Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
33 (2017), 147–164
www.emis.de/journals
ISSN 1786-0091

PLANARITY IN VAGUE GRAPHS WITH APPLICATION

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Abstract. In lots of practical applications with a graph structure, there may exist crossing between edges. Crossing between edges is not allowed in crisp planar graph. Crossing of edges can be considered in a vague multigraph with certain amount of vague planarity value. This is why the notion of vague multiset is introduced. Then vague multigraphs, vague planar graphs, vague strong edges, vague faces, strong vague faces are defined. The vague dual graph of a vague planar graph is also introduced. Several properties of vague planar graphs and vague dual graphs are also studied. An application of vague planar graph is also given.

1. Introduction

Graph theory is now a very important research topic due to its wide range of applications in data mining, image segmentation, clustering, image capturing, networking, communication, planning, scheduling, etc. Design of data structure, modeling of network topologies can be done using the concept of graph. Also, paths, walks and circuits are used to solve many problems of traveling salesman, database design, resource networking, etc. There are many real world applications like design problems for circuits, subways, utility lines with a graph structure in which crossing between edges is a nuisance. Crossing of two connections normally means that the communication lines must be run at different heights. This is not a big problem for electrical wires but it creates extra expenses for some types of lines, i.e. burying one subway tunnel under another. In particular, circuits are easier to manufacture if their connections can be constructed in fewer layers. These applications are designed using the concept of planar graphs. Several computational challenges like image segmentation or shape matching can also be solved by means of cuts of planar graph. In a city planning, subway tunnels, pipelines, metro lines, etc. are essential in twenty first century. There are chances of accident due to crossing. Also, the cost for crossing of routes in the underground is high while the underground

2010 *Mathematics Subject Classification.* 05C72, 05C76, 05C10, 03B52.

Key words and phrases. Vague graphs, vague multigraphs, vague planar graphs, considerable edges, faces of vague planar graphs, vague dual graphs.

routes reduce the traffic jam. Routes without crossing is preferable for safety, but due to the lack of space crossing of such lines are allowed. Crossing between one congested and one non-congested routes is more preferable compared to the crossing between two congested routes. The term "congested" has no definite meaning. We generally use "congested", "very congested", "highly congested" routes, etc. These terms are called linguistic terms and they have some membership values. A congested route may be termed as strong route and low congested route may be termed as weak route. Thus crossing between strong route and weak route is safer than the crossing between two strong routes. In other words, crossing between strong and weak route may be allowed in city planning with certain amount of safety. The terms "strong route" and "weak route" lead strong edge and weak edge of a vague graph respectively and the permission of crossing between strong and weak edges leads to the concept of vague planar graph.

Now a days, most mathematical models are developed using fuzzy sets to handle various types of systems containing elements of uncertainty. In 1993, Gau and Buehrer [4], introduced the notion of vague set theory as a generalization of Zadeh fuzzy set theory [32]. Vague sets are higher order fuzzy sets. Application of higher order fuzzy sets makes the solution-procedure more complex, but if the complexity on computation-time, computation-volume, or memory-space are not matters of concern, then we can achieve better results. In a fuzzy set, each element is associated with a point-value selected from the unit interval [0, 1], which is termed as the grade of membership in the set. Instead of using point-based membership as in fuzzy sets, interval-based membership is used in a vague set. The interval-based membership in vague sets is more expressive in capturing vagueness of data. There are some interesting features for handling vague data that are unique to vague sets. For example, vague sets allow for a more intuitive graphical representation of vague data, which facilitates significantly better analysis in data relationships, incompleteness, and similarity measures. Considering the fuzzy relations between fuzzy sets, Rosenfeld [20] introduced the concept of fuzzy graphs in 1975 and later on developed the structure of fuzzy graphs obtaining analogous of several graph concepts. The concept of weak isomorphism, co-weak isomorphism and isomorphism between fuzzy graphs was introduced by Bhutani in [3]. The notion of fuzzy line graph was introduced by Mordeson in [15]. Mordeson and Nair [17] defined the complement of fuzzy graph and further studied by Sunitha and Kumar [23]. After that several researchers are working on fuzzy graphs such as [14, 16]. Samanta and Pal introduced several types of fuzzy graphs like fuzzy competition graphs [26, 27], fuzzy tolerance graphs [24], fuzzy threshold graphs [25], etc. As a generalization of fuzzy graphs some more work can be found on [5, 6, 7, 8, 9, 10, 11, 12, 13, 21, 22, 30]. Abdul and Jabbar et al. [1] introduced the concept of fuzzy planar graph. Nirmala et al. [18] defined special fuzzy planar graphs. Samanta et al. [28, 29] defined fuzzy planar graph assuming crossing of edges. In this paper, the notion of vague multiset is introduced. Then vague multigraphs, vague planar graphs, vague strong edges, vague faces, strong vague faces are defined. The vague dual graph of a vague planar graph is also introduced. Several properties of vague planar graphs and vague dual graphs are also studied.

2. Preliminaries

A finite graph is a graph $G = (V, E)$ where V and E are both finite sets. G is called an infinite graph if either V or E or both are infinite sets. Mostly in graph theory, the graphs discussed are finite. A multigraph [2] is a graph that may contain multiple edges between any two vertices, but it does not contain any self loops. A graph can be drawn in many different ways. A graph may or may not be drawn on a plane without crossing of edges.

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding [2]. A graph G is planar if it can be drawn in the plane with its edges only intersecting at vertices of G. So the graph is non-planar if it can not be drawn without crossing. A planar graph with cycles divides the plane into a set of regions, also called faces. The length of a face in a planar graph G is the length of the closed walk in G bounding the face. The portion of the plane lying outside a graph embedded in a plane is infinite region.

In crisp graph theory, the dual graph of a given planar graph G is a graph which has a vertex corresponding to each plane region of G, and the graph has an edge joining two neighboring regions for each edge in G, for a certain embedding of G.

Definition 2.1 ([4])**.** A vague set on a non-empty set X is a pair (t_A, f_A), where $t_A\colon X \to [0,1]$, $f_A\colon X \to [0,1]$ are true and false membership functions, respectively, such that $t_A(x) + f_A(x) \le 1$ for all $x \in X$.

In the above definition, $t_A(x)$ is considered as the lower bound for degree of membership of x in A(based on evidence for), and $f_A(x)$ is the lower bound for negation of membership of x in A(based on evidence against). Therefore, the degree of membership of x in the vague set A is characterized by the interval $[t_A(x), 1-f_A(x)]$. So, a vague set is a special case of interval-valued sets studied by many mathematicians and applied in many branches of mathematics. Vague sets also have many applications. The interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A and is denoted by $V_A(x)$. We denote zero vague and unit vague value by $\mathbf{0} = [\mathbf{0}, \mathbf{0}]$ and $\mathbf{1}$= $[1, 1]$, respectively.

It is worth to mention here that interval-valued fuzzy sets are not vague sets. In interval-valued fuzzy sets, an interval valued membership value is assigned to each element of the universe considering the "evidence for x" only, without considering "evidence against x". In vague sets both are independently proposed by the decision maker. This makes a major difference in the judgment about the grade of membership.

A vague relation is a further generalization of a fuzzy relation.

Definition 2.2 ([19])**.** Let X and Y be ordinary finite non-empty sets. We call a vague relation a vague subset of $X \times Y$, that is an expression R defined by $R = \{< (x, y), t_R(x, y), f_R(x, y) >: x \in X, y \in Y\}$, where $t_R \colon X \times Y \to [0, 1]$ and $f_R \colon X \times Y \to [0, 1]$, satisfies the condition $0 \le t_R(x, y) + f_R(x, y) \le 1$, for all $(x, y) \in X \times Y$.

Definition 2.3 ([19])**.** A vague relation B on a set V is a vague relation from V to V. If A is a vague set on a set V, then a vague relation B on A is a vague relation which satisfies $t_B(x, y) \le \min\{t_A(x), t_A(y)\}$ and $f_B(x, y) \ge \max\{t_B(x), t_B(y)\}$ for all $x, y \in V$.

Definition 2.4 ([19])**.** Let $G^* = (V, E)$ be a crisp graph. A pair $G = (V, A, B)$ is called a vague graph of G^*, where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on $E \subseteq V \times V$ such that $t_B(x, y) \le min\{t_A(x), t_A(y)\}$ and $f_B(x, y) \ge \max\{t_B(x), t_B(y)\}$ for each $(x, y) \in E$. We call A the vague vertex set of G and B as the vague edge set of G respectively.

A vague graph G is said to be strong if $t_B(u, v) = \min\{t_A(u), t_A(v)\}$ and $f_B(u, v) = \max\{f_A(u), f_A(v)\}$ for all $(u, v) \in E$.

G is said to be complete if $t_B(u, v) = \min\{t_A(u), t_A(v)\}$ and $f_B(u, v) = \max\{f_A(u), f_A(v)\}$ for all $u, v \in V$.

A vague graph $G = (V, A, B)$ is said to be bipartite if the vertex set V can be partitioned into two non-empty sets V_1 and V_2 such that $t_B(v_1, v_2) > 0$ and $f_B(v_1, v_2) > 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$.

3. Vague multiset

A (crisp) multiset over a non-empty set V is simply a mapping $d \colon V \to \mathbb{N}$ where $\mathbb{N}$ is the set of all natural numbers. Yager [31] first discussed fuzzy multiset, although he used the term "fuzzy bag". An element of a non-empty set V may occur more than once with possibly the same or different membership values. A natural generalization of this interpretation of multiset leads to the notion of fuzzy multiset, or fuzzy bag, over a non-empty set V as a mapping $\widetilde{C} \colon V \times [0, 1] \to \mathbb{N}$. The membership values of V are denoted as v_{μ^j}, $j = 1, 2, \ldots, p$ where $p = \max\{j : v_{\mu^j} \ne 0\}$. So the fuzzy multiset can be denoted as $M = \{(v, v_{\mu^j}), j = 1, 2, \ldots, p : v \in V\}$. Vague multiset is another generalization of multiset which is defined below.

Definition 3.1 (Vague multiset)**.** Let V be a nonempty set. Let $t^i_A \colon V \to [0, 1]$ and $f^i_A \colon V \to [0, 1]$ be the mappings such that $t^i_A(v) + f^i_A(v) \le 1$ for all $v \in V$, $i = 1, 2, \ldots, p$. The vague multiset on V is denoted by A and is defined as $\{(v, t^i_A(v), f^i_A(v)) : v \in V, i = 1, 2, \ldots, p\}$.

Example 3.2. Let $V = \{a, b, c, d\}$. Then one of the vague multisets on V is given by $(a, 0.5, 0.2)$, $(a, 0.4, 0.3)$, $(a, 0.6, 0.3)$ $(b, 0.7, 0.2)$, $(c, 0.5, 0.4)$, $(d, 0.4, 0.4)$, $(d, 0.5, 0.4)$.

4. Vague multigraph

In this section, we introduce the concept of vague multigraph using the notion of vague multiset.

Definition 4.1. Let V be a nonempty set and let $A = (t_A, f_A)$ be a vague set on V. Let $B = \{((u,v), t_B^i(u,v), f_B^i(u,v)), i = 1, 2, \ldots, p : (u,v) \in V \times V\}$ be a vague multiset of $V \times V$. Then $G = (V, A, B)$ is said to be vague multigraph if $t_B^i(u,v) \leq \min\{t_A(u), t_A(v)\}$ and $f_B^i(u,v) \geq \max\{f_A(u), f_A(v)\}$, for all $u, v \in V$, $i = 1, 2, \ldots, p$. Here, $(t_A(u), f_A(u))$ and $(t_B^i(u,v), f_B^i(u,v))$ represent the membership value of the vertex u and the membership value of the edge (u,v) in G respectively.

It may be noted that there may be more than one edge between the vertices u and v. $(t_B^i(u,v), f_B^i(u,v))$ denotes the true and false membership value of the i-th edge between the vertices u and v respectively and p represents the number of edges between the vertices u and v.

An example of vague multigraph is given below.

Example 4.2. Let $V = \{a, b\}$ be a set of vertices. Let $t_A(a) = 0.5$, $f_A(a) = 0.4$, $t_A(b) = 0.6$, $f_A(b) = 0.2$ and $t_B^1(a,b) = 0.4$, $f_B^1(a,b) = 0.5$, $t_B^2(a,b) = 0.2$, $f_B^2(a,b) = 0.7$, $t_B^3(a,b) = 0.5$, $f_B^3(a,b) = 0.4$, $t_B^4(a,b) = 0.3$, $f_B^4(a,b) = 0.6$.

Then $A = \{(a, 0.5, 0.4), (b, 0.6, 0.2)\}$ and
$B = \{((a,b), 0.4, 0.5), ((a,b), 0.2, 0.7), ((a,b), 0.5, 0.4), ((a,b), 0.3, 0.6)\}$. Therefore, $G = (V, A, B)$ is a vague multigraph (see Fig. 1).

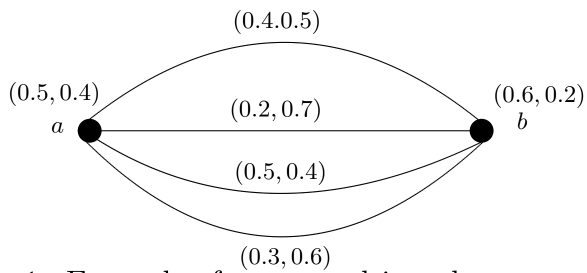

FIGURE 1. Example of vague multigraph.

The underlying crisp graph of a vague multigraph $G = (V, A, B)$ is denoted by $G = (V, E)$ where $V = \{u \in V : t_A(u) > 0 \text{ and } f_A(u) > 0\}$ and $B = \{(u,v) \in V \times V : t_B^i(u,v) > 0 \text{ and } f_B^i(u,v) > 0, i = 1, 2, \ldots, p\}$. A special type of vague multigraph is defined below.

5. Vague planar graphs

Planarity has an important significance in connection with wire lines, gas lines, water lines, printed circuits diagrams, etc. But, sometimes little crossing may be accepted to these designs of such lines or circuits. So the concept of vague planar graph is an important topic for these connections. A crisp graph is called non-planar if there is at least one crossing between the edges for all geometric representations of the graph. Suppose a crisp graph G has a crossing for a certain geometrical representation between two edges (u, v) and (w, x) and has another crossing between the edges (a, b) and (c, d). Now, when we think about the strength of the edges of a graph in real phenomena, some crossing causes a big problem and some are not. Assume that the edges (u, v), (w, x) and (a, b) are strong edges and (c, d) is a weak edge of G. In realistic view, the crossing between two strong edges (u, v) and (w, x) can not be considered in a planar graph, whereas the crossing between one strong edge (a, b) and one weak edge (c, d) may be considered. These linguistic words can be stated in a well-defined manner as follows: In vague concept, we say that each of the three strong edges (u, v), (w, x) and (a, b) have membership values near to $(1, 0)$ and the weak edge (c, d) has membership value near to $(0, 0)$. If we remove the edge (w, x), then the membership value of the edge (w, x) in the graph is taken as $(0, 0)$.

Let $G = (V, A, B)$ be a vague multigraph and for a certain geometrical representation, the graph has only one crossing between the edges $((w, x), t_B(w, x), f_B(w, x))$ and $((y, z), t_B(y, z), f_B(y, z))$. If $t_B(w, x) = 1$, $f_B(w, x) = 0$ and $t_B(y, z) = 0$, $f_B(y, z) = 0$, then we say that the graph has no crossing. Similarly, if $t_B(w, x)$ has value near to 1, $f_B(w, x)$ has value near to 0 or if $t_B(y, z)$ and $f_B(y, z)$ have value near to 0, the crossing will not be important for the planarity. If $t_B(w, x)$, $t_B(y, z)$ have value near to 1 and $f_B(w, x)$, $f_B(y, z)$ have value near to 0, then the crossing becomes very important for the planarity. So, if there is a crossing at a point between two edges, then we assign a value corresponding to the point, called intersecting value.

5.1. **Intersecting value in vague multigraph.** Let $G = (V, A, B)$ be a vague multigraph where $B = \{((u, v), t^i_B(u, v), f^i_B(u, v)), i = 1, 2, \ldots, p : (u, v) \in V \times V\}$. G is called vague complete multigraph if $t^i_B(u, v) = \min\{t_A(u), t_A(v)\}$ and $f^i_B(u, v) = \max\{f_A(u), f_A(v)\}$ for all $u, v \in V$ and $i = 1, 2, \ldots, p$.

Example 5.1. Let us consider the vague multigraph G as shown in Fig. 2. It is easy to see that G is a vague complete multigraph.

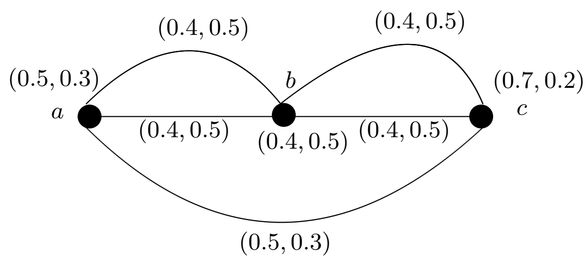

FIGURE 2. Vague complete multigraph

Now we define the strength of an edge $((u,v), t_B^i(u,v), f_B^i(u,v))$ which is defined by a value $I_{(u,v)} = (I^t_{(u,v)}, I^f_{(u,v)})$ where $I^t_{(u,v)} = \frac{t_B^i(u,v)}{\min\{t_A(u),t_A(v)\}}$ and $I^f_{(u,v)} = \frac{\max\{f_A(u),f_A(v)\}}{f_B^i(u,v)}$.

Definition 5.2. Let $G = (V, A, B)$ be a vague multigraph. An edge (u, v) in G is said to be vague strong if $I^t_{(u,v)} \geq 0.5$ and $I^f_{(u,v)} \leq 0.5$. Otherwise it is called vague weak.

In vague multigraph, when two edges intersect at a point, a value is assigned to that point in the following way.

Let in a vague multigraph $G = (V, A, B)$, B contains two edges $((u_1, v_1), t_B^i(u_1, v_1), f_B^i(u_1, v_1))$ and $((u_2, v_2), t_B^j(u_2, v_2), f_B^j(u_2, v_2))$ which intersect at a point P, where i and j are fixed integers. The intersecting value at the point P is given by $\mathcal{I}_P = (\mathcal{I}^t_P, \mathcal{I}^f_P)$ where $\mathcal{I}^t_P = \frac{I^t_{(u_1,v_1)}+I^t_{(u_2,v_2)}}{2}$ and $\mathcal{I}^f_P = \frac{I^f_{(u_1,v_1)}+I^f_{(u_2,v_2)}}{2}$.

In crisp sense, a planar graph has no crossing of edges, i.e. there is no intersection of edges. So, the 'planarity' of the planar graph is 'full'. Therefore, if the number of points of intersection in a vague multigraph increases, then the 'planarity' decreases. So, in vague multigraph, $\mathcal{I}_P$ is inversely proportional to the 'planarity'. Using these concept, the notion of vague planar graph is introduced below.

Definition 5.3 (Planarity of vague multigraph)**.** Let $G = (V, A, B)$ be a vague multigraph and for a certain geometrical representation $P_1, P_2, \ldots, P_k$ be the points of intersections between the edges. Then G is said to be vague planar graph with vague planarity value $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ where $\mathcal{P}_t = \frac{1}{1+\{\mathcal{I}^t_{P_1}+\mathcal{I}^t_{P_2}+\cdots+\mathcal{I}^t_{P_k}\}}$ and $\mathcal{P}_f = \frac{1}{1+\{\mathcal{I}^f_{P_1}+\mathcal{I}^f_{P_2}+\cdots+\mathcal{I}^f_{P_k}\}}$.

Clearly, $\mathcal{P}$ is bounded since $0 < \mathcal{P}_t \leq 1$ and $0 < \mathcal{P}_f \leq 1$.

If there is no point of intersection for a certain geometrical representation of vague planar graph, then its degree of vague planarity is $(1, 1)$. This is the case where the underlying crisp graph of this vague planar graph is the crisp planar graph. According the definition, every vague graph is a vague planar graph with some degree of vague planarity value.

Example 5.4. Let us consider a vague multigraph with two point of intersections P_1 and P_2 (see Fig. 3). P_1 is a point between the edges $((a, b), 0.3, 0.5)$ and $((c, d), 0.3, 0.5)$, P_2 is a point between the edges $((a, b), 0.4, 0.5)$ and $((c, d), 0.3, 0.5)$.

Now, for the edge $((a, b), 0.3, 0.5)$, $I_{(a,b)} = (0.6, 0.8)$, for the edge $((a, b), 0.4, 0.5)$, $I_{(a,b)} = (0.8, 0.8)$ and for the edge $((c, d), 0.3, 0.5)$, $I_{(c,d)} = (0.6, 0.6)$.

For the point P_1, the intersecting value is $\mathcal{I}_{P_1} = (\frac{0.6+0.6}{2}, \frac{0.8+0.6}{2}) = (0.6, 0.7)$ and for the point P_2, the intersecting value is $\mathcal{I}_{P_2} = (\frac{0.8+0.6}{2}, \frac{0.8+0.6}{2}) = (0.7, 0.7)$. So, the vague planarity value for the vague multigraph is $(\frac{1}{1+0.6+0.7}, \frac{1}{1+0.7+0.7}) = (0.435, 0.417)$.

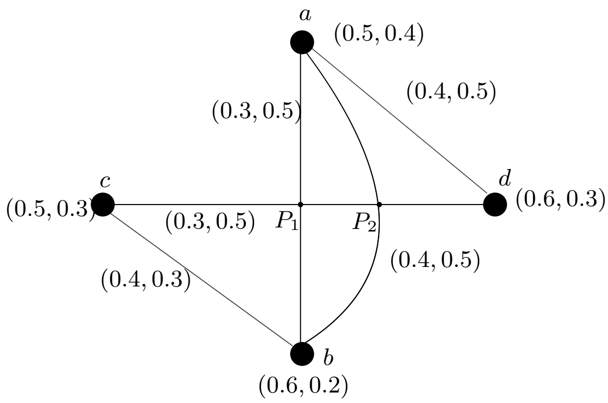

FIGURE 3. Example of vague planar graph with vague planarity (0.435, 0.471)

Now consider a vague complete multigraph whose vague planarity value is given by the following theorem.

Theorem 5.5. *Let $G = (V, A, B)$ be a vague complete multigraph. The vague planarity value $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ is given by $\mathcal{P}_t = \frac{1}{1+n_p}$ and $\mathcal{P}_f = \frac{1}{1+n_p}$, where n_p is the number of points of intersection between the edges in G.*

Proof. Since G is complete, we have $t_B^i(u, v) = \min\{t_A(u), t_A(v)\}$ and $f_B^i(u, v) = \max\{f_A(u), f_A(v)\}$ for all $u, v \in V$ and for $i = 1, 2, \ldots, p$. Let $P_1, P_2, \ldots, P_k$ be the points of intersection between the edges in G.

For an edge (u, v) in G, $I_{(u,v)}^t = \frac{t_B^i(u,v)}{\min\{t_A(u),t_A(v)\}} = 1$ and $I_{(u,v)}^f = \frac{\max\{f_A(u),f_A(v)\}}{f_B^i(u,v)} = 1$.

Therefore, for the point P_1 which is the point of intersection between the edges (a, b) and (c, d), the intersection value is $\mathcal{I}_{P_1} = (1, 1)$. Hence, $\mathcal{I}_{P_i} = (1, 1)$ for $i = 1, 2, \ldots, k$.

Now, $\mathcal{P}_t = \frac{1}{1+(\mathcal{I}_{P_1}^t+\mathcal{I}_{P_2}^t+\cdots+\mathcal{I}_{P_k}^t)} = \frac{1}{1+(1+1+\cdots+1)} = \frac{1}{1+n_p}$, where n_p is the number of points of intersection between the edges in G. Therefore, the vague planarity $\mathcal{P}$ is given by $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ where $\mathcal{P}_t = \mathcal{P}_f = \frac{1}{1+n_p}$. □

Theorem 5.6. *Let $G = (V, A, B)$ be a vague planar graph with vague planarity $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ is such that $\mathcal{P}_t > 0.5$ and $\mathcal{P}_f < 0.5$. Then the number of points of intersection between vague strong edges in G is at most one.*

Proof. If possible, let G has at least two points of intersection P_1 and P_2 between two vague strong edges in G.

Now, for any vague strong edge $((u, v), t_B^i(u, v), f_B^i(u, v))$, $I_{(u,v)}^t \geq 0.5$ and $I_{(u,v)}^f \leq 0.5$.

Thus, for any two intersecting strong edges $((u, v), t_B^i(u, v), f_B^i(u, v))$ and $((w, x), t_B^j(w, x), f_B^j(w, x))$, $\frac{I_{(u,v)}^t+I_{(w,x)}^t}{2} \geq 0.5$ and $\frac{I_{(u,v)}^f+I_{(w,x)}^f}{2} \leq 0.5$ i.e. $\mathcal{I}_{P_1}^t \geq 0.5$ and $\mathcal{I}_{P_1}^f \leq 0.5$.

Similarly, $\mathcal{I}^t_{P_2} \geq 0.5$ and $\mathcal{I}^f_{P_2} \leq 0.5$. Then, $1+\mathcal{I}^t_{P_1}+\mathcal{I}^t_{P_2} \geq 2$ and $1+\mathcal{I}^f_{P_1}+\mathcal{I}^f_{P_2} \leq 2$. Therefore, $\mathcal{P}_t = \frac{1}{1+\mathcal{I}^t_{P_1}+\mathcal{I}^t_{P_2}} \leq 0.5$ and $\mathcal{P}_f = \frac{1}{1+\mathcal{I}^f_{P_1}+\mathcal{I}^f_{P_2}} \geq 0.5$.

This is a contradiction since $\mathcal{P}_t > 0.5$ and $\mathcal{P}_f < 0.5$.

Hence, the number of points of intersection between vague strong edges cannot be two. Clearly, if the number of point of intersection of vague strong edges increases, then the vague planarity value decreases. Similarly, if the number of point of intersection of vague edges is one, then the vague planarity value $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ is such that $\mathcal{P}_t > 0.5$ and $\mathcal{P}_f < 0.5$. Any vague planar graph without any crossing between edges has vague planarity value $\mathcal{P}$ where $\mathcal{P}_t > 0.5$ and $\mathcal{P}_f < 0.5$. Thus, we conclude that the maximum number of point of intersection between vague strong edges in G is one. □

Next, let us now state a fundamental theorem of vague planar graph.

Theorem 5.7. *Let $G = (V, A, B)$ be a vague planar graph with vague planarity value $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$. If $\mathcal{P}_t \geq 0.67$ and $\mathcal{P}_f \leq 0.33$, then G does not contain any point of intersection between two vague strong edges.*

Proof. If possible, let P be a point of intersection between two vague strong edges $((u, v), t^i_B(u, v), f^i_B(u, v))$ and $((w, x), t^j_B(w, x), f^j_B(w, x))$.

For any vague strong edge $((u, v), t^i_B(u, v), f^i_B(u, v))$, we have $I^t_{(u,v)} \geq 0.5$ and $I^f_{(u,v)} \leq 0.5$.

For the minimum value of $I^t_{(u,v)}$, $I^t_{(w,x)}$ and for the maximum value of $I^f_{(u,v)}$, $I^f_{(w,x)}$, $\mathcal{I}^t_P = 0.5$ and $\mathcal{I}^f_P = 0.5$. Then, $\mathcal{P}_t = \frac{1}{1+0.5} < 0.67$ and $\mathcal{P}_f = \frac{1}{1+0.5} > 0.33$, a contradiction. Hence, G does not contain any point of intersection between vague strong edges. □

This theorem motivated us to introduce a special type of vague planar graph called strong vague planar graphs whose vague planarity value $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ is such that $\mathcal{P}_t \geq 0.67$, $\mathcal{P}_f \leq 0.33$. If the vague planarity is $(1, 1)$, then the geometrical representation of vague planar graph is similar to the crisp planar graph. In the above theorem, it is shown that, if the vague planarity $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ with $\mathcal{P}_t \geq 0.67$ and $\mathcal{P}_f \leq 0.33$, then there is no crossing between vague strong edges. In this case, if there is any point of intersection between edges, then the intersection is between vague weak edge and any other edge. The significance of vague weak edge is less compared to the vague strong edges. Thus, strong vague planar graph is more significant. If the vague planarity value increases, then the geometrical structure of planar graph tends to crisp planar graph.

Definition of strong vague planar graph is given below.

Definition 5.8. A vague planar graph G is said to be strong vague planar graph if the vague planarity value $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ of the graph is such that $\mathcal{P}_t \geq 0.67$, $\mathcal{P}_f \leq 0.33$.

Thus, depending on vague planarity value, the vague planar graph is divided into two groups namely, strong vague planar graph and weak vague planar graph.

Strength of an edge has an important role to model some types of project. If the strength of an edge is very small, then the edge may be ignored to design a project. So, the edges with sufficient strengths are very useful to design such projects. These edges are called considerable edges which is defined below.

Definition 5.9. Let $G = (V, A, B)$ be a vague planar graph. Let $0 < c < 0.5$ be a rational number. An edge $((u, v), t_B(u, v), f_B(u, v))$ is said to be considerable edge if $\frac{t_B(u,v)}{\min\{t_A(u),t_A(v)\}} \geq c$ and $\frac{\max\{f_A(u),f_A(v)\}}{f_B(u,v)} \leq c$. If an edge is not considerable, then it is called non-considerable edge.

For a vague multigraph G, a multi-edge $((u, v), t^i_B(u, v), f^i_B(u, v))$ is said to be considerable edge if $\frac{t^i_B(u,v)}{\min\{t_A(u),t_A(v)\}} \geq c$ and $\frac{\max\{f_A(u),f_A(v)\}}{f^i_B(u,v)} \leq c$ for all $i = 1, 2, \ldots, p$.

If $\frac{t_B(u,v)}{\min\{t_A(u),t_A(v)\}} \geq c$ and $\frac{\max\{f_A(u),f_A(v)\}}{f_B(u,v)} \leq c$ for all edges (u, v) of a vague graph G, then the number c is said to be considerable number of the vague graph. Considerable number of a vague graph may not be unique.

Clearly, for a specific value of c, there is a set of considerable edges and for different values of c, one can obtain different sets of considerable edges. Actually, c is a pre-assigned number for a specific application. For example, a social network (people, organization, etc) is taken as a vague vertex and the relationship between them is represented by vague edge. The amount of relationship (within $[0, 1]$) is taken as true and false membership degree of the vague edge. If we choose $c = 0.25$ for this network, then we get a set of considerable edge, say $\mathcal{C}$. This set consists of a group of people who have some considerable amount of relationship. The number of point of intersection between considerable edges can be determined from the following theorem.

Theorem 5.10. *Let G be vague planar graph with vague planarity value $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ be such that $\mathcal{P}_t \geq 0.5$ and $\mathcal{P}_f \leq 0.5$ and considerable number c. Then the number of point of intersection between considerable edges in G is at most $[\frac{1}{c}]$ or $\frac{1}{c}$ according as $\frac{1}{c}$ is not an integer or an integer respectively.*

Proof. Let $G = (V, A, B)$ be a vague planar graph where

$$B = \{((u, v), t^i_B(u, v), f^i_B(u, v)), i = 1, 2, \ldots, p : (u, v) \in V \times V\}.$$

Let $0 < c < 0.5$ be the considerable number. For any considerable edge $((u, v), t^i_B(u, v), f^i_B(u, v))$, we have $t^i_B(u, v) \geq c \min\{t_A(u), t_A(v)\}$ and $f^i_B(u, v) \geq \frac{1}{c} \max\{f_A(u), f_A(v)\}$.

This implies that, $I^t_{(u,v)} \geq c$ and $I^f_{(u,v)} \leq c$.

Let $P_1, P_2, \ldots, P_k$ be the k-points of intersection between the considerable edges. Also let, P_1 be the point of intersection between the considerable edges $((u_1, v_1), t^i_B(u_1, v_1), f^i_B(u_1, v_1))$ and $((u_2, v_2), t^j_B(u_2, v_2), f^j_B(u_2, v_2))$.

Then $\mathcal{I}^t_{P_1} = \frac{I^t_{(u_1,v_1)}+I^t_{(u_2,v_2)}}{2} \geq c$ and $\mathcal{I}^f_{P_1} = \frac{I^f_{(u_1,v_1)}+I^f_{(u_2,v_2)}}{2} \leq c$.

So, $\sum_{i=1}^{k} \mathcal{I}^t_{P_i} \geq kc$ and $\sum_{i=1}^{k} \mathcal{I}^f_{P_i} \leq kc$.

Hence, $\mathcal{P}_t \leq \frac{1}{1+kc}$ and $\mathcal{P}_f \geq \frac{1}{1+kc}$.

This imply that $0.5 \leq \mathcal{P}_t \leq \frac{1}{1+kc}$ and $\frac{1}{1+kc} \leq \mathcal{P}_f \leq 0.5$

i.e. $0.5 \leq \mathcal{P}_t \leq \frac{1}{1+kc} \leq \mathcal{P}_f \leq 0.5$

i.e. $0.5 = \frac{1}{1+kc} = \mathcal{P}_t = \mathcal{P}_f$

i.e. $k = \frac{1}{c}$.

Hence the values of k are given by

$$k = \begin{cases} [\frac{1}{c}], & \text{if } \frac{1}{c} \text{ is not an integer,} \\ \frac{1}{c}, & \text{if } \frac{1}{c} \text{ is an integer.} \end{cases}$$

This completes the proof. □

We know that, the complete graph (crisp) with five vertices K_5 and the complete bipartite graph with six vertices $K_{3,3}$ cannot be drawn without crossings. Therefore, any graph (crisp) containing K_5 or $K_{3,3}$ as a subgraph is non-planar.

Theorem 5.11. *Any complete vague graph of five vertices K_5 or complete bipartite vague graph of six vertices $K_{3,3}$ are not strong vague planar graph.*

Proof. Let $G = (V, A, B)$ be a complete vague graph of five vertices where $V = \{u, v, w, x, y\}$ and $B = \{((u, v), t_B(u, v), f_B(u, v)) : (u, v) \in V \times V\}$. For all $u, v \in V$, we have $t_B(u, v) = \min\{t_A(u), t_A(v)\}$ and $f_B(u, v) = \max\{f_A(u), f_A(v)\}$.

By Theorem 5.5, we have the vague planarity value of a complete vague graph is $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ where $\mathcal{P}_t = \frac{1}{1+n_p}$ and $\mathcal{P}_f = \frac{1}{1+n_p}$, n_p being the number of point of intersection of the edges in G.

We know that the geometric representation of the underlying crisp graph of a vague complete graph of five vertices is non-planar and one point of intersection cannot be avoided for any representation. So, $\mathcal{P}_f = 0.5 > 0.33$. Hence G is not strong vague planar graph.

Similarly, it can be proved that the complete bipartite vague graph $K_{3,3}$ is not a strong vague planar graph. □

6. Faces of vague planar graph

Face of a vague planar graph is an important parameter. Face of a vague planar graph is a region bounded by vague edges. Every vague face is characterized by vague edges in its boundary. If all the edges in the boundary of vague face have true and false membership values 1 and 0 respectively, it becomes crisp face. If one such edges is removed or has true and false 0 and 1 respectively, the vague face does not exist. So, the existence of a vague face depends on the minimum value of strength of vague edges in its boundary. Vague face and its true and false membership values of a vague graph are defined below.

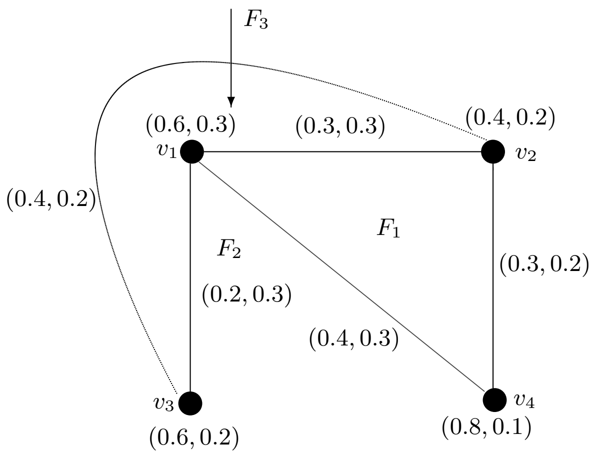

FIGURE 4. Example of vague planar graph with three vague faces

Definition 6.1. Let $G = (V, A, B)$ be a vague planar graph and

$$B = \{((u, v), t_B^i(u, v), f_B^i(u, v)), i = 1, 2, \ldots, p : (u, v) \in V \times V\}.$$

A vague face of G is a region bounded by the set of vague edges $E' \subseteq V \times V$, of a geometric representation of G. The strength of the face is (t_F, f_F) where $t_F = \min\{I^t_{(u,v)} : (u, v) \in E'\}$ and $f_F = \max\{I^f_{(u,v)} : (u, v) \in E'\}$.

Definition 6.2. A vague face is called strong vague face if $t_F > 0.5$ or $f_F < 0.5$ and weak vague face otherwise. Every vague planar graph has an infinite region which is called outer vague face. Other faces are called inner vague faces.

Example 6.3. Let us consider the vague planar graph as shown in the Fig. 4. Here, F_1, F_2, F_3 are three vague faces. The vague face F_1 is bounded by the edges $((v_1, v_2), 0.3, 0.3)$, $((v_2, v_4), 0.3, 0.2)$ and $((v_4, v_1), 0.4, 0.3)$ with strength $(0.67, 1)$.

Similarly, F_3 is a face bounded by the edges $((v_1, v_3), 0.2, 0.3)$, $((v_1, v_2), 0.3, 0.3)$ and $((v_2, v_3), 0.4, 0.2)$ with strength $(0.33, 1)$. F_2 is the outer face with strength $(0.33, 1)$. So, F_1 is strong vague face while F_2, F_3 are weak vague faces.

7. Vague dual graph

In this section, we introduce the concept of dual of a vague planar graph. In vague dual graph, vertices are corresponding to the strong vague faces and each edge in dual graph between two vertices is corresponding to each edge in the boundary between two vague faces of vague planar graph. The definition is given below.

Definition 7.1. Let $G = (V, A, B)$ be a vague planar graph where $B = \{((u, v), t_B^i(u, v), f_B^i(u, v)), i = 1, 2, \ldots, p : (u, v) \in V \times V\}$. Let $F_1, F_2, \ldots, F_k$

be the strong vague faces of G. The vague dual graph of G is a vague planar graph $G_1 = (V_1, A_1, B_1)$ where $V_1 = \{x_j, j = 1, 2, \ldots, k\}$, the vertex x_j of G_1 is correspond to the face F_j of G.

The true and false membership values of vertices are given by the mapping $A_1 = (t_{A_1}, f_{A_1})\colon V_1 \to [0,1] \times [0,1]$ such that $t_{A_1}(x_j) = \max\{t^i(u,v), i = 1, 2, \ldots, l : (u,v)$ is an edge of the boundary of the vague face $F_i\}$, and $f_{A_1}(x_j) = \min\{f^i(u,v), i = 1, 2, \ldots, l : (u,v)$ is an edge of the boundary of the vague face $F_i\}$.

There may exist more than one common edge between two vague faces F_i and F_j of G. Thus there may be more than one edge between two vertices x_i and x_j in the vague dual graph G_1. Let $t^l_B(x_i, x_j)$ and $f^l_B(x_i, x_j)$ denote the true and false membership values of the l-th edge between x_i and x_j. The true and false membership values of the edges of the vague dual graph are given by $t^l_{B_1}(x_i, x_j) = t^l_B(u,v)$, $f^l_{B_1}(x_i, x_j) = f^l_B(u,v)$ where (u,v) is a common edge between two vague faces F_i and F_j and $l = 1, 2, \ldots, t$; t being the number of common edges in the boundary between F_i and F_j or the number the edges between x_i and x_j.

If there is any strong pendant edge in the vague planar graph, then there will be a self-loop in G_1 corresponding to this pendant edge. The edge true and false membership value of the self-loop is equal to the true and false membership value of the pendant edge. Vague dual graph of vague planar graph does not contain any point of intersection of edges for a certain representation, so it is a vague planar graph with vague planarity value $(1, 1)$.

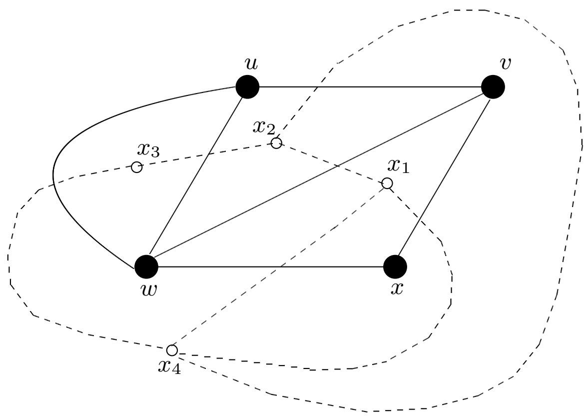

FIGURE 5. Vague planar graph and it's vague dual graph

Next, we give an example of a vague dual graph of a vague planar graph which are shown in Fig. 5. We assume that the black filled circles and the lines represent the vertices and edges of the vague planar graph while the empty circles and the dotted lines represent the vertices and edges of vague dual graph corresponding to the vague planar graph.

Example 7.2. Let us now consider a vague planar graph $G = (V, A, B)$ as shown in Fig. 5, where $V = \{u, v, w, x\}$,
$A = \{(u, 0.7, 0.1), (v, 0.6, 0.2), (w, 0.8, 0.1), (x, 0.9, 0.1)\}$, and
$B = \{((u, v), 0.6, 0.2), ((v, x), 0.5, 0.25), ((w, x), 0.7, 0.2), ((u, w), 0.7, 0.1), ((u, w), 0.6, 0.15), ((v, w), 0.5, 0.22)\}$.
The vague planar graph has the following faces:

(i) the vague face F_1 is bounded by $((v, w), 0.5, 0.22)$, $((w, x), 0.7, 0.2)$, $((v, x), 0.5, 0.25)$,
(ii) the vague face F_2 is bounded by $((u, v), 0.6, 0.2)$, $((u, w), 0.7, 0.1)$, $((v, w), 0.5, 0.22)$,
(iii) the vague face F_3 is bounded by $((u, w), 0.7, 0.1)$, $((u, w), 0.6, 0.15)$, and
(iv) the outer vague face F_4 is surrounded by $((u, v), 0.6, 0.2)$, $((v, x), 0.5, 0.25)$, $((w, x), 0.7, 0.2)$, $((u, w), 0.6, 0.15)$.

Since all the faces are strong vague faces, for each strong vague faces, we consider a vertex for the vague dual graph. Thus the vertex set V_1 of the vague dual graph is $V_1 = \{x_1, x_2, x_3, x_4\}$, where the vertex x_i corresponds to the strong vague face F_i, $i = 1, 2, 3, 4$.

Now, the true and false membership values of the vertex set V_1 are calculated below:
$t_{A_1}(x_1) = \max\{0.5, 0.7, 0.5\} = 0.7$, $f_{A_1}(x_1) = \min\{0.22, 0.2, 0.25\} = 0.2$,
$t_{A_1}(x_2) = \max\{0.6, 0.7, 0.5\} = 0.7$, $f_{A_1}(x_2) = \min\{0.2, 0.1, 0.22\} = 0.1$,
$t_{A_1}(x_3) = \max\{0.7, 0.6\} = 0.7$, $f_{A_1}(x_3) = \min\{0.1, 0.15\} = 0.1$,
$t_{A_1}(x_4) = \max\{0.6, 0.5, 0.7, 0.6\} = 0.7$, $f_{A_1}(x_4) = \min\{0.2, 0.25, 0.2, 0.15\} = 0.15$.

There are two common edges (w, x) and (v, x) between the faces F_1 and F_4 in G. Therefore, there exist two edges between x_1 and x_4 in the vague dual graph. The true and false membership values of these edges are given by
$t_{B_1}(x_1, x_4) = t_B(w, x) = 0.7$, $f_{B_1}(x_1, x_4) = f_B(w, x) = 0.2$,
$t_{B_1}(x_1, x_4) = t_B(v, x) = 0.5$, $f_{B_1}(x_1, x_4) = f_B(v, x) = 0.25$.
The true and false membership values of other edges of the vague dual graph are calculated as
$t_{B_1}(x_1, x_2) = t_B(v, w) = 0.5$, $f_{B_1}(x_1, x_2) = f_B(v, w) = 0.22$,
$t_{B_1}(x_2, x_3) = t_B(u, w) = 0.7$, $f_{B_1}(x_2, x_3) = t_B(u, w)$=0.1,
$t_{B_1}(x_2, x_4) = t_B(u, v)$=0.6, $f_{B_1}(x_2, x_4) = t_B(u, v)$=0.2,
$t_{B_1}(x_3, x_4) = t_B(u, w)$=0.6, $f_{B_1}(x_3, x_4) = t_B(u, w)$=0.15.
Thus, the edge set of the vague dual graph is
$B_1 = \{((x_1, x_2), 0.5, 0.22), ((x_2, x_3), 0.7, 0.1), ((x_2, x_4), 0.6, 0.2), ((x_3, x_4), 0.6, 0.15), ((x_1, x_4), 0.7, 0.2), ((x_1, x_4), 0.6, 0.2)\}$.

Now, we have the following observations.

Theorem 7.3. *Let $G = (V, A, B)$ be a vague planar graph whose number of vertices, number of edges and number of strong vague faces denoted by n, e and f respectively. Let G_1 be the vague dual graph of G. Then*

(i) *the number of vertices of* G_1 *is equal to* f,
(ii) *the number of edges of* G_1 *is equal to* e,
(iii) *the number of vague faces of* G_1 *is equal to* n.

Proof. Proof of (*i*), (*ii*) and (*iii*) follows from the definition of vague dual graph. □

The number of strong vague faces in vague dual graph of a vague planar graph is always less than or equal to the number of vertices of vague planar graph, since all faces of vague dual graph may not be strong vague faces.

Theorem 7.4. *Let* $G = (V, A, B)$ *be a vague planar graph having no weak vague edges and* G_1 *be the vague dual graph of* G*. Then the true and false membership values of the vague edges of* G_1 *are equal to the true and false membership values of the vague planar graph of* G.

Proof. Obvious. □

8. Application of vague planar graphs

We consider the above network consisting of nine important cities (vertices) in a country connected by railway lines (see Fig.6). Now, we consider the crowdness of the railway lines connecting cities. Crowdness of a railway line is a vague quantity. The amount of crowdness depends on the decision makers mentality, habits, natures, etc., i.e. completely depends on the decision makers. The measurement of crowdness using point based membership as in fuzzy sets is a difficult one for the decision maker. So, we consider the amount of crowdness as an interval based membership in vague sets, since it is more expressive in capturing vagueness of data.

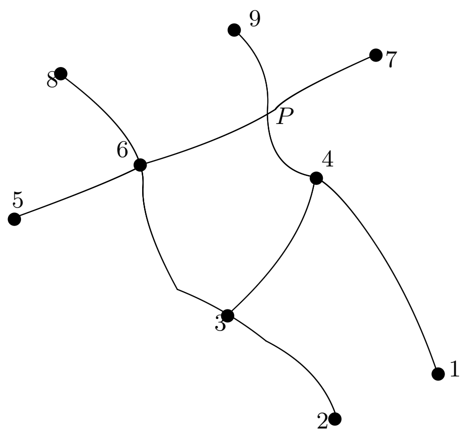

Figure 6. A network of railway lines connecting cities.

We assume that, the crowdness of the railways line (i, j) connecting the cities i and j as follows:
This is very clear that if the crossing of railways increases, the possibility of crowdness increases. So, the traveling time will increase significantly. Thus,

Table 1. The crowdness of the network of Fig. 6

Railwaylines	(1,4)	(2,3)	(3,4)	(3,6)	(4,9)	(5,6)
Crowdness	(0.6,0.3)	(0.5,0.2)	(0.4,0.5)	(0.4,0.2)	(0.8,0.1)	(0.6,0.2)
Railwaylines	(6,7)	(6,8)	(7,6)			
Crowdness	(0.7,0.2)	(0.7,0.3)	(0.3,0.2)			

it is natural to construct the railway lines in such a way that the number of crossing decreases, i.e. the vague planarity value increases. This is why the measurement of vague planarity value is important.

In the network of Fig. 6, there are only one crossing between the railway lines $(4,9)$ and $(6,7)$. To model the given network as a vague planar graph, we consider the true and false membership values of each vertices as 1 and 0 respectively. Then we have the following.

$I^t_{(4,9)} = 0.8$, $I^f_{(4,9)} = 0$, $I^t_{(6,7)} = 0.7$ and $I^f_{(6,7)} = 0$.

Therefore, the intersecting value at the point P, the intersection between the railway lines $(4,9)$ and $(6,7)$ is $\mathcal{I}_P = (\frac{I^t_{(4,9)}+I^t_{(6,7)}}{2}, \frac{I^f_{(4,9)}+I^f_{(6,7)}}{2}) = (0.75, 0)$.

So, the vague planarity value $\mathcal{P} = (\mathcal{P}_t, \mathcal{P}_f)$ is given by $\mathcal{P}_t = \frac{1}{1+0.75} = 0.57$, $\mathcal{P}_f = 1$. Therefore, the vague planarity value of the network of Fig. 6 is $(0.57, 1)$, which is far from vague planarity $(0.67, 0.33)$. Hence, it is likely to be crowded due to the crossing of railway lines.

9. Conclusions

Planarity is important in connecting the wire lines, gas lines, water lines, printed circuit designs, etc. But, sometimes little crossing may be allowed for such design. These graph theoretic problems may be vague or uncertain in some aspects. It is quiet natural to deal with the vagueness using the concepts of vague sets compared to fuzzy sets. Therefore, the concept of vague sets is applied to multigraphs and planar graphs. The edges of a vague multigraph may be vague weak or vague strong. Using the concept of vague weak edge, vague planar graph is introduced where an edge may intersect with other edges. This facility is not available in a crisp planar graph. Since the role of vague weak edge is not significant, therefore the intersection between a vague weak edge with any edge is less important. This motivates us to allow the intersection of edges in vague planar graph. We define a new term called vague planarity value of a vague planar graph. If the vague planarity value of a vague graph is $(1,1)$, then no edges crosses other. This leads to the crisp planar graph. Hence, the vague planarity value measures the planarity of a vague graph. Strong vague planar graphs are introduced. Also face of a vague graph is defined. Many new theorems of vague planar graph have been proved in this paper. These theories will be helpful to improve algorithms in different fields including computer vision, image segmentation, etc.

Acknowledgements. Financial support for the first author offered under the Innovative Research Scheme, UGC, New Delhi, India (Ref. No.VU/Innovative/ Sc/15/2015) is thankfully acknowledged. We are highly thankful to the editor in chief and reviewers for their valuable comments and suggestions to improve the paper.

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Received July 8, 2016.

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