can be identified with pairs consisting of a k -algebra A and a k -algebra homomorphism $\eta : R^e \rightarrow A$. Via this identification, the morphisms of monoids correspond to k -algebra homomorphisms $f : A \rightarrow A'$ such that $f\eta = \eta'$.

Let H be a weak bialgebra over a field. Then its ‘right’ subalgebra $R := \square^R(H)$ is a separable Frobenius algebra and there is an algebra homomorphism $R^e \rightarrow H$, $s \otimes r \mapsto s\overline{\square}^L(r)$. Thus H is in particular a monoid in $(\mathbf{bim}(R^e), \circ, I)$.

4.2. The second monoidal structure. As we have seen in the previous paragraph, the multiplication of a weak bialgebra H , with base algebra $R := \square^R(H)$, is R^e -balanced with respect to the R^e -actions $(s \otimes r)h(s' \otimes r') = s\overline{\square}^L(r)hs'\overline{\square}^L(r')$ on H . The comultiplication of H factorizes through another R^e -module tensor product with respect to the twisted R^e -actions $(s \otimes r) \cdot h \cdot (s' \otimes r') = r'\square^L(r)hs'\square^L(s)$ on H (note the occurrence of the Nakayama automorphism $\square^R \square^L$ of R in $\square^L = \overline{\square}^L \square^R \square^L$).

By this motivation, for any separable Frobenius algebra R , define an automorphism functor $F : \mathbf{bim}(R^e) \rightarrow \mathbf{bim}(R^e)$ as follows. For any R^e -bimodule M , the underlying vector space of $F(M)$ is M endowed with the R^e -actions

$$(4.3) \quad (s \otimes r) \cdot m = (1 \otimes \theta(r))m(1 \otimes s) \quad m \cdot (s \otimes r) = (r \otimes 1)m(s \otimes 1).$$

This gives the object map of the functor F . On morphisms F acts as the identity map. The (strict) inverse of F sends an R^e -bimodule M to the R^e -bimodule whose underlying vector space is M , and whose actions are

$$(s \otimes r) \cdot m = (1 \otimes \theta^{-1}(r))m(1 \otimes s) \quad m \cdot (s \otimes r) = (r \otimes 1)m(s \otimes 1),$$

where juxtaposition denotes the original untwisted actions.

Any automorphism functor on a monoidal category can be used to twist the monoidal structure. In particular, we can use the above functor F to twist the monoidal category $(\mathbf{bim}(R^e), \circ, I)$ to a new monoidal category $(\mathbf{bim}(R^e), \bullet, J)$. The new monoidal product and unit are

$$M \bullet N = F^{-1}(F(M) \circ F(N)) \quad \text{and} \quad J = F^{-1}(I).$$

The underlying vector space of the R^e -bimodule $M \bullet N$ is the tensor product $F(M) \otimes_{R^e} F(N)$; that is, the factor space of $M \otimes N$ with respect to the relations

$$(4.4) \quad \{(r \otimes 1)m(s \otimes 1) \otimes n - m \otimes (1 \otimes \theta(r))n(1 \otimes s)\}.$$

The R^e -bimodule structure of $M \bullet N$ comes out as

$$(4.5) \quad (s \otimes r) \cdot (m \bullet n) = (1 \otimes \theta^{-1}(r)) \cdot m \bullet n \cdot (1 \otimes s) = (1 \otimes r)m \bullet (s \otimes 1)n$$

and, analogously,

$$(4.6) \quad (m \bullet n) \cdot (s \otimes r) = m(1 \otimes r) \bullet n(s \otimes 1).$$

The R^e -bimodule structure of $J = R^e$ is given by the actions

$$(4.7) \quad (s \otimes r) \cdot (x \otimes y) = x \otimes sy\theta^{-1}(r) \quad \text{and} \quad (x \otimes y) \cdot (s \otimes r) = rxs \otimes y.$$

The left and right unit constraints for the monoidal product \bullet are given by

$$(4.8) \quad \begin{aligned} \lambda_M^\bullet : J \bullet M &\rightarrow M, & (x \otimes y) \bullet m &\mapsto (x \otimes y) \cdot m = (1 \otimes \theta(y))m(1 \otimes x) \\ \rho_M^\bullet : M \bullet J &\rightarrow M, & m \bullet (x \otimes y) &\mapsto m \cdot (x \otimes y) = (y \otimes 1)m(x \otimes 1). \end{aligned}$$

The canonical epimorphism

$$\pi_{M,N}^\bullet : M \otimes N \rightarrow M \bullet N, \quad m \otimes n \mapsto m \bullet n$$

is a homomorphism of R^e -bimodules if $M \otimes N$ is considered as the bimodule $F^{-1}(F(M) \otimes F(N))$. Since R^e is a separable Frobenius algebra, $\pi_{M,N}^\bullet$ admits an R^e -bimodule section

$$(4.9) \quad \iota_{M,N}^\bullet : M \bullet N \rightarrow M \otimes N, \quad m \bullet n \mapsto m \cdot (e_j \otimes f_i) \otimes (f_j \otimes e_i) \cdot n = (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j).$$

Thus $M \bullet N$ is isomorphic to the R^e -subbimodule

$$\iota_{M,N}^\bullet(M \bullet N) = (e_i \otimes 1)M(e_j \otimes 1) \otimes (1 \otimes f_i)N(1 \otimes f_j)$$

of $M \otimes N$ (with the aforementioned structure). Alternatively,

$$M \bullet N \cong \{ x \in M \otimes N : \iota_{M,N}^\bullet \pi_{M,N}^\bullet(x) = x \}.$$

Note that by (4.8), (4.9), (4.7), (1.20), (1.24) and (1.18), the following diagrams commute, for any R^e -bimodule M .

$$(4.10) \quad \begin{array}{ccc} J \bullet M & \xrightarrow{\lambda_M^\bullet} & M \\ \downarrow \iota_{J,M}^\bullet & & \parallel \\ J \otimes M & \xrightarrow{\psi \otimes \psi \otimes M} & M \end{array} \quad \begin{array}{ccc} M \bullet J & \xrightarrow{\rho_M^\bullet} & M \\ \downarrow \iota_{M,J}^\bullet & & \parallel \\ M \otimes J & \xrightarrow{M \otimes \psi \otimes \psi} & M \end{array}$$

Theorem 4.3. $(\mathbf{bim}(R^e), \circ, I, \bullet, J)$ possesses the structure of a duoidal category.

Proof. Given R^e -bimodules A, B, C, D , we define the interchange law (1.1) by

$$(4.11) \quad \gamma((a \bullet b) \circ (c \bullet d)) = (a(e_i \otimes 1) \circ c) \bullet (b \circ (1 \otimes f_i)d)$$

and the morphisms in (1.2) by

$$(4.12) \quad \begin{aligned} \tau &: I \rightarrow J, & (x \otimes y) &\mapsto (yf_i \otimes xe_i) \\ \mu_J &: J \circ J \rightarrow J, & (x \otimes y) \circ (p \otimes q) &\mapsto \psi(xq)p \otimes y \\ \Delta_I &: I \rightarrow I \bullet I, & (x \otimes y) &\mapsto (1 \otimes y) \bullet (x \otimes 1). \end{aligned}$$

In order to show that γ is well defined, we should check that the map

$$\tilde{\gamma} : A \otimes B \otimes C \otimes D \rightarrow (A \circ C) \bullet (B \circ D), \quad a \otimes b \otimes c \otimes d \mapsto (a(e_i \otimes 1) \circ c) \bullet (b \circ (1 \otimes f_i)d)$$

is R^e -balanced in all of the three occurring tensor products. This follows by computations of the kind

$$\begin{aligned}
 \tilde{\gamma}[a \cdot (1 \otimes r) \otimes b \otimes c \otimes d] &= [(r \otimes 1)a(e_i \otimes 1) \circ c] \bullet [b \circ (1 \otimes f_i)d] \\
 &= [(a(e_i \otimes 1) \circ c) \cdot (1 \otimes r)] \bullet [b \circ (1 \otimes f_i)d] \\
 &= [a(e_i \otimes 1) \circ c] \bullet [(1 \otimes r) \cdot (b \circ (1 \otimes f_i)d)] \\
 &= [a(e_i \otimes 1) \circ c] \bullet [(1 \otimes \theta(r))b \circ (1 \otimes f_i)d] \\
 &= \tilde{\gamma}[a \otimes (1 \otimes r) \cdot b \otimes c \otimes d],
 \end{aligned}$$

and similarly in the other five cases. By similar steps one can also see that μ_J is well defined and that γ, τ, μ_J and Δ_I are morphisms of R^e -bimodules. For instance, let us show that τ is a morphism of left R^e -modules; the compatibilities of the other maps with the respective R^e -actions are checked similarly.

$$\begin{aligned}
 (s \otimes r) \cdot \tau(x \otimes y) &= (s \otimes r) \cdot (yf_i \otimes xe_i) \stackrel{(4.7)}{=} yf_i \otimes sxe_i \theta^{-1}(r) \\
 &\stackrel{(1.22)}{=} yrf_i \otimes sxe_i = \tau(sx \otimes yr) = \tau((s \otimes r)(x \otimes y))
 \end{aligned}$$

We turn to checking the compatibility between both monoidal structures. This amounts to showing that the just defined maps satisfy the associativity, unitality and compatibility of units conditions from Section 1.1. The computations are fairly straightforward so we only illustrate them on some chosen examples. For example, coassociativity of Δ_I and associativity of μ_J are obvious. The (left) counitality of Δ_I and the (left) unitality of μ_J are checked by the computations

$$\begin{array}{ccc}
 x \otimes y \xrightarrow{\Delta_I} (1 \otimes y) \bullet (x \otimes 1) & & (x \otimes y) \circ (u \otimes v) \xrightarrow{\tau \circ \text{id}} (yf_i \otimes xe_i) \circ (u \otimes v) \\
 \begin{array}{c} (1.24) \\ (1.25) \end{array} \parallel & \downarrow \tau \bullet \text{id} & \downarrow \lambda_J^\circ \\
 x \otimes yf_j \theta(e_j) \xleftarrow{\lambda_I^\circ} (yf_j \otimes e_j) \bullet (x \otimes 1) & & u \otimes xv \theta^{-1}(y) \xleftarrow[(1.18)(1.22)]{\psi} \psi(yf_i v) u \otimes xe_i,
 \end{array}$$

for any $x, y, u, v \in R$. Commutativity of (1.7) and (1.8) is immediate from (4.2), (4.5) and (4.6). The following computation, for any R^e -bimodules A and B , and any $a \in A$ and $b \in B$, proves the commutativity of (1.9). Commutativity of (1.10) is proven analogously.

$$\begin{array}{ccc}
 a \bullet b \xrightarrow{(\lambda_{A \bullet B}^\circ)^{-1}} (1 \otimes 1) \circ (a \bullet b) & & \\
 \downarrow (\lambda_A^\circ)^{-1} \bullet (\lambda_B^\circ)^{-1} & & \downarrow \Delta_{I \circ \text{id}} \\
 ((1 \otimes 1) \circ a) \bullet ((1 \otimes 1) \circ b) & & \\
 \parallel & & \\
 ((1 \otimes 1) \circ a) \cdot (1 \otimes e_j) \bullet (1 \otimes \theta^{-1}(f_j)) \cdot ((1 \otimes 1) \circ b) & & \\
 \parallel & & \\
 ((e_j \otimes 1) \circ a) \bullet ((1 \otimes 1) \circ (1 \otimes f_j)b) \xleftarrow{\gamma} ((1 \otimes 1) \bullet (1 \otimes 1)) \circ (a \bullet b)
 \end{array}$$

Reading from the top to the bottom, the first equality at the bottom left corner follows by (1.24) and (1.25). The computation below, for any R^e -bimodules A and B , and any $a \in A$ and $b \in B$, verifies the commutativity of (1.11), and (1.12) is checked analogously.

$$\begin{array}{ccc}
 a \circ b \xrightarrow{(\lambda_{A \circ B}^\circ)^{-1}} ((1 \otimes 1) \bullet a) \circ ((1 \otimes 1) \circ b) & & \\
 \downarrow (\lambda_{A \circ B}^\circ)^{-1} & & \downarrow \gamma \\
 (1 \otimes 1) \bullet (a \circ b) \xleftarrow[(1.18)]{\psi} \psi(e_i)(1 \otimes 1) \bullet (a \circ (1 \otimes f_i)b) \xleftarrow[\mu_J \bullet \text{id}]{} ((e_i \otimes 1) \circ (1 \otimes 1)) \bullet (a \circ (1 \otimes f_i)b)
 \end{array}$$

□

Remark 4.4. For a separable Frobenius (co)algebra R , and R^e -bi(co)modules M and N , the R -(co)module tensor product

$$M \otimes N / \{m(s \otimes 1) \otimes n - m \otimes n(1 \otimes s)\} \cong \{m(e_j \otimes 1) \otimes n(1 \otimes f_j)\}$$

is an R -bimodule via

$$r[m(e_j \otimes 1) \otimes n(1 \otimes f_j)]r' = (r \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes r')n(1 \otimes f_j).$$

The monoidal product $M \bullet N$ in Theorem 4.3 can be interpreted as the center of this bimodule. That is, \bullet is isomorphic to the so-called *Takeuchi product* over R [25].

The Takeuchi product of R^e -bimodules is defined for *any ring* R . However, at this level of generality it does not define a monoidal product on the category of R^e -bimodules (only a lax monoidal one, see [12]). It is a consequence of the separable Frobenius structure of R that allows us to write the Takeuchi product over it as a (co)module tensor product, what is more, as a split (co)equalizer.

Remark 4.5. For any commutative algebra R over a field k , a duoidal category $\mathbf{bim}(R)$ of R -bimodules was constructed in [3, Example 6.18]. Although the constructions in [3, Example 6.18] and in the current section are similar in flavor, they yield inequivalent categories for a commutative separable Frobenius k -algebra R (in which case both can be applied). Indeed, an equivalence $\mathbf{bim}(R) \cong \mathbf{bim}(R^e)$ would imply the Morita equivalence of R^e and $R^e \otimes R^e$; hence $R^e \cong R \cong k$. To say a bit more about the relationship between the categories $\mathbf{bim}(R)$ and $\mathbf{bim}(R^e)$, let R be a commutative separable Frobenius k -algebra. Any R -bimodule M with left and right actions $r \otimes m \mapsto r \triangleright m$ and $m \otimes r \mapsto m \triangleleft r$ can be regarded as an $R^e \cong R \otimes R$ -bimodule putting $(s \otimes r)m := r \triangleright m \triangleleft s =: m(s \otimes r)$. This is the object map of a fully faithful embedding (acting on the morphisms as the identity map) from the category $\mathbf{bim}(R)$ in [3, Example 6.18] to the category $\mathbf{bim}(R^e)$ in Theorem 4.3 — but it is not an equivalence. It is strict monoidal with respect to the monoidal products \diamond in [3, Example 6.18] and \circ in Theorem 4.3 — but not with respect to \star in [3, Example 6.18] and \bullet in Theorem 4.3. In fact, it takes the monoidal product \star to \bullet but it does not preserve its monoidal unit. The image of the \star -monoidal unit R in [3, Example 6.18] does not serve as a \bullet -monoidal unit in our $\mathbf{bim}(R^e)$, while our \bullet -monoidal unit R^e does not lie in the image of the above embedding $\mathbf{bim}(R) \rightarrow \mathbf{bim}(R^e)$.

Remark 4.6. Recall (from [3, Appendix C.5.3]) that for a commutative k -algebra R , the duoidal category $\mathbf{bim}(R)$ in [3, Example 6.18] arises via the so-called ‘looping principle’. This means the following. If $(\mathcal{V}, \times, 1)$ is a monoidal 2-category and \mathcal{C} is a \mathcal{V} -enriched bicategory, then for any object R of \mathcal{C} , the hom object $\mathcal{C}(R, R)$ is a pseudo-monoid in \mathcal{V} . By [3, Appendix C.2.4], duoidal categories can be regarded as pseudo-monoids in the monoidal 2-category \mathbf{coMon} of monoidal categories, comonoidal functors and comonoidal natural transformations (with monoidal structure provided by the Cartesian product). So via the looping principle, hom objects in a \mathbf{coMon} -enriched bicategory are duoidal categories.

Below we claim that also the duoidal category $\mathbf{bim}(R^e)$ in Theorem 4.3 can be obtained via the looping principle. For this purpose, we sketch the construction of a \mathbf{coMon} -enriched bicategory \mathcal{C} whose objects are separable Frobenius k -algebras, and for any object R , $\mathcal{C}(R, R) \cong \mathbf{bim}(R^e)$. For any separable Frobenius k -algebras S and R , let $\mathcal{C}(R, S)$ be the category of R^e - S^e -bimodules. As in (4.3), we can regard any R^e - S^e -bimodule M as an $S \otimes R$ -bimodule via the actions

$$(s \otimes r) \cdot m \cdot (s' \otimes r') = (r' \otimes \theta(r))m(s' \otimes s).$$

Hence $\mathcal{C}(R, S)$ is a monoidal category via the $S \otimes R$ -module tensor product

$$M \bullet N := M \otimes N / \{(r \otimes 1)m(s \otimes 1) \otimes n - m \otimes (1 \otimes \theta(r))n(1 \otimes s)\},$$

cf. (4.4). The product $M \bullet N$ is an R^e - S^e -bimodule as in (4.5-4.6). The monoidal unit is $R \otimes S$ with the actions $(r \otimes r')(x \otimes y)(s \otimes s') = rx\theta^{-1}(r') \otimes s'ys$ (which becomes isomorphic to the R^e -bimodule J in (4.7) if $S = R$). For any separable Frobenius k -algebra R , there is a comonoidal functor I_R from the singleton category $\mathbf{1}$ to $\mathcal{C}(R, R)$, sending the single object of $\mathbf{1}$ to the R^e -bimodule I in (4.1). Its comonoidal structure is given (up-to isomorphism) by the R^e -bimodule maps $\tau : I \rightarrow J$ and $\Delta_I : I \rightarrow I \bullet I$ in (4.12). Coassociativity and counitality of this comonoidal functor follows by coassociativity and counitality of Δ_I . Furthermore, for any separable Frobenius k -algebras S , R and T , there is a comonoidal functor $\circ_{S,R,T} : \mathcal{C}(S, R) \times \mathcal{C}(R, T) \rightarrow \mathcal{C}(S, T)$ given by the usual R^e -module tensor product. Its comonoidal structure is given by the maps

$$\begin{aligned} (S \otimes R) \circ (R \otimes T) &\rightarrow S \otimes T, & (s \otimes r) \circ (r' \otimes t) &\mapsto \psi(rr')(s \otimes t) \quad \text{and} \\ (A \bullet B) \circ (C \bullet D) &\rightarrow (A \circ C) \bullet (B \circ D), & (a \bullet b) \circ (c \bullet d) &\mapsto (a(e_i \otimes 1) \circ c) \bullet (b \circ (1 \otimes f_i)d), \end{aligned}$$

for any S^e - R^e -bimodules A and B and R^e - T^e -bimodules C and D (compare them with μ_J in (4.12) and γ in (4.11)). These maps are checked to be bimodule maps in the same way as μ_J and γ are in the proof of Theorem 4.3. Naturality of the binary part is immediate. Coassociativity and counitality of the comonoidal functor $\circ_{S,R,T}$ is verified by the same computations used to check that (1.8), (1.11) and (1.12) hold in the proof of Theorem 4.3. The unitors and the associator for the module tensor product \circ give rise to 2-cells

$$\begin{array}{ccccc}
\mathcal{C}(R, S) \times \mathbf{1} & \xleftarrow{\cong} & \mathcal{C}(R, S) & \xrightarrow{\cong} & \mathbf{1} \times \mathcal{C}(R, S) \\
\downarrow \mathcal{C}(R, S) \times I_S & \Rightarrow & \parallel & \Leftarrow & \downarrow I_R \times \mathcal{C}(R, S) \\
\mathcal{C}(R, S) \times \mathcal{C}(S, S) & \xrightarrow{\circ_{R,S,S}} & \mathcal{C}(R, S) & \xleftarrow{\circ_{R,R,S}} & \mathcal{C}(R, R) \times \mathcal{C}(R, S)
\end{array}$$

$$\begin{array}{ccc}
(\mathcal{C}(Z, R) \times \mathcal{C}(R, S)) \times \mathcal{C}(S, T) & \xrightarrow{\cong} & \mathcal{C}(Z, R) \times (\mathcal{C}(R, S) \times \mathcal{C}(S, T)) \\
\downarrow \circ_{Z,R,S} \times \mathcal{C}(S, T) & & \downarrow \mathcal{C}(Z, R) \times \circ_{R,S,T} \\
\mathcal{C}(Z, S) \times \mathcal{C}(S, T) & \Rightarrow & \mathcal{C}(Z, R) \times \mathcal{C}(R, T) \\
\downarrow \circ_{Z,S,T} & & \downarrow \circ_{Z,R,T} \\
\mathcal{C}(Z, T) & \xlongequal{\quad} & \mathcal{C}(Z, T)
\end{array}$$

in **coMon**, for any separable Frobenius algebras S, R, Z, T . Indeed, they are shown to be comonoidal natural transformations by computations similar to those verifying the associativity and the unitality of μ_J and the validity of (1.7), (1.9) and (1.10) in the proof of Theorem 4.3. They clearly obey the Mac Lane type coherence conditions. This proves that \mathcal{C} is a **coMon**-enriched bicategory hence $\mathcal{C}(R, R) \cong \mathbf{bim}(R^e)$ is a duoidal category.

4.7. The functor $\mathbf{bim}(-^e)$. Let R and R' be separable Frobenius (co)algebras. Associated to any coalgebra homomorphism $q : R \rightarrow R'$, there is a functor $\mathbf{bim}(q^e) : \mathbf{bim}(R^e) \rightarrow \mathbf{bim}(R'^e)$ (where $q^e : R^e \rightarrow R'^e$ is defined by $q^e(s \otimes r) = q(s) \otimes q(r)$). It acts on the morphisms as the identity map. It takes an R^e -bi(co)module P with coactions $\lambda : P \rightarrow R^e \otimes P$ and $\rho : P \rightarrow P \otimes R^e$ to the R'^e -bi(co)module P with the coactions $(q^e \otimes P)\lambda$ and $(P \otimes q^e)\rho$. The R'^e -actions on P are induced from the R^e -actions by the dual forms of q ; that is, by the algebra maps

$$\tilde{q} : R' \rightarrow R, \quad r' \mapsto \psi'(r' q(e_i)) f_i \quad \text{and} \quad \hat{q} : R' \rightarrow R, \quad r' \mapsto e_i \psi'(q(f_i) r')$$

as

$$(r' \otimes s') p(u' \otimes v') = (\tilde{q}(r') \otimes \hat{q}(s')) p(\hat{q}(u') \otimes \tilde{q}(v')), \quad \text{for } p \in P, \quad r', s', u', v' \in R'.$$

Note that

$$(4.13) \quad \hat{q}(e'_i) \otimes f'_i = e_j \otimes q(f_j) \quad \text{and} \quad e'_i \otimes \tilde{q}(f'_i) = q(e_j) \otimes f_j.$$

The maps \tilde{q} and \hat{q} are equal if and only if q commutes with the Nakayama automorphisms; that is, $\theta' q = q \theta$.

Proposition 4.8. *Let R and R' be separable Frobenius (co)algebras and $q : R \rightarrow R'$ be a coalgebra homomorphism which commutes with the Nakayama automorphisms. The induced functor $\mathbf{bim}(q^e) : \mathbf{bim}(R^e) \rightarrow \mathbf{bim}(R'^e)$ is comonoidal with respect to both monoidal structures.*

Proof. The coalgebra homomorphisms $q : R \rightarrow R'$ are in bijective correspondence with the algebra homomorphisms $\tilde{q} : R' \rightarrow R$ via transposition (or duality)

$$q \mapsto \tilde{q} = \psi'(-q(e_i)) f_i \quad \tilde{q} \mapsto q = e'_i \psi(\tilde{q}(f'_i) -).$$

In particular, the Nakayama automorphism and its dual $\tilde{\theta}$ satisfy

$$\tilde{\theta}(r) = \psi(r \theta(e_i)) f_i = \psi(r f_i) e_i = \theta^{-1}(r),$$

cf. (1.24) and (1.20). Thus the assumption $\theta'q = q\theta$ can be written equivalently as $\tilde{q}\theta' = \theta\tilde{q}$.

The candidate for the binary part of the comonoidal structure with respect to \circ is the $(R^e\text{-bimodule})$ map

$$\text{bim}(q^e)_2^\circ : \text{bim}(q^e)(M \circ N) \rightarrow \text{bim}(q^e)M \circ' \text{bim}(q^e)N, \quad m \circ n \mapsto m(e_i \otimes f_j) \circ' (f_i \otimes e_j)n.$$

It is evidently coassociative. The nullary part of the \circ -comonoidal structure is

$$\text{bim}(q^e)_0^\circ = q^e : R^e \rightarrow R'^e, \quad x \otimes y \mapsto q(x) \otimes q(y).$$

Its R'^e -bimodule map property; that is,

$$sq(x)s' \otimes rq(y)r' = q(\tilde{q}(s)x\tilde{q}(s')) \otimes q(\tilde{q}(r)y\tilde{q}(r'))$$

is proven by

$$(4.14) \quad \begin{aligned} q(\tilde{q}(r')x\tilde{q}(s')) &= \psi'(r'q(e_i))q(f_i x e_j)\psi'(q(f_j)s') = \psi'(r'q(e_i))q(f_i x)e'_k\psi'(f'_k s') \\ &= \psi'(r'q(e_i))q(f_i x)s' = \psi'(r'e'_i)f'_i q(x)s' = r'q(x)s', \end{aligned}$$

for all $r', s' \in R'$, where in the second and the penultimate equalities we used that q is comultiplicative. Right counitality; that is, commutativity of

$$\begin{array}{ccc} \text{bim}(q^e)M \xrightarrow{\text{bim}(q^e)((\rho_M^\circ)^{-1})} \text{bim}(q^e)(M \circ I) & & m \xrightarrow{\quad} m \circ (1 \otimes 1) \\ \text{bim}(q^e)M \downarrow & & \parallel \\ \text{bim}(q^e)M & & m(e_i \otimes f_j)(\tilde{q}q(f_i) \otimes \tilde{q}q(e_j)) \\ \rho_{\text{bim}(q^e)M}^{\circ'} \uparrow & & \uparrow \\ \text{bim}(q^e)M \circ' I' \leftarrow \text{bim}(q^e)M \circ' \text{bim}(q^e)I & & m(e_i \otimes f_j) \circ' (q(f_i) \otimes q(e_j)) \leftarrow m(e_i \otimes f_j) \circ' (f_i \otimes e_j) \\ \text{bim}(q^e)M \circ' \text{bim}(q^e)_0^\circ & & \end{array}$$

follows from

$$(4.15) \quad e_j \tilde{q}q(f_j) \stackrel{(4.13)}{=} \tilde{q}(e'_i)\tilde{q}(f'_i) = \tilde{q}(e'_i f'_i) = \tilde{q}(1') = 1 \quad \text{and} \quad \tilde{q}q(e_j)f_j \stackrel{(4.13)}{=} \tilde{q}(e'_i)\tilde{q}(f'_i) = \tilde{q}(e'_i f'_i) = \tilde{q}(1') = 1,$$

where in both cases, in the second and the last equalities we used that \tilde{q} is an algebra homomorphism. A similar computation shows counitality on the other side.

The binary part of the \bullet -comonoidal structure is given by the R^e -bimodule maps

$$\text{bim}(q^e)_2^\bullet : \text{bim}(q^e)(M \bullet N) \rightarrow \text{bim}(q^e)M \bullet' \text{bim}(q^e)N, \quad m \bullet n \mapsto (e_i \otimes 1)m(e_j \otimes 1) \bullet' (1 \otimes f_i)n(1 \otimes f_j).$$

Its coassociativity is obvious. The nullary part is given by $\text{bim}(q^e)_0^\bullet = q^e : R^e \rightarrow R'^e$. Its R'^e -bilinearity; that is,

$$r'q(x)s' \otimes sq(y)\theta'^{-1}(r) = q(\tilde{q}(r')x\tilde{q}(s')) \otimes q(\tilde{q}(s)y\theta'^{-1}\tilde{q}(r))$$

follows by (4.14) and $\tilde{q}\theta' = \theta\tilde{q}$. Right counitality; that is, commutativity of

$$\begin{array}{ccc} \text{bim}(q^e)M \xrightarrow{\text{bim}(q^e)((\rho_M^\bullet)^{-1})} \text{bim}(q^e)(M \bullet J) & & m \xrightarrow{\quad} m \bullet (1 \otimes 1) \\ \text{bim}(q^e)M \downarrow & & \parallel \\ \text{bim}(q^e)M & & (\tilde{q}q(e_i)f_i \otimes 1)m(e_j \tilde{q}q(f_j) \otimes 1) \\ \rho_{\text{bim}(q^e)M}^{\bullet'} \uparrow & & \uparrow \\ \text{bim}(q^e)M \bullet' J' \leftarrow \text{bim}(q^e)M \bullet' \text{bim}(q^e)J & & (f_i \otimes 1)m(e_j \otimes 1) \bullet' (q(f_j) \otimes q(e_i)) \leftarrow (f_i \otimes 1)m(e_j \otimes 1) \bullet' (f_j \otimes e_i) \\ \text{bim}(q^e)M \bullet' \text{bim}(q^e)_0^\bullet & & \end{array}$$

follows by (4.15) and similarly on the other side. \square

By Theorem 4.3 and Proposition 4.8 there is a functor $\text{bim}(-^e) : \text{sfr} \rightarrow \text{duo}$. Our next aim is to describe the corresponding category $\text{bmd}(\text{bim}(-^e))$ as a category of weak bialgebras over k . We begin with identifying in the next two paragraphs the objects of $\text{bmd}(\text{bim}(-^e))$ with weak bialgebras; that is, the bimonoids in $\text{bim}(R^e)$ with weak bialgebras of 'right' subalgebras isomorphic to R .

4.9. From weak bialgebras to bimonoids. Let $(H, \mu, \eta, \Delta, \epsilon)$ be a weak bialgebra over a field k . By [20, Proposition 4.2], $R := \square^R(H)$ is a separable Frobenius algebra with $1_1 \otimes \square^R(1_2) \in R \otimes R$ as separability idempotent, and the restriction of the counit ϵ to R as the corresponding Frobenius functional. In this case, the Nakayama automorphism and its inverse (see (1.20)) are the (co)restrictions of $\square^R \square^L$ and $\square^R \square^L$ to R , respectively. In this paragraph we equip H with the structure of a bimonoid in $\mathbf{bim}(R^e)$.

First we construct on H a monoid structure in $\mathbf{bim}(R^e)$. By [5, Lemma 2.2], this amounts to the construction of an algebra homomorphism $R^e \rightarrow H$: Since $\square^R(H)$ and $\square^L(H)$ are commuting subalgebras in H , and since \square^L restricts to an anti-algebra isomorphism between them (cf. [6, Proposition 1.18]), it follows that the map $\tilde{\eta} : R^e \rightarrow H$ defined as $\tilde{\eta}(s \otimes r) = s \square^L(r)$ is a desired homomorphism of algebras. It induces an R^e -bimodule structure on H . We denote the resulting actions by juxtaposition. By virtue of [5, Lemma 2.2], the multiplication μ factorizes through an R^e -bilinear associative multiplication $\tilde{\mu} : H \circ H \rightarrow H$ with unit $\tilde{\eta}$, so that $(H, \tilde{\mu}, \tilde{\eta})$ has a structure of monoid in $\mathbf{bim}(R^e)$.

In order to equip H with the structure of a comonoid in $\mathbf{bim}(R^e)$, note that $\Delta : H \rightarrow H \otimes H$ factorizes through $H \bullet H$ (via the inclusion $\iota_{H,H}^\bullet : H \bullet H \rightarrow H \otimes H$). That is, for any $h \in H$,

$$(4.16) \quad \begin{aligned} \Delta(h) &= h_1 \otimes h_2 = 1_1 h_1 1_{1'} \otimes 1_2 h_2 1_{2'} = 1_1 h_1 1_{1'} \otimes \square^L \square^R(1_2) h_2 \square^L \square^R(1_{2'}) \\ &= (1_1 \otimes 1) h_1 (1_{1'} \otimes 1) \otimes (1 \otimes \square^R(1_2)) h_2 (1 \otimes \square^R(1_{2'})) = \iota_{H,H}^\bullet(h_1 \bullet h_2), \end{aligned}$$

where (1.32) and (1.34) has been used. As the comultiplication for the bimonoid associated to the weak bialgebra H , consider the corestriction $\tilde{\Delta} : H \rightarrow H \bullet H$ of Δ . As the counit, put $\tilde{\epsilon} = (\square^R \otimes \square^R) \Delta = (\square^R \otimes \square^R) \Delta^{op} : H \rightarrow R \otimes R$. (The two forms are equal, indeed, since for any $h \in H$,

$$\begin{aligned} \square^R(h_1) \otimes \square^R(h_2) &= 1_1 \epsilon(h_1 1_2) \otimes 1_{1'} \epsilon(1_2' h_2) = 1_1 \otimes 1_{1'} \epsilon(1_2' h_1 2) \\ &= 1_1 \epsilon(h_2 1_2) \otimes 1_{1'} \epsilon(1_2' h_1) = \square^R(h_2) \otimes \square^R(h_1). \end{aligned}$$

The comultiplication $\tilde{\Delta}$ is R^e -bilinear by the R -module map properties of Δ , cf. (1.38). Right R^e -linearity of $\tilde{\epsilon}$ follows by

$$\begin{aligned} \tilde{\epsilon}(h(s \otimes r)) &= \tilde{\epsilon}(hs \square^L(r)) = 1_1 \otimes 1_{1'} \epsilon(1_2' h s \square^L(r) 1_2) = r 1_1 \otimes 1_{1'} \epsilon(1_2' h s 1_2) \\ &\stackrel{(1.33)}{=} r 1_1 \otimes 1_{1'} \epsilon(1_2' h 1_2 s) \stackrel{(1.37)}{=} r 1_1 \otimes 1_{1'} \epsilon(1_2' h 1_2 \square^L(s)) \\ &= r 1_1 s \otimes 1_{1'} \epsilon(1_2' h 1_2) \stackrel{(4.7)}{=} \tilde{\epsilon}(h) \cdot (s \otimes r), \end{aligned}$$

for $h \in H$ and $s \otimes r \in R^e$. The third and the sixth equalities follow by the Frobenius property of R : just apply $\text{id} \otimes \square^L$ to the identities

$$1_1 \otimes \square^R(1_2) \square^R(h) = \square^R(h) 1_1 \otimes \square^R(1_2) \quad \text{and} \quad 1_1 \otimes \square^R \square^L(h) \square^R(1_2) = 1_1 \square^R(h) \otimes \square^R(1_2)$$

holding true for all $h \in H$ by (1.21-1.22). Left R^e -linearity of $\tilde{\epsilon}$ is checked symmetrically. Coassociativity of $\tilde{\Delta}$ is obvious. The computation

$$h_1 \cdot (\square^R(h_2) \otimes \square^R(h_3)) = (\square^R(h_3) \otimes 1) h_1 (\square^R(h_2) \otimes 1) = \square^R(h_3) h_1 \square^R(h_2) = h,$$

for any $h \in H$, shows its right counitality and left counitality is checked symmetrically. This proves that $(H, \tilde{\Delta}, \tilde{\epsilon})$ is a comonoid in $\mathbf{bim}(R^e)$.

Our next aim is to show that the compatibility conditions — expressed by diagrams (1.13), (1.14), (1.15) and (1.16) — hold between the above monoid and comonoid structures of H . Commutativity of (1.13) follows by commutativity of

$$\begin{array}{ccccc} h \circ h' & \xrightarrow{\tilde{\Delta} \circ \tilde{\Delta}} & (h_1 \bullet h_2) \circ (h'_1 \bullet h'_2) & \xrightarrow{\gamma} & (h_1 1_1 \circ h'_1) \bullet (h_2 \circ 1_2 h'_2) \\ \tilde{\mu} \downarrow & & & & \downarrow \tilde{\mu} \bullet \tilde{\mu} \\ hh' & \xrightarrow{\tilde{\Delta}} & (hh')_1 \bullet (hh')_2 & \xlongequal{\quad} & h_1 1_1 h'_1 \bullet h_2 1_2 h'_2, \end{array}$$

for any $h, h' \in H$, where the equality in the bottom row follows by the comultiplicativity and the unitality of the multiplication $\mu : H \otimes H \rightarrow H$. Commutativity of (1.14) follows by commutativity of

$$\begin{array}{ccc} h \circ h' & \xrightarrow{\tilde{\epsilon} \circ \tilde{\epsilon}} & (\sqcap^R(h_1) \otimes \overline{\sqcap}^R(h_2)) \circ (\sqcap^R(h'_1) \otimes \overline{\sqcap}^R(h'_2)) \\ \tilde{\mu} \downarrow & & \downarrow \mu_J \\ hh' & \xrightarrow{\tilde{\epsilon}} & \sqcap^R((hh')_1) \otimes \overline{\sqcap}^R((hh')_2) \equiv \epsilon(\sqcap^R(h_1) \overline{\sqcap}^R(h'_2)) \sqcap^R(h'_1) \otimes \overline{\sqcap}^R(h_2), \end{array}$$

for any $h, h' \in H$. In order to verify the equality in the bottom row, observe that for all $h, h' \in H$,

$$\overline{\sqcap}^R(hh') \stackrel{(1.37)}{=} \overline{\sqcap}^R(h \overline{\sqcap}^R(h')) \stackrel{(1.32)}{=} \overline{\sqcap}^R(h \overline{\sqcap}^R \sqcap^L(h')) \stackrel{(1.37)}{=} \overline{\sqcap}^R(h \sqcap^L(h')).$$

Using this identity in the penultimate equality,

$$\begin{aligned} \epsilon(\sqcap^R(h_1) \overline{\sqcap}^R(h'_2)) \sqcap^R(h'_1) \otimes \overline{\sqcap}^R(h_2) &\stackrel{(1.37)}{=} \epsilon(\sqcap^R(h_1) h'_2) \sqcap^R(h'_1) \otimes \overline{\sqcap}^R(h_2) \\ &\stackrel{(1.38)}{=} \sqcap^R(\sqcap^R(h_1) h'_2) \otimes \overline{\sqcap}^R(h_2) \stackrel{(1.37)}{=} \sqcap^R(h_1 h'_2) \otimes \overline{\sqcap}^R(h_2) \\ &= \sqcap^R(h_1 1_1 h'_2) \otimes \overline{\sqcap}^R(h_2 1_2) \stackrel{(1.38)}{=} \sqcap^R(h_1 h'_1) \otimes \overline{\sqcap}^R(h_2 \sqcap^L(h'_2)) \\ &= \sqcap^R(h_1 h'_1) \otimes \overline{\sqcap}^R(h_2 h'_2) = \sqcap^R((hh')_1) \otimes \overline{\sqcap}^R((hh')_2). \end{aligned}$$

Commutativity of (1.15) and (1.16) follows by commutativity of

$$\begin{array}{ccc} s \otimes r & \xrightarrow{\tilde{\eta}} & s \overline{\sqcap}^L(r) \\ \Delta_I \downarrow & & \downarrow \tilde{\Delta} \\ (1 \otimes r) \bullet (s \otimes 1) & \xrightarrow{\tilde{\eta} \bullet \tilde{\eta}} & \overline{\sqcap}^L(r) \bullet s \stackrel{(1.36)}{=} \overline{\sqcap}^L(r) \cdot (1_1 \otimes 1) \bullet (\sqcap^R(1_2) \otimes 1) \cdot s \equiv \overline{\sqcap}^L(r) 1_1 \bullet s 1_2 \end{array}$$

and

$$\begin{array}{ccc} s \otimes r & \xrightarrow{\tilde{\eta}} & s \overline{\sqcap}^L(r) \\ \tau \downarrow & & \downarrow \tilde{\epsilon} \\ r \sqcap^R(1_2) \otimes s 1_1 & \equiv & r 1_1 \otimes s \overline{\sqcap}^R(1_2) \stackrel{(1.35)}{=} \sqcap^R(\overline{\sqcap}^L(r) 1_1) \otimes \overline{\sqcap}^R(s 1_2), \end{array}$$

respectively, for any $s, r \in R$. Therefore, we conclude that $(H, \tilde{\mu}, \tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon})$ is a bimonoid in $\mathbf{bim}(R^e)$.

4.10. From bimonoids to weak bialgebras. Take now a bimonoid $(H, \tilde{\mu}, \tilde{\Delta}, \tilde{\eta}, \tilde{\epsilon})$ in $\mathbf{bim}(R^e)$, for some separable Frobenius (co)algebra R over the field k . In this paragraph we equip H with the structure of a weak bialgebra over k , whose ‘right’ subalgebra is isomorphic to R .

First we construct an associative and unital k -algebra structure on A , via the multiplication and the unit defined by

$$\mu : H \otimes H \xrightarrow{\pi_{H,H}^\circ} H \circ H \xrightarrow{\tilde{\mu}} H \quad \text{and} \quad \eta : k \xrightarrow{\eta_{R^e}} R^e \xrightarrow{\tilde{\eta}} H,$$

where η_{R^e} denotes the unit of the k -algebra R^e .

Next, we can make H to be a k -coalgebra via the comultiplication and the counit

$$\Delta : H \xrightarrow{\tilde{\Delta}} H \bullet H \xrightarrow{\iota_{H,H}^\bullet} H \otimes H \quad \text{and} \quad \epsilon : H \xrightarrow{\tilde{\epsilon}} R^e \xrightarrow{\psi \otimes \psi} k.$$

Indeed, Δ is evidently coassociative and it is counital by commutativity of

$$\begin{array}{ccccc}
 H & \xrightarrow{\tilde{\Delta}} & H \bullet H & \xrightarrow{\iota_{H,H}^\bullet} & H \otimes H \\
 & \searrow & \downarrow \tilde{\epsilon} \bullet H & \downarrow \iota_{R^e,H}^\bullet & \downarrow \tilde{\epsilon} \otimes H \\
 & & R^e \bullet H & \xrightarrow{\iota_{R^e,H}^\bullet} & R^e \otimes H \\
 & & & \searrow \lambda_H^\bullet & \downarrow (\psi \otimes \psi) \otimes H \\
 & & & & H
 \end{array}$$

— where the triangle at the bottom right commutes by (4.10) — and similarly on the other side.

Our next aim is to show that the above algebra and coalgebra structures of H combine into a weak bialgebra. In doing so, we use both Sweedler notations $\tilde{\Delta}(h) = h_{\bar{1}} \bullet h_{\bar{2}}$ and $\Delta(h) = h_1 \otimes h_2$, for any $h \in H$.

We begin with checking the multiplicativity of the comultiplication Δ ; that is, axiom (1.27). For any $h \in H$, $\Delta(h) = \iota_{H,H}^\bullet \tilde{\Delta}(h) = (e_j \otimes 1) h_{\bar{1}} (e_i \otimes 1) \otimes (1 \otimes f_j) h_{\bar{2}} (1 \otimes f_i)$, hence

$$\begin{aligned}
 \Delta(h) \Delta(h') &= (e_j \otimes 1) h_{\bar{1}} (e_i \otimes 1) h'_{\bar{1}} (e_k \otimes 1) \otimes (1 \otimes f_j) h_{\bar{2}} (1 \otimes f_i) h'_{\bar{2}} (1 \otimes f_k) \\
 &= \iota_{H,H}^\bullet (\tilde{\mu} \bullet \tilde{\mu}) \gamma(\tilde{\Delta} \circ \tilde{\Delta}) \pi_{H,H}^\circ(h \otimes h') \stackrel{(1.13)}{=} \iota_{H,H}^\bullet \tilde{\Delta} \tilde{\mu} \pi_{H,H}^\circ(h \otimes h') = \Delta \mu(h \otimes h') = \Delta(hh'),
 \end{aligned}$$

for all $h, h' \in H$.

Next we check axiom (1.29), expressing weak comultiplicativity of the unit. From (1.15) on the bimonoid H it follows that

$$(4.17) \quad \Delta \tilde{\eta}(r \otimes s) = \iota_{H,H}^\bullet \tilde{\Delta} \tilde{\eta}(r \otimes s) = \iota_{H,H}^\bullet (\tilde{\eta} \bullet \tilde{\eta}) \Delta_I(r \otimes s) = \tilde{\eta}(e_i \otimes s) \otimes \tilde{\eta}(r \otimes f_i).$$

With this identity at hand, the weak comultiplicativity of the unit is checked by

$$\begin{aligned}
 (H \otimes \Delta) \Delta(1) &= \tilde{\eta}(e_i \otimes 1) \otimes \Delta \tilde{\eta}(1 \otimes f_i) = \tilde{\eta}(e_i \otimes 1) \otimes \tilde{\eta}(e_j \otimes f_i) \otimes \tilde{\eta}(1 \otimes f_j) \\
 &= \tilde{\eta}(e_i \otimes 1) \otimes \tilde{\eta}(1 \otimes f_i) \tilde{\eta}(e_j \otimes 1) \otimes \tilde{\eta}(1 \otimes f_j) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)).
 \end{aligned}$$

Since $\tilde{\eta}(1 \otimes r) \tilde{\eta}(s \otimes 1) = \tilde{\eta}(s \otimes r) = \tilde{\eta}(s \otimes 1) \tilde{\eta}(1 \otimes r)$, for all $s, r \in R$, also $(1 \otimes \Delta(1))(\Delta(1) \otimes 1) = (H \otimes \Delta) \Delta(1)$.

Finally, we check that axiom (1.30) — expressing weak multiplicativity of the counit — holds. This starts with proving the equality

$$(4.18) \quad \tilde{\epsilon} = (\sqcap \otimes \sqbar) \iota_{H,H}^\bullet \tilde{\Delta} = (\sqcap \otimes \sqbar) \Delta$$

in terms of the maps

$$\sqcap := (H \xrightarrow{\tilde{\epsilon}} R \otimes R^{op} \xrightarrow{R \otimes \psi} R) \quad \text{and} \quad \sqbar := (H \xrightarrow{\tilde{\epsilon}} R \otimes R^{op} \xrightarrow{\psi \otimes R^{op}} R^{op}).$$

Equality (4.18) is proven by commutativity of the following diagram, noting that ρ_H^\bullet is an isomorphism.

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\tilde{\epsilon} \otimes \tilde{\epsilon}} & R^e \otimes R^e & \xrightarrow{R \otimes \psi \otimes \psi \otimes R^{op}} & R^e \\
 \uparrow \iota_{H,H}^\bullet & & \uparrow \iota_{R^e,R^e}^\bullet & & \parallel \\
 H \bullet H & \xrightarrow{H \bullet \tilde{\epsilon}} & H \bullet R^e & \xrightarrow{\tilde{\epsilon} \bullet R^e} & R^e \bullet R^e \xrightarrow{\rho_{R^e}^\bullet} R^e \\
 \uparrow \tilde{\Delta} & \nearrow \rho_H^\bullet & & & \\
 H & & & &
 \end{array}$$

The bottom-right region and the top-left region commute by the R^e -bimodule map property of $\tilde{\epsilon}$. The bottom-left region commutes by counitality $\tilde{\Delta}$. Commutativity of the top-right region follows similarly to (4.10). From (1.14) on the bimonoid H and from (4.18), it follows that

$$(4.19) \quad \sqcap((hh')_1) \otimes \sqbar((hh')_2) = \psi(\sqcap(h_1) \sqbar(h'_2)) \sqcap(h'_1) \otimes \sqbar(h_2),$$

for all $h, h' \in H$. Using the R^e -bilinearity of $\tilde{\epsilon}$ together with (4.18) in the second equality,

$$\begin{aligned} \epsilon(h1_1)\epsilon(1_2h') &= (\psi \otimes \psi)\tilde{\epsilon}(h(e_i \otimes 1))(\psi \otimes \psi)\tilde{\epsilon}((1 \otimes f_i)h') \\ &= \psi(\square(h_1)e_i)\psi\overline{\square}(h_2)\psi \square(h'_1)\psi(\overline{\square}(h'_2)\theta^{-1}(f_i)) \\ &\stackrel{(1.18)(1.19)}{=} \psi(\square(h_1)\overline{\square}(h'_2))\psi \square(h'_1)\psi\overline{\square}(h_2) \\ &\stackrel{(4.19)}{=} \psi \square((hh')_1)\psi\overline{\square}((hh')_2) \stackrel{(4.18)}{=} (\psi \otimes \psi)\tilde{\epsilon}(hh') = \epsilon(hh'), \end{aligned}$$

where in the first equality we used (4.17) and that the multiplication μ of the k -algebra H is R^e -balanced and R^e -bilinear. A symmetrical computation verifies $\epsilon(h1_2)\epsilon(1_1h') = \epsilon(hh')$, for all $h, h' \in H$.

We have so far constructed a weak bialgebra structure on H . It remains to check that $\square^R(H)$ is isomorphic to the given separable Frobenius algebra R . With this purpose, consider the map

$$(4.20) \quad \sigma : R \rightarrow H, \quad r \mapsto \tilde{\eta}(r \otimes 1).$$

Since for any $s, r \in R$

$$(4.21) \quad \epsilon\tilde{\eta}(r \otimes s) = (\psi \otimes \psi)\tilde{\epsilon}\tilde{\eta}(r \otimes s) = (\psi \otimes \psi)\tau(r \otimes s) = \psi(sf_i)\psi(re_i) = \psi(sr),$$

and by (4.17),

$$\square^R\sigma(r) = \tilde{\eta}(e_i \otimes 1)\epsilon\tilde{\eta}(r \otimes f_i) \stackrel{(4.21)}{=} \tilde{\eta}(e_i \otimes 1)\psi(f_i r) = \tilde{\eta}(r \otimes 1) = \sigma(r).$$

This proves that σ corestricts to a map $R \rightarrow \square^R(H)$, to be denoted also by σ . This restricted map $\sigma : R \rightarrow \square^R(H)$ is our candidate to establish the desired isomorphism of separable Frobenius algebras. Since $\tilde{\eta}$ is a k -algebra homomorphism, so is σ . Comultiplicativity of σ is proven using the identity

$$(4.22) \quad \square^R\tilde{\eta}(1 \otimes r) = \tilde{\eta}(e_i \otimes 1)\epsilon\tilde{\eta}(1 \otimes f_i r) \stackrel{(4.21)}{=} \tilde{\eta}(e_i \otimes 1)\psi(f_i r) = \tilde{\eta}(r \otimes 1).$$

With this identity at hand,

$$\sigma(r)\tilde{\eta}(e_i \otimes 1) \otimes \square^R\tilde{\eta}(1 \otimes f_i) \stackrel{(4.22)}{=} \tilde{\eta}(re_i \otimes 1) \otimes \tilde{\eta}(f_i \otimes 1) = \sigma(re_i) \otimes \sigma(f_i).$$

Finally, σ is also counital by applying (4.21) for $s = 1$. Since any map between Frobenius algebras, which is both an algebra and a coalgebra homomorphism, is an isomorphism (cf. [19, Proposition A.3]), this proves that σ is an isomorphism of separable Frobenius algebras.

Theorem 4.11. *Let R be a separable Frobenius (co)algebra over a field k . A bimonoid in the duoidal category $\mathbf{bim}(R^e)$ in Theorem 4.3 is, equivalently, a weak bialgebra over k whose right subalgebra is isomorphic to R (as a separable Frobenius (co)algebra).*

Proof. In light of Paragraphs 4.9 and 4.10, we only have to prove the bijectivity of the correspondence described in them. Starting with a weak bialgebra $(H, \mu, \eta, \Delta, \epsilon)$, and applying to it the above constructions, the resulting weak bialgebra has the same structure as H , as the following computations show. The resulting multiplication is the unique map which yields $\mu\pi_{H,H}^\circ$ if composed with $\pi_{H,H}^\circ$. Hence it is equal to μ . The resulting unit map multiplies an element of k by $1\overline{\square}^L(1) = 1$ hence it is equal to η . The resulting comultiplication is equal to Δ by (4.16). The resulting counit sends $h \in H$ to

$$(\epsilon|_R \square^R \otimes \epsilon|_R \overline{\square}^R)\Delta(h) = (\epsilon \otimes \epsilon)\Delta(h) = \epsilon(h).$$

Conversely, consider a bimonoid $(H, \tilde{\mu}, \tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon})$ in $\mathbf{bim}(R^e)$ and the bimonoid obtained by applying to it the constructions in Paragraphs 4.9 and 4.10. By construction, they have identical multiplications and comultiplications. Concerning the unit and the counit, note that in the weak bialgebra in Paragraph 4.10,

$$\begin{aligned} \square^R(h) &= \tilde{\eta}(e_i \otimes 1)(\psi \otimes \psi)\tilde{\epsilon}(h\tilde{\eta}(1 \otimes f_i)) \\ &= \tilde{\eta}(e_i \otimes 1)\psi(f_i \square(h_1))\psi\overline{\square}(h_2) = \tilde{\eta}(\square(h_1) \otimes 1)\epsilon(h_2) = \tilde{\eta}(\square(h) \otimes 1), \end{aligned}$$

for all $h \in H$, where in the second equality we used the right R^e -linearity of $\tilde{\epsilon}$, (4.18) and (4.7); and in the penultimate equality we used (1.18) and $\psi\overline{\square} = \epsilon$. By similar computations also the idempotent maps

$\overline{\Pi}^L$ and $\overline{\Pi}^R$ — in the weak bialgebra associated in Paragraph 4.10 to the bimonoid $(H, \tilde{\mu}, \tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon})$ — can be expressed as

$$\overline{\Pi}^L = \tilde{\eta}(1 \otimes \Pi(-)) \quad \text{and} \quad \overline{\Pi}^R = \tilde{\eta}(\Pi(-) \otimes 1).$$

So the counits differ by the isomorphism $\sigma \otimes \sigma$ by (4.18). Finally, in the bimonoid obtained by applying both constructions, the unit map takes $s \otimes r \in R \otimes R^{op}$ to $\sigma(s)\overline{\Pi}^L\sigma(r) = \tilde{\eta}(s \otimes 1)\tilde{\eta}(1 \otimes r) = \tilde{\eta}(s \otimes r)$. \square

By Theorem 4.11, an object of $\mathbf{bmd}(\mathbf{bim}(-^e))$ is given by a weak bialgebra. We make no notational distinction between a weak bialgebra H and the corresponding bimonoid in the bi(co)module category $\mathbf{bim}(R^e)$, where R is the ‘right’ subalgebra $\Pi^R(H)$.

By [22, 20], a weak bialgebra of ‘right’ subalgebra R can be regarded as a right R -bialgebroid (or ‘ \times_R -bialgebra’ in [25]) supplemented by a separable Frobenius structure on R . However, since for arbitrary algebras R we cannot equip the category of R^e -bimodules with a duoidal structure (see Remark 4.4), we cannot extend Theorem 4.11 to interpret arbitrary bialgebroids as bimonoids in an appropriate duoidal category.

Theorem 4.12. *Let H and H' be weak bialgebras with respective right subalgebras R and R' . A morphism in $\mathbf{bmd}(\mathbf{bim}(-^e))$ from (R, H) to (R', H') is, equivalently, a coalgebra map $Q : H \rightarrow H'$, rendering commutative the diagrams*

$$\begin{array}{ccccccc} H & \xrightarrow{Q} & H' & & H & \xrightarrow{Q} & H' \\ \downarrow \Pi^R & & \downarrow \Pi'^R & & \downarrow \Pi^R \Pi^L & & \downarrow \Pi'^R \Pi'^L \\ H & \xrightarrow{Q} & H' & & H & \xrightarrow{Q} & H' \\ & & & & & & \\ H \otimes H & \xrightarrow{E} & H \otimes H & \xrightarrow{Q \otimes Q} & H' \otimes H' & & \\ \downarrow \mu & & \downarrow \mu' & & \downarrow \mu' & & \\ H & \xrightarrow{Q} & H' & & H' & & \end{array}$$

where $E(h \otimes h') := h1_1 \otimes \Pi^R(1_2)h'$.

Proof. Let us take first a morphism in $\mathbf{bmd}(\mathbf{bim}(-^e))$, and see that it obeys the properties in the claim. A morphism in $\mathbf{bmd}(\mathbf{bim}(-^e))$ is given by a morphism $q : R \rightarrow R'$ in \mathbf{sfr} and a morphism $Q : \mathbf{bim}(q^e)H \rightarrow H'$ in $\mathbf{bim}(R'^e)$, rendering commutative the four diagrams in part (b) of Lemma 2.1.

Let us see first that Q is a coalgebra map. In order to prove that it is comultiplicative, we need to see that the top row of

$$\begin{array}{ccccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{\pi_{H,H}^\bullet} & H \bullet H & \xrightarrow{\mathbf{bim}(q^e)_2^\bullet} & H \bullet' H & \xrightarrow{\iota_{H,H}^\bullet} & H \otimes H \\ \downarrow Q & & & & & & \downarrow Q \bullet' Q & & \downarrow Q \otimes Q \\ H' & \xrightarrow{\Delta'} & H' \otimes H' & \xrightarrow{\pi_{H',H'}^\bullet} & H' \bullet' H' & \xrightarrow{\iota_{H',H'}^\bullet} & H' \otimes H' & & \\ & & & & & & & & \end{array}$$

(2.1) (4.16)

is equal to the comultiplication Δ of H . Computing its value on $h \in H$, we get

$$(\tilde{q}(e'_k)e_i \otimes 1)h_1(e_j\tilde{q}(e'_l) \otimes 1) \otimes (1 \otimes f_i\tilde{q}(f'_k))h_2(1 \otimes \tilde{q}(f'_l)f_j).$$

It is equal to

$$(e_i \otimes 1)h_1(e_j \otimes 1) \otimes (1 \otimes f_i)h_2(1 \otimes f_j) = 1_1h_11_{1'} \otimes 1_2h_21_{2'} = \Delta(h)$$

by

$$(4.23) \quad \begin{aligned} \tilde{q}(e'_k)e_i \otimes f_i\tilde{q}(f'_k) &= \tilde{q}(e'_k)\tilde{q}(f'_k)e_i \otimes f_i = \tilde{q}(e'_kf'_k)e_i \otimes f_i = \tilde{q}(1')e_i \otimes f_i = e_i \otimes f_i \quad \text{and} \\ e_j\tilde{q}(e'_l) \otimes \tilde{q}(f'_l)f_j &= e_j \otimes \tilde{q}(f'_l)\theta\tilde{q}(e'_l)f_j = e_j \otimes \tilde{q}(f'_l)\tilde{q}\theta'(e'_l)f_j = e_j \otimes \tilde{q}(e'_l)\tilde{q}(f'_l)f_j = e_j \otimes f_j. \end{aligned}$$

This proves comultiplicativity of Q . In order to see that Q is counital as well, observe that condition (2.2) takes now the form

$$\begin{array}{ccccc} H & \xrightarrow{(\sqcap^R \otimes \overline{\sqcap}^R) \Delta} & R \otimes R^{op} & \xrightarrow{q \otimes q^{op}} & R' \otimes R'^{op} \\ Q \downarrow & & & & \parallel \\ H' & \xrightarrow{(\sqcap'^R \otimes \overline{\sqcap}'^R) \Delta'} & & & R' \otimes R'^{op}. \end{array}$$

Composing both paths around it with $R' \otimes \epsilon'_{|R'}$ and with $\epsilon'_{|R'} \otimes R'$, respectively, we obtain

$$(4.24) \quad \sqcap'^R Q(h) = q \sqcap^R(h) \quad \text{and} \quad \overline{\sqcap}'^R Q(h) = q \overline{\sqcap}^R(h);$$

and composing either one of these equalities with $\epsilon'_{|R'}$ we have the counitality of Q proven.

Let us check now that Q satisfies the required weak multiplicativity condition; that is, it renders commutative the last diagram in the claim. Since (q, Q) is a morphism in $\text{bmd}(\text{bim}(-^e))$ by assumption, it renders commutative diagram (2.3) for any R^e -bimodules A and B . Evaluating both paths around it on an arbitrary element $(a \bullet h) \circ (b \bullet h')$, and using the commutativity of \tilde{q} with the Nakayama automorphisms, the R'^e -bilinearity of Q , (1.24) and that \tilde{q} is an algebra map together with (1.21), yields the equivalent form

$$(4.25) \quad \begin{aligned} & ((e_i \otimes 1)a(e_j e_p \otimes f_l) \circ' (f_p \otimes e_l)b(e_{j'} \otimes 1)) \bullet' Q((1 \otimes f_i)h(1 \otimes f_j)h'(1 \otimes f_{j'})) = \\ & ((e_i \otimes 1)a(e_j \otimes f_l) \circ' (e_{i'} \otimes e_l)b(e_{j'} \otimes 1)) \bullet' Q((1 \otimes f_i)h(e_k \otimes f_j))Q((f_k \otimes f_{i'})h'(1 \otimes f_{j'})) \end{aligned}$$

of condition (2.3) on (q, Q) . Taking $A = B = R^e \otimes R^e$ with the R^e -actions

$$(r \otimes s)((x \otimes y) \otimes (v \otimes w))(r' \otimes s') := (rx \otimes ys) \otimes (vr' \otimes s'w),$$

putting $a = b = 1 \otimes 1 \otimes 1 \otimes 1$, and applying $(\iota^{o'} \otimes H')\iota^{\bullet'}$ to the resulting equality, by the R'^e -bilinearity of Q and (4.23) we obtain

$$\begin{aligned} & e_i \otimes 1 \otimes e_j e_p \otimes f_l \otimes f_p \otimes e_l \otimes e_{j'} \otimes 1 \otimes Q((1 \otimes f_i)h(1 \otimes f_j)h'(1 \otimes f_{j'})) \\ &= e_i \otimes 1 \otimes e_j \otimes f_l \otimes e_{i'} \otimes e_l \otimes e_{j'} \otimes 1 \otimes Q((1 \otimes f_i)h(e_k \otimes f_j))Q((f_k \otimes f_{i'})h'(1 \otimes f_{j'})). \end{aligned}$$

Applying ψ to the first, third, fifth and seventh tensorands in the last equality, we get

$$1 \otimes f_l \otimes e_l \otimes 1 \otimes Q(hh') = 1 \otimes f_l \otimes e_l \otimes 1 \otimes Q(h(e_k \otimes 1))Q((f_k \otimes 1)h').$$

This is equivalent to

$$Q(hh') = Q(h1_1)Q(\sqcap^R(1_2)h'),$$

that is, commutativity of the last diagram in the claim.

Next we check that q can be uniquely reconstructed from Q — namely, it is the (co)restriction to $R \rightarrow R'$ of $Q : H \rightarrow H'$. Evaluating the equal paths around

$$\begin{array}{ccccccc} R^e & \xrightarrow{q^e} & R'^e & \xrightarrow{\Delta_{I'}} & R'^e \bullet' R'^e & \xrightarrow{\iota_{R'^e, R'^e}^{\bullet'}} & R'^e \otimes R'^e \\ \Delta_I \downarrow & & & & \downarrow R'^e \bullet' \tilde{\eta}' & & \downarrow R'^e \otimes \tilde{\eta}' \\ R^e \bullet' R^e & & & & & & \\ \text{bim}(q^e)_2 \downarrow & & & & & & \\ R^e \bullet' R^e & \xrightarrow{R^e \bullet' \tilde{\eta}} & R^e \bullet' H & \xrightarrow{q^e \bullet' Q} & R'^e \bullet' H' & & \\ \iota_{R^e, R^e}^{\bullet'} \downarrow & & \downarrow \iota_{R^e, H}^{\bullet'} & & \searrow \iota_{R'^e, H'}^{\bullet'} & & \downarrow \\ R^e \otimes R^e & \xrightarrow{R^e \otimes \tilde{\eta}} & R^e \otimes H & \xrightarrow{q^e \otimes Q} & R'^e \otimes H' & & \end{array} \quad (2.4)$$

at $1_1 \otimes \sqcap^R(1_2)r \in R^e$, and composing the result with $\epsilon'_{|R'} \otimes \epsilon'_{|R'} \otimes H'$, we conclude $q(r) = Q(r)$.

Comparing this identity $q(r) = Q(r)$ with (4.24), the compatibility of Q with \sqcap^R and \sqcap^R — that is, commutativity of the first two diagrams in the claim — follows. Commutativity of the third diagram — that is, compatibility of Q with $\sqcap^R \sqcap^L$ — is equivalent to the assumed commutativity of q with the Nakayama automorphisms.

Conversely, assume that $Q : H \rightarrow H'$ is a coalgebra map rendering commutative the four diagrams in the statement. We construct its mate $q : R \rightarrow R'$ together with whom they constitute a morphism in $\mathbf{bmd}(\mathbf{bim}(-^e))$.

By commutativity of any of the first two diagrams, Q restricts to a map $q : R \rightarrow R'$. Let us see that the restriction $q : R \rightarrow R'$ of Q is a morphism in \mathbf{sfr} . First of all, that it is a coalgebra map. Take $y \in R$. Since Q respects the counits, $\epsilon'_{R'} q(y) = \epsilon' Q(y) = \epsilon_{|R}(y)$. Moreover, q is comultiplicative by

$$\begin{aligned} Q(y)1'_1 \otimes \sqcap'^R(1'_2) &\stackrel{(1.38)}{=} Q(y)_1 \otimes \sqcap'^R(Q(y)_2) = Q(y_1) \otimes \sqcap'^R Q(y_2) \\ &= Q(y_1) \otimes Q \sqcap^R(y_2) \stackrel{(1.38)}{=} Q(y1_1) \otimes Q \sqcap^R(1_2). \end{aligned}$$

By commutativity of the third diagram in the claim, q commutes with the Nakayama automorphisms. Hence it is a morphism in \mathbf{sfr} , as needed.

In order for Q to be a morphism in $\mathbf{bim}(R'^e)$, it has to be an R'^e -bimodule map. We check that it is a right R' -module map; its compatibility with the other three R' -actions is similarly proven.

$$\begin{aligned} Q(h\tilde{q}(r')) &= Q(h1_1)\epsilon'(q \sqcap^R(1_2)r') \stackrel{(1.38)}{=} Q(h_1)\epsilon'(q \sqcap^R(h_2)r') \\ &= Q(h_1)\epsilon'(\sqcap'^R Q(h_2)r') \stackrel{(1.37)}{=} Q(h_1)\epsilon'(Q(h_2)r') = Q(h)_1\epsilon'(Q(h)_2r') = Q(h)r', \end{aligned}$$

illustrating that Q is a morphism in $\mathbf{bim}(R'^e)$.

It remains to show that the morphisms $q : R \rightarrow R'$ in \mathbf{sfr} and $Q : H \rightarrow H'$ in $\mathbf{bim}(R'^e)$ obey the conditions in part (b) of Lemma 2.1. The following commutative diagrams show that (2.1) and (2.2) hold.

$$\begin{array}{ccc} H & \xrightarrow{\tilde{\Delta}} & H \bullet H \xrightarrow{\mathbf{bim}(q^e)_2} H \bullet' H \\ & \searrow \Delta & \downarrow \pi_{H,H} \bullet' \\ & & H \otimes H \\ & \searrow \Delta' & \downarrow Q \otimes Q \\ H' & \xrightarrow{\tilde{\Delta}'} & H' \otimes H' \xrightarrow{\pi_{H',H'} \bullet'} H' \bullet' H' \\ & \searrow \Delta' & \downarrow Q \bullet' Q \end{array} \quad \begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \xrightarrow{\sqcap^R \otimes \sqcap^R} R \otimes R^{op} \\ & \searrow Q & \downarrow Q \otimes Q \\ & & H' \otimes H' \xrightarrow{\sqcap'^R \otimes \sqcap'^R} R' \otimes R'^{op} \\ & \searrow Q & \downarrow q \otimes q^{op} \end{array}$$

Commutativity of diagram (2.3) was seen to be equivalent to (4.25). It holds by the following computation, for all $h, h' \in H$, $a \in A$ and $b \in B$ for any R^e -bimodules A and B .

$$\begin{aligned} &((e_i \otimes 1)a(e_j \otimes f_i) \circ' (e_{i'} \otimes e_l)b(e_{j'} \otimes 1)) \bullet' Q((1 \otimes f_i)h(e_k \otimes f_j))Q((f_k \otimes f_{i'})h'(1 \otimes f_{j'})) \\ &= ((e_i \otimes 1)a(e_j \otimes f_i) \circ' (e_{i'} \otimes e_l)b(e_{j'} \otimes 1)) \bullet' Q((1 \otimes f_i)h(1 \otimes f_j)(1 \otimes f_{i'})h'(1 \otimes f_{j'})) \\ &= ((e_i \otimes 1)a(e_j e_{i'} \otimes f_i) \circ' (f_{i'} \otimes e_l)b(e_{j'} \otimes 1)) \bullet' Q((1 \otimes f_i)h(1 \otimes f_j)h'(1 \otimes f_{j'})) \end{aligned}$$

In the first equality we used the weak multiplicativity of Q , holding true by assumption. In the second equality we used (1.24) and (1.22).

Commutativity of diagram (2.4) is checked by the computation

$$\begin{aligned}
(q(1_1) \otimes q(y)) \bullet' Q\tilde{\eta}(x \otimes \square^R(1_2)) &= (1 \otimes q(y)) \bullet' Q\tilde{\eta}(x \otimes \square^R(1_2))(1 \otimes q(1_1)) \\
&= (1 \otimes q(y)) \bullet' Q(\tilde{\eta}(x \otimes \square^R(1_2))(1 \otimes \tilde{q}q(1_1))) \\
&= (1 \otimes q(y)) \bullet' Q\tilde{\eta}(x \otimes \tilde{q}q(1_1) \square^R(1_2)) \\
&= (1 \otimes q(y)) \bullet' Q\tilde{\eta}(x \otimes \tilde{q}(1'_1) \tilde{q} \square'^R(1'_2)) \\
&= (1 \otimes q(y)) \bullet' Q\tilde{\eta}(x \otimes 1) \\
&= (1 \otimes q(y)) \bullet' \tilde{\eta}'(q(x) \otimes 1),
\end{aligned}$$

for any $x, y \in R$. In the first equality we used the definition of \bullet' (cf. (4.4)). In the second and third equalities we used the right R'^e -linearity of Q and the right R^e -linearity of $\tilde{\eta}$, respectively. In the fourth equality we used (4.13); in the penultimate equality we used that \tilde{q} is an algebra map together with (1.36); and in the last equality we used that Q restricts to q on R . \square

We conclude by Theorem 4.11 and Theorem 4.12 that the category $\mathbf{bmd}(\mathbf{bim}(-^e))$ has weak bialgebras as its objects and morphisms as in Theorem 4.12. Thus we can regard it as the category of weak bialgebras and introduce the notation \mathbf{wba} for it.

Applying results from [24], we know from Lemma 2.1 that the morphisms in \mathbf{wba} are closed under the composition. But it is also easy to see this directly. Indeed, if both morphisms $Q : H \rightarrow H'$ and $Q' : H' \rightarrow H''$ render commutative the first three diagrams in Theorem 4.12, then so does their composite evidently. If Q and Q' make commutative the last diagram Theorem 4.12, then so does their composite since Q is a morphism of R' -bimodules and (4.23) holds: for any $h, h' \in H$,

$$\begin{aligned}
Q'Q(hh') &= Q'[Q(h1_1)Q(\square^R(1_2)h')] = Q'[Q(h1_1)1'_1]Q'[\square'^R(1'_2)Q(\square^R(1_2)h')] \\
&= Q'Q(h1_1\tilde{q}(1'_1))Q'Q(\tilde{q}\square'^R(1'_2)\square^R(1_2)h') \stackrel{(4.23)}{=} Q'Q(h1_1)Q'Q(\square^R(1_2)h').
\end{aligned}$$

While the notion of weak bialgebra is self-dual, the morphisms in Theorem 4.12 are not. (They are coalgebra homomorphisms but not algebra homomorphisms.) The dual counterpart of \mathbf{wba} ; that is, a category of weak bialgebras with the dual notion of morphisms, would be obtained from a construction based on a symmetric form of Definition 2.3 (see the discussion in Remark 2.4).

The morphisms in Theorem 4.12 look different from all other kinds of morphisms between weak bialgebras discussed previously in [23, Section 1.4]. However, if we restrict to morphisms $Q : H \rightarrow H'$ whose (co)restriction $q : \square^R(H) \rightarrow \square'^R(H')$ is the identity map, they are in particular unit preserving $\square^R(H) = \square'^R(H')$ -bimodule maps; hence also homomorphisms of algebras (see also Remark 2.2). That is to say, they are ‘*strict morphisms*’ of weak bialgebras in the sense of [23, Section 1.4]. For usual (non-weak) bialgebras H and H' over the field k , any morphism $H \rightarrow H'$ in Theorem 4.12 (co)restricts to the identity map $\square^R(H) \cong k \rightarrow \square'^R(H') \cong k$. Hence \mathbf{wba} contains the usual category of k -bialgebras — in which morphisms are algebra and coalgebra homomorphisms — as a full subcategory.

5. THE “FREE VECTOR SPACE” FUNCTOR.

Let k be a field. For any small category A with finite object set X , let kA denote the free k -vector space spanned by the set of morphisms in A . Consider the unique k -coalgebra structure (kA, Δ, ϵ) for which the elements of A are group-like, that is, $\Delta(a) = a \otimes a$, $\epsilon(a) = 1$ for all $a \in A$. Let $t : X \leftarrow A \rightarrow X : s$ be the target and source maps, respectively, in the category A . The vector space kA is an algebra with the multiplication determined by the rule $ab = \delta_{s(a), t(b)} a \cdot b$, where δ denotes Kronecker’s ‘delta operator’ and \cdot is the composition in A . The unit of kA is given by $1 = \sum_{x \in X} x$ (where x denotes also the identity morphism at x). With these algebra and coalgebra structures kA turns out to be a weak bialgebra over k (see for example [5, Section 3.2.2, page 187], or [18, Section 2.5] for the case when A is a groupoid hence kA is a weak Hopf algebra). For any $a \in A$, we get $\square^R(a) = s(a) = \overline{\square}^L(a)$ and $\overline{\square}^R(a) = t(a) = \square^L(a)$.

This assignment gives the object map of a functor $\mathbf{k} : \mathbf{cat} \rightarrow \mathbf{wba}$ — from the category \mathbf{cat} of small categories with finitely many objects to \mathbf{wba} — as Proposition 5.1 shows.

Proposition 5.1. *Let A and A' be small categories with finite object sets X and X' , respectively. For any functor $f : A \rightarrow A'$, the linear extension $kf : kA \rightarrow kA'$ is a morphism in \mathbf{wba} .*

Proof. First, note that kf is a morphism of k -coalgebras because it sends group-like elements to group-like elements; and group-like elements provide a basis in kA . We need to show that the four diagrams in Theorem 4.12 commute for $Q = kf$. As for the first two concerns, for any basis element $a \in A$

$$\begin{aligned} (kf) \sqcap^R(a) &= (kf)s(a) = fs(a) = s'f(a) = \sqcap'^R f(a) = \sqcap'^R(kf)(a) \\ (kf) \overline{\sqcap}^R(a) &= (kf)t(a) = ft(a) = t'f(a) = \overline{\sqcap}'^R f(a) = \overline{\sqcap}'^R(kf)(a). \end{aligned}$$

The commutativity of the third diagram in Theorem 4.12 becomes redundant by $\sqcap^L = \overline{\sqcap}^R$. In order to check that the fourth diagram commutes, let us first note that any element in the range of the map $E = (-)1_1 \otimes \sqcap^R(1_2)(-) : kA \otimes kA \rightarrow kA \otimes kA$ is of the form

$$\sum_{x \in X} \left(\sum_{a \in A} \lambda_a x \right) \otimes x \left(\sum_{a' \in A} \lambda_{a'} a' \right) = \sum_{x \in X} \left(\sum_{a : s(a)=x} \lambda_a a \right) \otimes \left(\sum_{a' : t(a')=x} \lambda_{a'} a' \right) = \sum_{a, a' : s(a)=t(a')} \lambda_a \lambda_{a'} a \otimes a',$$

and if $s(a) = t(a')$ then

$$(kf)\mu(a \otimes a') = f(a \cdot a') = f(a) \cdot f(a') = \mu'(kf \otimes kf)(a \otimes a').$$

□

6. ON GROUP-LIKE ELEMENTS IN A WEAK BIALGEBRA.

In forthcoming Section 7 we are going to construct the right adjoint \mathbf{g} of the “free vector space” functor \mathbf{k} in Section 5. Recall that for any small category A , the set of morphisms is in a bijective correspondence with the set of functors from the interval category $\mathbf{2} = \begin{array}{ccc} & \circ & \\ \circ & \xrightarrow{a} & \circ \end{array}$ to A . So if the right adjoint \mathbf{g} of \mathbf{k} exists, then for any weak bialgebra H over the field k , the set of morphisms in $\mathbf{g}(H)$ is isomorphic to $\mathbf{cat}(\mathbf{2}, \mathbf{g}(H)) \cong \mathbf{wba}(k\mathbf{2}, H)$. This motivates the study of the set $\mathbf{wba}(k\mathbf{2}, H)$ for any weak bialgebra H , with the aim of finding the way to look at it as the set of morphisms in an appropriate category.

Definition 6.1. For any weak bialgebra H , define the subset

$$\mathbf{g}(H) := \{g \in H : \Delta(g) = g \otimes g, \epsilon(g) = 1, \Delta \sqcap^R(g) = \sqcap^R(g) \otimes \sqcap^R(g), \Delta \overline{\sqcap}^R(g) = \overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g)\}$$

of the set of group-like elements in H .

Remark 6.2. Let us stress that for a general weak bialgebra H , the set $\mathbf{g}(H)$ is strictly smaller than the set $\{g \in H : \Delta(g) = g \otimes g, \epsilon(g) = 1\}$ of group-like elements.

For example, let us consider the free k -vector space on the basis provided by the morphisms of the interval category $\mathbf{2}$. It is a weak bialgebra via the dual of the weak bialgebra structure in Section 5. In terms of Kronecker’s delta, it has the unique multiplication such that $pq = \delta_{p,q}p$, for all $p, q \in \{S, T, a\}$, the unit $S + T + a$, the unique comultiplication for which

$$\Delta(S) = S \otimes S, \quad \Delta(T) = T \otimes T, \quad \Delta(a) = T \otimes a + a \otimes S$$

and the unique counit for which $\epsilon(S) = \epsilon(T) = 1$ and $\epsilon(a) = 0$. In this weak bialgebra

$$\sqcap^R(S) = \overline{\sqcap}^R(S) = S + a \quad \sqcap^R(T) = \overline{\sqcap}^R(T) = T \quad \sqcap^R(a) = \overline{\sqcap}^R(a) = 0.$$

Thus there are two group-like elements S and T but only T belongs to $\mathbf{g}(k\mathbf{2})$.

As we shall see below, there are some distinguished classes of weak bialgebras H , however, in which $\mathbf{g}(H)$ coincides with the set of group-like elements in H .

In contrast to usual bialgebras, where the unit element is always group-like, there are weak bialgebras H in which the set of group-like elements (and therefore the subset $\mathbf{g}(H)$) is empty. Consider, for example, the groupoid with two objects S and T and only one non-identity isomorphism $a : S \rightarrow T$. The free k -vector space on the basis provided by its morphisms, is a weak bialgebra via the dual of the weak bialgebra structure in Section 5. It has the unique multiplication such that $pq = \delta_{p,q}p$, for all $p, q \in \{S, T, a, a^{-1}\}$, the unit $S + T + a + a^{-1}$, the unique comultiplication for which

$$\Delta(S) = S \otimes S + a^{-1} \otimes a, \quad \Delta(T) = T \otimes T + a \otimes a^{-1}, \quad \Delta(a) = T \otimes a + a \otimes S, \quad \Delta(a^{-1}) = S \otimes a^{-1} + a^{-1} \otimes T,$$

and the unique counit for which $\epsilon(S) = \epsilon(T) = 1$ and $\epsilon(a) = \epsilon(a^{-1}) = 0$. In this weak bialgebra there is no group-like element.

Lemma 6.3. *For a weak bialgebra H , any element $g \in H$ such that $\Delta(g) = g \otimes g$ obeys the following identities.*

- (i) $g \sqcap^R(g) = g = \overline{\sqcap}^R(g)g$ and $\sqcap^L(g)g = g = g\overline{\sqcap}^L(g)$.
- (ii) All elements $\sqcap^R(g)$, $\overline{\sqcap}^R(g)$, $\sqcap^L(g)$, $\overline{\sqcap}^L(g)$ are idempotent.
- (iii) If in addition $g \in \mathfrak{g}(H)$, then $\sqcap^R \sqcap^L(g) = \overline{\sqcap}^R(g)$ and $\sqcap^L \sqcap^R(g) = \overline{\sqcap}^L(g)$.

Proof. The equalities in (i) follow from $\Delta(g) = g \otimes g$ and (1.36). The statements in (ii) are obtained by applying \sqcap^R , $\overline{\sqcap}^R$, \sqcap^L and $\overline{\sqcap}^L$, respectively, to the equalities in (i), and taking into account the module map properties (1.35). For $g \in \mathfrak{g}(H)$,

$$(6.1) \quad \overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g) = \Delta \overline{\sqcap}^R(g) = 1_1 \otimes 1_2 \overline{\sqcap}^R(g).$$

Applying to both sides $\text{id} \otimes \sqcap^R$ and multiplying on the right the result by $g \otimes 1$, by the application of part (i) we get

$$g \otimes \overline{\sqcap}^R(g) = 1_1 g \otimes \sqcap^R(1_2) \overline{\sqcap}^R(g).$$

Application of $\epsilon \otimes \text{id}$ to both sides of this equality yields

$$(6.2) \quad \overline{\sqcap}^R(g) = \sqcap^R \sqcap^L(g) \overline{\sqcap}^R(g).$$

On the other hand, applying to both sides of (6.1) $\sqcap^L \otimes \sqcap^R \sqcap^L$ and multiplying on the right the result by $g \otimes 1$, we obtain

$$g \otimes \sqcap^R \sqcap^L(g) = 1_2 g \otimes \sqcap^R \sqcap^L(g) 1_1,$$

where we used (1.32), part (i), (1.34), (1.35), anti-multiplicativity of $\sqcap^R : \sqcap^L(H) \rightarrow \sqcap^R(H)$, and (1.34). Thus by applying $\epsilon \otimes \text{id}$, we get

$$(6.3) \quad \sqcap^R \sqcap^L(g) = \sqcap^R \sqcap^L(g) \overline{\sqcap}^R(g).$$

Comparing (6.2) and (6.3), we conclude on the first equality in (iii). The other equality in (iii) is proven symmetrically. \square

Proposition 6.4. *For a cocommutative weak bialgebra H , the set of group-like elements and the set $\mathfrak{g}(H)$ are equal; that is, $\mathfrak{g}(H) = \{g \in H : \Delta(g) = g \otimes g, \epsilon(g) = 1\}$.*

Proof. It follows immediately from the cocommutativity of H that $\sqcap^L = \overline{\sqcap}^R$ and $\sqcap^R = \overline{\sqcap}^L$, so that $\sqcap^R(H)$ and $\sqcap^L(H)$ are coinciding commutative separable Frobenius subalgebras in H , with separability element $1_1 \otimes 1_2$. Hence if $\Delta(g) = g \otimes g$, then

$$\begin{aligned} \Delta \sqcap^R(g) &= 1_1 \otimes \sqcap^R(g) 1_2 = 1_1 \otimes \sqcap^R(g) \sqcap^R(g) 1_2 \\ &= \sqcap^R(g) 1_1 \otimes \sqcap^R(g) 1_2 = \sqcap^R(g 1_1) \otimes \sqcap^R(g 1_2) = \sqcap^R(g) \otimes \sqcap^R(g). \end{aligned}$$

In the first equality we used (1.38) and in the second one we used part (ii) of Lemma 6.3. In the third equality we used that $\Delta(1)$ is a separability element for the commutative algebra $\sqcap^R(H)$. In the fourth equality we used $\Delta(1) \in \sqcap^R(H) \otimes \sqcap^R(H)$ and (1.35). In the last equality we used the multiplicativity of the comultiplication (cf. (1.27)) and that $\Delta(g) = g \otimes g$. The identity $\Delta \overline{\sqcap}^R(g) = \overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g)$ follows symmetrically. \square

Lemma 6.5. *Let H be a weak Hopf algebra and $g \in H$ such that $\Delta(g) = g \otimes g$. Then the following assertions hold.*

- (i) $\Delta \sqcap^L(g) = \sqcap^L(g) \otimes \sqcap^L(g)$ and $\Delta \sqcap^R(g) = \sqcap^R(g) \otimes \sqcap^R(g)$.
- (ii) $S^2 \sqcap^R(g) = \sqcap^R(g)$ and $S^2 \sqcap^L(g) = \sqcap^L(g)$.
- (iii) $\overline{\sqcap}^L(g) = \sqcap^R(g)$ and $\sqcap^L(g) = \overline{\sqcap}^R(g)$.
- (iv) $S^2(g) = g$.
- (v) $\sqcap^R S(g) = \overline{\sqcap}^R(g)$ and $\overline{\sqcap}^R S(g) = \sqcap^R(g)$; $\sqcap^L S(g) = \overline{\sqcap}^L(g)$ and $\overline{\sqcap}^L S(g) = \sqcap^L(g)$.
- (vi) $\Delta \overline{\sqcap}^L(g) = \overline{\sqcap}^L(g) \otimes \overline{\sqcap}^L(g)$ and $\Delta \overline{\sqcap}^R(g) = \overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g)$.

Proof. (i). Since $\Delta(g) = g \otimes g$, it follows by the multiplicativity of Δ in (1.27) and the anti-co-multiplicativity of S in (1.40) that

$$\Delta \sqcap^L(g) = \Delta(g_1 S(g_2)) = \Delta(g S(g)) = g_1 S(g_2) \otimes g_2 S(g_1) = g S(g) \otimes g S(g) = \sqcap^L(g) \otimes \sqcap^L(g),$$

and symmetrically for $\sqcap^R(g)$.

(ii). By the weak Hopf algebra axioms (1.39) and part (i),

$$\sqcap^L \sqcap^R(g) = \sqcap^R(g)_1 S(\sqcap^R(g)_2) = \sqcap^R(g) S \sqcap^R(g) \stackrel{(1.41)}{=} \sqcap^R(g) \sqcap^L \sqcap^R(g).$$

Symmetrically,

$$\sqcap^R(g) = \sqcap^R \sqcap^R(g) = S(\sqcap^R(g)_1) \sqcap^R(g)_2 = S \sqcap^R(g) \sqcap^R(g) \stackrel{(1.41)}{=} \sqcap^L \sqcap^R(g) \sqcap^R(g).$$

The right hand sides are equal by (1.33), proving

$$(6.4) \quad \sqcap^L \sqcap^R(g) = \sqcap^R(g).$$

Applying \sqcap^R to both sides of (6.4) and using (1.41), we conclude on $S^2 \sqcap^R(g) = \sqcap^R(g)$. The other equality is proven symmetrically.

(iii). By (6.4) and some weak Hopf algebra identities in Section 1,

$$\sqcap^R(g) \stackrel{(6.4)}{=} \sqcap^L \sqcap^R(g) \stackrel{(1.32)}{=} \overline{\sqcap}^L \sqcap^L \sqcap^R(g) \stackrel{(6.4)}{=} \overline{\sqcap}^L \sqcap^R(g) \stackrel{(1.32)}{=} \overline{\sqcap}^L(g).$$

The other equality is proven symmetrically.

(iv). If $\Delta(g) = g \otimes g$, then

$$(6.5) \quad g S(g) g = g_1 S(g_2) g_3 = g_1 \sqcap^R(g_2) = g.$$

Hence

$$\begin{aligned} g &= g S(g) g = g S(g S(g) g) = g S(g) S^2(g) S(g) g = g_1 S(g_2) S^2(g) S(g_1) g_2 \\ &= \sqcap^L(g) S^2(g) \sqcap^R(g) = S^2 \sqcap^L(g) S^2(g) S^2 \sqcap^R(g) = S^2(\sqcap^L(g) g \sqcap^R(g)) = S^2(g). \end{aligned}$$

In the first and the second equalities we used (6.5). In the third and the penultimate equalities we used anti-multiplicativity of S , cf. (1.40). In the fourth equality we used $\Delta(g) = g \otimes g$, in the fifth equality we used the weak Hopf algebra axioms (1.39) and in the sixth equality we used part (ii). The last equality follows by part (i) of Lemma 6.3.

(v). The first claim follows by $\overline{\sqcap}^R(g) = \overline{\sqcap}^R S^2(g) = \sqcap^R S(g)$, cf. part (iv) and (1.41). The second claim is immediate by (1.41). The remaining two claims follow symmetrically.

(vi). This is immediate by parts (i) and (iii). \square

From parts (i) and (vi) of Lemma 6.5 we obtain the following.

Corollary 6.6. *In any weak Hopf algebra H , $\mathbf{g}(H) = \{g \in H : \Delta(g) = g \otimes g, \epsilon(g) = 1\}$.*

Our motivation of the study of the set $\mathbf{g}(H)$ in a weak bialgebra H comes from the following.

Proposition 6.7. *For any weak bialgebra H over a field k , there is a bijection between the sets $\mathbf{wba}(k\mathbf{2}, H)$ and $\mathbf{g}(H)$.*

Proof. Let $\gamma \in \mathbf{wba}(k\mathbf{2}, H)$ and consider $g := \gamma(a)$ (where a stands for the only non-identity morphism in $\mathbf{2}$). Let us see that $g \in \mathbf{g}(H)$:

$$\begin{aligned} \Delta(g) &= \Delta\gamma(a) = (\gamma \otimes \gamma)\Delta_{k\mathbf{2}}(a) = \gamma(a) \otimes \gamma(a) = g \otimes g, \\ \epsilon(g) &= \epsilon\gamma(a) = \epsilon_{k\mathbf{2}}(a) = 1, \\ \Delta \sqcap^R(g) &= \Delta \sqcap^R \gamma(a) = (\gamma \otimes \gamma)\Delta_{k\mathbf{2}} \sqcap_{k\mathbf{2}}^R(a) = \gamma \sqcap_{k\mathbf{2}}^R(a) \otimes \gamma \sqcap_{k\mathbf{2}}^R(a) \\ &= \sqcap^R \gamma(a) \otimes \sqcap^R \gamma(a) = \sqcap^R(g) \otimes \sqcap^R(g), \\ \Delta \overline{\sqcap}^R(g) &= \Delta \overline{\sqcap}^R \gamma(a) = (\gamma \otimes \gamma)\Delta_{k\mathbf{2}} \overline{\sqcap}_{k\mathbf{2}}^R(a) = \gamma \overline{\sqcap}_{k\mathbf{2}}^R(a) \otimes \gamma \overline{\sqcap}_{k\mathbf{2}}^R(a) \\ &= \overline{\sqcap}^R \gamma(a) \otimes \overline{\sqcap}^R \gamma(a) = \overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g). \end{aligned}$$

Conversely, let $g \in \mathbf{g}(H)$ and consider the linear map $\gamma : k\mathbf{2} \rightarrow H$, given by

$$\gamma(S) = \sqcap^R(g), \quad \gamma(T) = \overline{\sqcap}^R(g), \quad \gamma(a) = g,$$

(where S and T are the objects of the category $\mathbf{2}$ and the same symbols stand for their unit morphisms). By Theorem 4.12, to check that γ is a morphism in $\mathbf{wba}(k\mathbf{2}, H)$ it should be proven first that γ is a coalgebra map. This follows by noting that — since $\epsilon \sqcap^R = \epsilon$ and $\epsilon \overline{\sqcap}^R = \epsilon$ — for any morphism c in $\mathbf{2}$,

$$\Delta\gamma(c) = \gamma(c) \otimes \gamma(c) = (\gamma \otimes \gamma)\Delta_{k\mathbf{2}}(c) \quad \text{and} \quad \epsilon\gamma(c) = \epsilon(g) = 1 = \epsilon_{k\mathbf{2}}(c).$$

Next, γ can be seen to commute with \sqcap^R as

$$\begin{aligned} \sqcap^R\gamma(S) &= \sqcap^R\sqcap^R(g) = \sqcap^R(g) = \gamma(S) = \gamma\sqcap_{k\mathbf{2}}^R(S) \\ \sqcap^R\gamma(T) &= \sqcap^R\overline{\sqcap}^R(g) = \overline{\sqcap}^R(g) = \gamma(T) = \gamma\sqcap_{k\mathbf{2}}^R(T) \\ \sqcap^R\gamma(a) &= \sqcap^R(g) = \gamma(S) = \gamma s(a) = \gamma\sqcap_{k\mathbf{2}}^R(a). \end{aligned}$$

Commutativity with $\overline{\sqcap}^R$ is checked symmetrically. Commutativity with the Nakayama automorphism $\sqcap^R\sqcap^L$ follows by part (iii) of Lemma 6.3 as

$$\begin{aligned} \sqcap^R\sqcap^L\gamma(S) &= \sqcap^R\sqcap^L\sqcap^R(g) = \sqcap^R\overline{\sqcap}^L(g) \stackrel{(1.32)}{=} \sqcap^R(g) = \gamma(S) = \gamma\sqcap_{k\mathbf{2}}^R\sqcap_{k\mathbf{2}}^L(S) \\ \sqcap^R\sqcap^L\gamma(T) &= \sqcap^R\sqcap^L\overline{\sqcap}^R(g) \stackrel{(1.32)}{=} \sqcap^R\sqcap^L(g) = \overline{\sqcap}^R(g) = \gamma(T) = \gamma\sqcap_{k\mathbf{2}}^R\sqcap_{k\mathbf{2}}^L(T) \\ \sqcap^R\sqcap^L\gamma(a) &= \sqcap^R\sqcap^L(g) = \overline{\sqcap}^R(g) = \gamma(T) = \gamma\sqcap_{k\mathbf{2}}^R\sqcap_{k\mathbf{2}}^L(g). \end{aligned}$$

Finally, the weak multiplicativity condition in Theorem 4.12 translates to four equalities in parts (i) and (ii) of Lemma 6.3, see

$$\begin{aligned} \gamma(S)\gamma(S) &= \sqcap^R(g)\sqcap^R(g) = \sqcap^R(g) = \gamma(S) & \gamma(a)\gamma(S) &= g\sqcap^R(g) = g = \gamma(a) \\ \gamma(T)\gamma(T) &= \overline{\sqcap}^R(g)\overline{\sqcap}^R(g) = \overline{\sqcap}^R(g) = \gamma(T) & \gamma(T)\gamma(a) &= \overline{\sqcap}^R(g)g = g = \gamma(a). \end{aligned}$$

These constructions clearly yield mutually inverse maps between the sets $\mathbf{g}(H)$ and $\mathbf{wba}(k\mathbf{2}, H)$. \square

Proposition 6.8. *For any weak bialgebra H , there is a category with morphism set $\mathbf{g}(H)$ in Definition 6.1. The object set is $\{r \in \sqcap^R(H) = \overline{\sqcap}^R(H) : \Delta(r) = r \otimes r, \epsilon(r) = 1\}$ and the identity morphisms are given by the evident inclusion into $\mathbf{g}(H)$. The source map is given by the restriction of \sqcap^R and the target map is given by the restriction of $\overline{\sqcap}^R$. The composition is given by the restriction of the multiplication in H .*

Proof. First we check that $\mathbf{g}(H)$ is closed under the composition. Let $g, g' \in \mathbf{g}(H)$ such that $\sqcap^R(g) = \overline{\sqcap}^R(g')$. Then

$$\begin{aligned} \Delta(gg') &= \Delta(g)\Delta(g') = (g \otimes g)(g' \otimes g') = gg' \otimes gg' \quad \text{and} \\ \epsilon(gg') &= \epsilon(g\overline{\sqcap}^R(g')) = \epsilon(g\sqcap^R(g)) = \epsilon(g) = 1. \end{aligned}$$

Since

$$(6.6) \quad \begin{aligned} \sqcap^R(gg') &\stackrel{(1.37)}{=} \sqcap^R(\sqcap^R(g)g') = \sqcap^R(\overline{\sqcap}^R(g')g') = \sqcap^R(g') \quad \text{and} \\ \overline{\sqcap}^R(gg') &\stackrel{(1.37)}{=} \overline{\sqcap}^R(g\overline{\sqcap}^R(g')) = \overline{\sqcap}^R(g\sqcap^R(g)) = \overline{\sqcap}^R(g), \end{aligned}$$

also

$$\begin{aligned} \Delta\sqcap^R(gg') &= \Delta\sqcap^R(g') = \sqcap^R(g') \otimes \sqcap^R(g') = \sqcap^R(gg') \otimes \sqcap^R(gg') \quad \text{and} \\ \Delta\overline{\sqcap}^R(gg') &= \Delta\overline{\sqcap}^R(g) = \overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g) = \overline{\sqcap}^R(gg') \otimes \overline{\sqcap}^R(gg') \end{aligned}$$

hold and we conclude that $gg' \in \mathbf{g}(H)$. Associativity of the composition is evident because of associativity of the multiplication. The object set is clearly a subset of the morphism set; and for any $g \in \mathbf{g}(H)$, both $\sqcap^R(g)$ and $\overline{\sqcap}^R(g)$ belong to the object set. The restrictions of \sqcap^R and $\overline{\sqcap}^R$ give the source and target maps, respectively, by part (i) of Lemma 6.3. It follows by (6.6) that the composition is compatible with the source and target maps. \square

The category in Proposition 6.8 is also denoted by $\mathbf{g}(H)$.

Remark 6.9. For an arbitrary weak bialgebra H , the construction of the category $\mathbf{g}(H)$ in Proposition 6.8 is not symmetric under the simultaneous replacements $\sqcap^R \leftrightarrow \overline{\sqcap}^L$, $\overline{\sqcap}^R \leftrightarrow \sqcap^L$. This is a consequence of the choice we made in the definition of morphisms between bimonoids (so in particular in the definition of morphisms in \mathbf{wba}), see Remark 2.4. In light of part (iii) of Lemma 6.5, the symmetry of the category $\mathbf{g}(H)$ under the simultaneous replacements $\sqcap^R \leftrightarrow \overline{\sqcap}^L$, $\overline{\sqcap}^R \leftrightarrow \sqcap^L$ is restored whenever H is a weak Hopf algebra.

Proposition 6.10. *Any morphism $H \rightarrow H'$ in \mathbf{wba} restricts to a functor $\mathbf{g}(H) \rightarrow \mathbf{g}(H')$.*

Proof. Let $Q : H \rightarrow H'$ be a morphism in \mathbf{wba} . First we need to see that it restricts to a map $\mathbf{g}(Q) = Q|_{\mathbf{g}(H)} : \mathbf{g}(H) \rightarrow \mathbf{g}(H')$. Since Q is in particular a coalgebra map, it follows for all $g \in \mathbf{g}(H)$ that

$$\Delta'Q(g) = (Q \otimes Q)\Delta(g) = Q(g) \otimes Q(g) \quad \text{and} \quad \epsilon'Q(g) = \epsilon(g) = 1.$$

Since Q commutes also with \sqcap^R and $\overline{\sqcap}^R$,

$$\begin{aligned} \Delta' \sqcap'^R Q(g) &= (Q \otimes Q)\Delta \sqcap^R(g) = Q \sqcap^R(g) \otimes Q \sqcap^R(g) = \sqcap'^R Q(g) \otimes \sqcap'^R Q(g) \\ \Delta' \overline{\sqcap}'^R Q(g) &= (Q \otimes Q)\Delta \overline{\sqcap}^R(g) = Q \overline{\sqcap}^R(g) \otimes Q \overline{\sqcap}^R(g) = \overline{\sqcap}'^R Q(g) \otimes \overline{\sqcap}'^R Q(g). \end{aligned}$$

This proves $Q(g) \in \mathbf{g}(H')$. Also from the compatibility of Q with \sqcap^R and $\overline{\sqcap}^R$, it follows that $\mathbf{g}(Q)$ respects the source and target maps as well as the unit morphisms. It preserves the composition by the weak multiplicativity condition; that is, by

$$\begin{aligned} Q(gg') &= Q(g1_1)Q(\sqcap^R(1_2)g') \stackrel{(1.38)}{=} Q(g_1)Q(\sqcap^R(g_2)g') \\ &= Q(g)Q(\sqcap^R(g)g') = Q(g)Q(\overline{\sqcap}^R(g')g') = Q(g)Q(g'), \end{aligned}$$

for all $g, g' \in \mathbf{g}(H)$ such that $\sqcap^R(g) = \overline{\sqcap}^R(g')$. \square

Clearly, the group-like elements in any coalgebra over a field are linearly independent, see [1, Theorem 2.1.2]. Hence the elements of $\mathbf{g}(H)$ in a weak bialgebra H are linearly independent. Since the right subalgebra $\sqcap^R(H)$ of H is finite dimensional, this proves that the cardinality of the object set of $\mathbf{g}(H)$ — that is, of the set $\mathbf{g}(H) \cap \sqcap^R(H)$ — is finite. So we conclude by Proposition 6.8 and Proposition 6.10 that there is a functor \mathbf{g} from \mathbf{wba} to the category \mathbf{cat} of small categories with finitely many objects.

7. THE RIGHT ADJOINT OF THE “FREE VECTOR SPACE” FUNCTOR.

The aim of this section is to show that the functor \mathbf{g} in Section 6 is right adjoint of the “free vector space” functor \mathbf{k} in Section 5. That is, to prove the following.

Theorem 7.1. *For any small category A with finitely many objects, and for any weak bialgebra H over a given field k , there is a bijection $\mathbf{wba}(\mathbf{k}(A), H) \cong \mathbf{cat}(A, \mathbf{g}(H))$ which is natural in A and H . Moreover, the image of $1_{\mathbf{k}(-)}$ under this bijection (that is, the unit of the adjunction $\mathbf{k} \dashv \mathbf{g}$) is a natural isomorphism.*

Proof. We use the same symbol A to denote the set of morphisms in the category A .

First we show that the to-be-unit of the adjunction $\mathbf{k} \dashv \mathbf{g}$ is a natural isomorphism. That is, for any category A (with finitely many objects) the functor $A \rightarrow \mathbf{gk}(A)$, $a \mapsto a$ is an isomorphism. This amounts to checking its bijectivity on the sets of morphisms. Injectivity is obvious. In order to see its surjectivity, let us take some $p \in \mathbf{gk}(A)$. Let us write $p = \sum_{a \in A} \lambda_a a$, with $\lambda_a \in k$ non-zero at most for finitely many $a \in A$. Then from the requirement that p is group-like,

$$\Delta(p) = p \otimes p = \sum_{a, b \in A} \lambda_a \lambda_b a \otimes b.$$

On the other hand, by linearity of Δ ,

$$\Delta(p) = \sum_{a \in A} \lambda_a \Delta(a) = \sum_{a \in A} \lambda_a a \otimes a.$$

Since $\{a \otimes b\}_{a,b \in A}$ is a linearly independent subset in $kA \otimes kA$, we conclude that λ_a is non-zero at most for one element $a \in A$. On the other hand, since

$$1 = \epsilon(p) = \lambda_a \epsilon(a) = \lambda_a,$$

we have $p = a \in A$.

We claim next that the desired bijection $\phi_{A,H} : \mathbf{wba}(k(A), H) \rightarrow \mathbf{cat}(A, \mathbf{g}(H))$ takes any morphism $Q : kA \rightarrow H$ to $Q|_A$, its restriction to $A \cong \mathbf{gk}(A)$. By Proposition 6.10, Q restricts to a functor $A \cong \mathbf{gk}(A) \rightarrow \mathbf{g}(H)$; so that $\phi_{A,H}$ is well defined. Naturality of $\phi_{A,H}$ is evident. Since A is a basis of the vector space kA , the map $\phi_{A,H}$ is injective. In order to show surjectivity of $\phi_{A,H}$, consider some functor $h : A \rightarrow \mathbf{g}(H)$. Since A is a basis of the vector space kA , it can be extended to a unique linear map $\tilde{h} : kA \rightarrow H$. Let us see that \tilde{h} is a morphism of weak bialgebras and hence $h = \phi_{A,H}(\tilde{h})$. For any $a \in A$, $h(a) \in \mathbf{g}(H)$ so $\Delta h(a) = h(a) \otimes h(a)$ and $\epsilon h(a) = 1$. Thus h extends to a coalgebra map \tilde{h} . The weak multiplicativity of \tilde{h} follows from the fact that h preserves the composition. Indeed, for $a, b \in A$,

$$\tilde{h}(a(1_{kA})_1)\tilde{h}(\square^R((1_{kA})_2)b) = \delta_{s(a),t(b)}h(a)h(b) = \delta_{s(a),t(b)}h(a.b) = \tilde{h}(ab).$$

Since h preserves the source and target maps, \tilde{h} commutes with \square^R and $\overline{\square}^R$. Finally, by part (iii) of Lemma 6.3,

$$\square^R \square^L h(a) = \overline{\square}^R h(a) = ht(a) = h \square_{kA}^R \square_{kA}^L(a) \quad \forall a \in A,$$

hence $\square^R \square^L \tilde{h} = \tilde{h} \square_{kA}^R \square_{kA}^L$ follows by linearity. \square

The counit of the above adjunction $k \dashv \mathbf{g}$ is not an isomorphism in general (as it is not so for usual, non-weak bialgebras; see for example [1]). Consider for example the weak bialgebra on the vector space $k\mathbf{2}$ from Remark 6.2. This weak bialgebra $k\mathbf{2}$ is three dimensional, while applying to it the functor \mathbf{kg} we get a one dimensional weak bialgebra. So they cannot be isomorphic. Another counterexample was kindly suggested by the referee: For any (non-zero) weak bialgebra H for which there are no group-like elements in $\square^R(H)$, $\mathbf{kg}(H)$ is the zero dimensional weak bialgebra.

Proposition 7.2. *The component $\phi_{\mathbf{g}(H),H}^{-1}(\mathbf{g}(H)) : \mathbf{kg}(H) \rightarrow H$ of the counit of the adjunction $k \dashv \mathbf{g} : \mathbf{wba} \rightarrow \mathbf{cat}$ is an isomorphism if and only if H is a pointed cosemisimple weak bialgebra.*

Proof. Assume that H is a pointed cosemisimple weak bialgebra; that is, that $H \cong k\{g \in H : \Delta(g) = g \otimes g, \epsilon(g) = 1\}$. Since then H is cocommutative, it follows by Proposition 6.4 that $H \cong \mathbf{kg}(H)$. The converse is clear since $\mathbf{kg}(H)$ is obviously a pointed cosemisimple coalgebra. \square

Corollary 7.3. *The functors k and \mathbf{g} induce an equivalence between the category of all small categories with finitely many objects, and the full subcategory of \mathbf{wba} of all pointed cosemisimple weak bialgebras over a given field k .*

Since over an algebraically closed field every cocommutative coalgebra is pointed (see for example [1, Theorem 2.3.3]), we get the following alternative form of Corollary 7.3.

Corollary 7.4. *If k is an algebraically closed field, then the functors k and \mathbf{g} induce an equivalence between the category of all small categories with finitely many objects, and the full subcategory of \mathbf{wba} of all cocommutative cosemisimple weak bialgebras.*

8. RESTRICTION TO HOPF BIMONIDS.

The aim of this section is to study and compare the full subcategories of Hopf monoids in the categories in Section 3 and in Section 4.

Definition 8.1. (cf. [4, pages 193-194]) Let $(\mathbf{C}, \circ, I, \bullet, J)$ be a duoidal category. We say that a bimonoid H in \mathbf{C} is a *Hopf monoid* if the induced monoidal comonad $(-) \bullet H$ is a right Hopf comonad; that is, if

$$(A \bullet H) \circ (B \bullet H) \xrightarrow{(A \bullet \Delta) \circ (B \bullet H)} (A \bullet H \bullet H) \circ (B \bullet H) \xrightarrow{\gamma} ((A \bullet H) \circ B) \bullet (H \circ H) \xrightarrow{((A \bullet H) \circ B) \bullet \mu} ((A \bullet H) \circ B) \bullet H$$

— to be denoted by $\beta_{A,B}$ — is a natural isomorphism.

Proposition 8.2. *For any set X , a Hopf monoid in $\mathbf{span}(X)$ is precisely a groupoid with object set X .*

Proof. Let H be a Hopf monoid in $\mathbf{span}(X)$ and consider the induced monoidal comonad $(-)\bullet H$. By assumption, the map (8.1) is an isomorphism for any objects A, B in $\mathbf{span}(X)$. So in particular, for $A = B = J = X \times X$, it is an isomorphism from $((X \times X) \bullet H) \circ ((X \times X) \bullet H) \cong H \circ H$ to $((X \times X) \bullet H) \circ (X \times X) \bullet H \cong \{(h, h') \in H \times H : t(h) = t(h')\}$. It sends (h, h') to (h, hh') . We can write its inverse in the form $(h, h') \mapsto (l(h, h'), r(h, h'))$, in terms of some maps l and r from $H \times H$ to H satisfying the conditions

$$\begin{aligned} sl(h, h') &= tr(h, h') \\ sr(h, h') &= s(h') \\ tl(h, h') &= t(h) \\ l(h, h') &= h \end{aligned} \tag{8.2}$$

$$l(h, h')r(h, h') = h' \tag{8.3}$$

for all $h, h' \in H$ such that $t(h) = t(h')$ and

$$r(h, hh') = h' \tag{8.4}$$

for all $h, h' \in H$ such that $s(h) = t(h')$. Using (8.2) to simplify (8.3) and substituting $h' = t(h)$ in it, we obtain

$$hr(h, t(h)) = t(h) \tag{8.5}$$

so that $r(h, t(h))$ is a right inverse of h . As the following computation proves, it is also its left inverse.

$$r(h, t(h))h \stackrel{(8.4)}{=} r(h, hr(h, t(h)))h \stackrel{(8.5)}{=} r(h, h) \stackrel{(8.4)}{=} s(h).$$

Since this construction is valid for every $h \in H$, we showed that H is a groupoid.

Conversely, if H is a groupoid with object set X , then $\beta_{A,B} : (a, h, b, h') \mapsto (a, h, b, hh')$ is an isomorphism with the inverse $\beta_{A,B}^{-1} : (a, h, b, h') \mapsto (a, h, b, h^{-1}h')$. Therefore, by Definition 8.1, H is a Hopf monoid. \square

Proposition 8.3. *For any separable Frobenius (co)algebra R , a Hopf monoid in $\mathbf{bim}(R^e)$ is precisely a weak Hopf algebra with right subalgebra isomorphic to R .*

Proof. By Theorem 4.11, a bimonoid in $\mathbf{bim}(R^e)$ is precisely a weak bialgebra H whose right subalgebra is isomorphic to R . Assume that H is a weak Hopf algebra with the antipode $S : H \rightarrow H$. Then (8.1) — which takes now the explicit form

$$\beta_{A,B}((a \bullet h) \circ (b \bullet h')) = ((a \bullet h_1) \circ b) \bullet h_2 h'$$

— is an isomorphism with the inverse

$$\beta_{A,B}^{-1}(((a \bullet h) \circ b) \bullet h') = (a \bullet h_1) \circ (b \bullet S(h_2)h').$$

This map is checked to be well-defined — that is, R^e -balanced in all of the occurring tensor products — by computations similar to those in the proof of Theorem 4.3. Moreover,

$$\begin{aligned} \beta_{A,B}^{-1}\beta_{A,B}((a \bullet h) \circ (b \bullet h')) &= (a \bullet h_1) \circ (b \bullet S(h_2)h_3 h') \stackrel{(1.39)}{=} (a \bullet h_1) \circ (b \bullet \square^R(h_2)h') \\ &\stackrel{(4.5)}{=} (a \bullet h_1) \circ (\square^R(h_2) \otimes 1) \cdot (b \bullet h') \\ &= (a \bullet h_1) \cdot (\square^R(h_2) \otimes 1) \circ (b \bullet h') \\ &\stackrel{(4.6)}{=} (a \bullet h_1 \square^R(h_2)) \circ (b \bullet h') \stackrel{(1.36)}{=} (a \bullet h) \circ (b \bullet h'). \end{aligned}$$

A similar computation verifies $\beta_{A,B}\beta_{A,B}^{-1} = \text{id}$.

Conversely, assume that $\beta_{A,B}$ is an isomorphism, for any objects A, B in $\mathbf{bim}(R^e)$. Then it is an isomorphism, in particular, for $A = B = R^e \otimes R^e$ with the R^e -actions

$$(r \otimes s)((x \otimes y) \otimes (v \otimes w))(r' \otimes s') := (rx \otimes ys) \otimes (vr' \otimes s'w).$$

Using the isomorphisms

$$\begin{aligned}
R \otimes R \otimes R \otimes H 1_1 \otimes \square^R(1_2)H &\rightarrow ((R^e \otimes R^e) \bullet H) \circ ((R^e \otimes R^e) \bullet H) \\
x \otimes y \otimes z \otimes h 1_1 \otimes \square^R(1_2)h' &\mapsto (((1 \otimes x) \otimes (1 \otimes 1)) \bullet h) \circ (((1 \otimes y) \otimes (1 \otimes z)) \bullet h') \\
\\
R \otimes R \otimes R \otimes 1_1 H \otimes 1_2 H &\rightarrow (((R^e \otimes R^e) \bullet H) \circ (R^e \otimes R^e)) \bullet H \\
x \otimes y \otimes z \otimes 1_1 h \otimes 1_2 h' &\mapsto (((1 \otimes x) \otimes (1 \otimes 1)) \bullet h) \circ ((1 \otimes y) \otimes (1 \otimes z)) \bullet h',
\end{aligned}$$

we obtain that

$$\begin{aligned}
R \otimes R \otimes R \otimes H 1_1 \otimes \square^R(1_2)H &\rightarrow R \otimes R \otimes R \otimes 1_1 H \otimes 1_2 H \\
x \otimes y \otimes z \otimes h 1_1 \otimes \square^R(1_2)h' &\mapsto x \otimes y \otimes z \otimes h_1 \otimes h_2 h'
\end{aligned}$$

is an isomorphism. Then also the Galois map $H 1_1 \otimes \square^R(1_2)H \rightarrow 1_1 H \otimes 1_2 H$, $h 1_1 \otimes \square^R(1_2)h' \mapsto h_1 \otimes h_2 h'$ is an isomorphism. This means equivalently that H is a weak Hopf algebra (see [20, Corollary 6.2] for the details of this equivalent characterization of weak Hopf algebras among weak bialgebras). \square

Let us take the full subcategory **grp** of groupoids in the category of small categories with finitely many objects. The morphisms in **grp** are functors (so that they are compatible with the inverse operation on the morphisms). Similarly, let us take the full subcategory **wha** of weak Hopf algebras in **wba**. Its morphisms are the coalgebra maps $H \rightarrow H'$ rendering commutative the diagrams in Theorem 4.12. Note that there is no reason to expect that all of them will be compatible with the antipodes (that is, the equality $S'Q = QS$ will hold). In fact, compatibility with the antipodes is equivalent to $\square'^L Q = Q \square^L$ holding true.

Theorem 8.4. *The adjunction in Section 7 restricts to an iso unit adjunction between **grp** and **wha**.*

Proof. First we check that $\mathbf{k} : \mathbf{cat} \rightarrow \mathbf{wba}$ restricts to a functor $\mathbf{grp} \rightarrow \mathbf{wha}$. If A is a groupoid, then kA has a weak Hopf algebra structure via the antipode $S : kA \rightarrow kA$, sending every $a \in A$ to a^{-1} . The antipode axioms hold by

$$\begin{aligned}
a_1 S(a_2) &= a S(a) = a.a^{-1} = t(a) = \square_{kA}^L(a) \\
S(a_1) a_2 &= S(a) a = a^{-1}.a = s(a) = \square_{kA}^R(a) \\
S(a_1) a_2 S(a_3) &= S(a) a S(a) = a^{-1}.a.a^{-1} = a^{-1} = S(a),
\end{aligned}$$

see [18, Section 2.5]. On the other hand, also $\mathbf{g} : \mathbf{wba} \rightarrow \mathbf{cat}$ restricts to a functor $\mathbf{wha} \rightarrow \mathbf{grp}$. That is, if H is a weak Hopf algebra, then $\mathbf{g}(H)$ is a groupoid (with many finitely objects) with the inverse operation $\mathbf{g}(H) \rightarrow \mathbf{g}(H)$, $g \mapsto S(g)$. In order to see that $S(g)$ is an element of $\mathbf{g}(H)$ indeed, note that $\Delta S(g) = (S \otimes S) \Delta^{op}(g) = S(g) \otimes S(g)$ and $\epsilon S(h) = \epsilon(h) = 1$ follow from the fact that S is an anti-coalgebra map. By part (v) of Lemma 6.5 also the other two conditions on elements of $\mathbf{g}(H)$ hold true and the to-be-inverse operation $g \mapsto S(g)$ is compatible with the source and target maps. Moreover, it works as an inverse by

$$\begin{aligned}
g.g^{-1} &= gS(g) = g_1 S(g_2) = \square^L(g) = \overline{\square}^R(g) = t_{\mathbf{g}(H)}(g) \quad \text{and} \\
g^{-1}.g &= S(g)g = S(g_1)g_2 = \square^R(g) = s_{\mathbf{g}(H)}(g),
\end{aligned}$$

where the penultimate equality in the first line follows by part (iii) of Lemma 6.5. \square

The following corollaries are immediate consequences of Corollary 7.3 and Corollary 7.4, respectively.

Corollary 8.5. *The functors \mathbf{k} and \mathbf{g} induce an equivalence between the category of all small groupoids with finitely many objects, and the full subcategory of **wha** of all pointed cosemisimple weak Hopf algebras over a given field k .*

Corollary 8.6. *If k is an algebraically closed field, then the functors \mathbf{k} and \mathbf{g} induce an equivalence between the category of all small groupoids with finitely many objects, and the full subcategory of **wha** of all cocommutative cosemisimple weak Hopf algebras.*

Example 8.7. Assume k to be a field of characteristic 0, and let N be a positive integer. The ‘algebraic quantum torus’; that is, the algebra $H = k\langle U, V, V^{-1} | U^N = 1, VU = qUV \rangle$, with $q \in k$ such that $q^N = 1$, is a double crossed product weak Hopf algebra of the group Hopf algebra $k\langle V, V^{-1} \rangle$ and the N -dimensional weak Hopf algebra $B := k\langle U | U^N = 1 \rangle$ with the comultiplication

$$\Delta(U^n) = \frac{1}{N} \sum_{j=1}^N (U^{j+n} \otimes U^{-j}),$$

the counit defined by $\epsilon(1) = N, \epsilon(U^n) = 0$ if $U^n \neq 1$ and the antipode $S = \text{id}$ (see [7, Example 9]).

For any N th root of unity $\omega \in k$ (possibly, different from q), we have a group-like element $g_\omega = \frac{1}{N} \sum_{j=1}^N \omega^j U^j$ of B . Thus, if k contains a primitive N th root of unity (so that the set $T := \{\omega \in k : \omega^N = 1\}$ has N elements) then, as coalgebras,

$$B = \bigoplus_{\omega \in T} k g_\omega \quad \text{and} \quad H = \bigoplus_{\omega \in T, m \in \mathbb{Z}} k g_\omega V^m.$$

We deduce from Corollary 8.5 that in this case H is isomorphic to the groupoid weak Hopf algebra kg , where $\mathbf{g} = \{g_\omega V^m | \omega \in T, m \in \mathbb{Z}\}$. This groupoid has N objects $\{g_\omega | \omega \in T\}$, but it is not finite. Since $g_\omega g_{\omega'} = 0$ if $\omega \neq \omega'$, and $g_\omega^2 = g_\omega$, we get that two morphisms $g_\omega V^m, g_{\omega'} V^n$ of \mathbf{g} are composable if and only if $\omega = \nu q^m$, and, in this case, $g_\omega V^m g_{\nu} V^n = g_\omega V^{m+n}$.

REFERENCES

- [1] E. Abe, *Hopf algebras*. University Press, Cambridge, 1980, ISBN 0 521 22240 0.
- [2] L. Abrams, *Modules, comodules and cotensor over Frobenius algebras*, J. Algebra **219** (1999), 201–213.
- [3] M. Aguiar and S. Mahajan, *Monoidal Functors, Species and Hopf Algebras*. CRM Monograph Series **29**, American Math. Soc. Providence, 2010. Electronically available at: <http://www.math.tamu.edu/~maguiar/a.pdf>.
- [4] T. Booker and R. Street, *Tannaka duality and convolution for duoidal categories*, Theory Appl. Categ. **28** (2013), no. 6, 166–205.
- [5] G. Böhm, *Hopf Algebroids*. in: Handbook of Algebra Vol **6**, M. Hazewinkel (ed.), pp. 173–236, Elsevier 2009.
- [6] G. Böhm, S. Caenepeel and K. Janssen, *Weak bialgebras and monoidal categories*, Comm. Algebra **39** (2011), 4584–4607.
- [7] G. Böhm and J. Gómez-Torrecillas, *On the double crossed product of weak Hopf algebras*, in: ‘Hopf Algebras and Tensor Categories’. N. Andruskiewitsch et al. (eds.), Contemp. Math. **585** pp. 153–173, AMS Providence 2013.
- [8] G. Böhm, F. Nill and K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^* -structure*, J. Algebra **221** (1999), 385–438.
- [9] A. Bruguières, S. Lack and A. Virelizier, *Hopf monads on monoidal categories*, Adv. in Math. **227** (2011), 745–800.
- [10] S. Caenepeel and M. De Lombaerde, *A categorical approach to Turaev’s Hopf group-coalgebras*, Comm. Algebra **34** (2006), 2631–2657.
- [11] S. Caenepeel and I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra **39** (2011), 2216–2240.
- [12] B.J. Day and R. Street, *Lax monoids, pseudo-operads, and convolution*, in: “Diagrammatic Morphisms and Applications”, D.E. Radford, F.J.O. Souza and D.N. Yetter (eds.) Contemp. Math. **318** pp. 75–96, AMS Providence 2003.
- [13] F. De Meyer and E. Ingraham, *Separable algebras over a commutative ring*. Lecture Notes in Mathematics **181**, Springer-Verlag, New York 1971.
- [14] J.H. Lu, *Hopf algebroids and quantum groupoids*, Internat. J. Math. **7** (1996), 47–70.
- [15] S. Mac Lane, *Categories for the working mathematician*. Second edition, Springer Verlag New York, 1998.
- [16] A. Makhlof and S.D. Silvestrov, *Hom-algebras and Hom-coalgebras*, J. Algebra Appl. **9** (2010), 553–589.
- [17] I. Moerdijk, *Monads on tensor categories*, J. Pure and Appl. Algebra **168** (2002), 189–208.
- [18] D. Nikshych and L. Vainerman, *Finite quantum groupoids and their applications*, in: ‘New Directions in Hopf Algebras’. S. Montgomery and H.-J. Schneider (eds.) pp. 211–262, Math. Sci. Res. Inst. Publ. **43**, 2002.
- [19] C. Pastro and R. Street, *Weak Hopf monoids in braided monoidal categories*, Algebra and Number Theory **3** (2009), 149–207.
- [20] P. Schauenburg, *Weak Hopf algebras and quantum groupoids*, in: ‘Noncommutative Geometry and Quantum Groups’. P.M. Hajac and W. Pusz (eds.), Banach Center Publ. **61** pp. 171–188, 2003.
- [21] R. Street, *The formal theory of monads*, J. Pure and Appl. Algebra **2** (1972), 149–168.
- [22] K. Szlachányi, *Finite quantum groupoids and inclusions of finite type*, in: ‘Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects’. R. Longo (ed.), Fields Inst. Comm. **30** pp. 393–407, AMS Providence 2001.
- [23] K. Szlachányi, *Galois actions by finite quantum groupoids*, in: ‘Locally Compact Quantum Groups and Groupoids’. L. Vainerman (ed.), pp. 105–126, IRMA Lectures in Mathematics and Theoretical Physics **2**, Walter de Gruyter, Berlin-New York, 2003.

- [24] K. Szlachányi, *The monoidal Eilenberg-Moore construction and bialgebroids*, J. Pure Appl. Algebra **182** (2003), 287-315.
- [25] M. Takeuchi, *Groups of algebras over $A \otimes \overline{A}$* , J. Math. Soc. Japan **29** no. 3 (1977), 459-492.
- [26] V.G. Turaev, *Homotopy field theory in dimension 3 and crossed group-categories*, preprint available at <http://arxiv.org/abs/math/0005291>.
- [27] P. Vecsernyés, *Larson-Sweedler theorem and the role of grouplike elements in weak Hopf algebras*, J. Algebra **270** no. 2 (2003), 471-520.

WIGNER RESEARCH CENTRE FOR PHYSICS, BUDAPEST, H-1525 BUDAPEST 114, P.O.B. 49, HUNGARY
E-mail address: `boh.m.gabriella@wigner.mta.hu`

DEPARTAMENTO DE ÁLGEBRA, UNIVERSIDAD DE GRANADA, E-18071 GRANADA, SPAIN
E-mail address: `gomezj@ugr.es`

DEPARTAMENTO DE ÁLGEBRA, UNIVERSIDAD DE GRANADA, E-18071 GRANADA, SPAIN
E-mail address: `esperanza@ugr.es`