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ON OSCILLATING SOLUTIONS OF DIFFERENTIAL EQUATIONS OF FIRST ORDER WITH RETARDED ARGUMENT WITH EXPONENTIAL NONLINEARITY

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Abstract. We obtain some estimations for solutions of nonlinear delay differential equation.

In this work, we study the behavior of oscillating solutions of differential equations of the first order lag with a power-law nonlinearity. The abundance of applications is stimulating a rapid development of the theory of differential equations with deviating argument. The solutions of such equations have special properties which do not have corresponding differential equations without deviating argument [6], [7], [3], [4], [5], [1], [2].

It is shown that if the value of the $\Phi_0 = \sup\limits_{(-\infty,A]} |\varphi(x)|$, where $\varphi(x)$ — initial function, A — the starting point, small enough, then the oscillating solutions of the considered equation is limited, and when exponent $\alpha > 1$ are dampted, if kernel $r(x,s)$ does not decrease on s. Thus, the greater the delay Δ_0, the less Φ_0.

Considers the equation

$$y'(x) = \int_0^\infty y^\alpha(x-s)dr(x,s) \quad (A \leqslant x < \infty), \tag{1}$$

where the number $\alpha > 0$, $(-1)^\alpha = -1$. The integration is on s for a fixed x, the integral is understood in sense of Stieltjes. The kernel $r(x,s)$ is defined when $x \in [A,\infty)$, $s \in [0,\infty)$ and ensures the existence and uniqueness of the solution $y(x)$ of the equation (1) on $[A,\infty)$ with the initial condition $y(x) = \varphi(x)$ $(-\infty, A]$, where $\varphi(x)$ — continuous on $(-\infty, A]$ function (we use description, introduced in [6, § 6]). So, for example, if the kernel $r(x,s)$ satisfies the conditions imposed on the kernels in [6, pp. 12, 60–63].

For each fixed $x \in [A,\infty)$ function $r(x,s)$ is constant for sufficiently large s. The value of $r(x,s)$ under such s define as $r(x,\infty)$.

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Supremum of those s, for which $r(x,s) \neq r(x,\infty)$, define as $\Delta(x)$. Let

$$\Delta_0 = \sup_{[A,\infty)} \Delta(x), \quad M_0 = \sup_{[A,\infty)} \bigvee_{s=0}^{\infty} r(x,s),$$

$$\Phi_0 = \sup_{(-\infty,A]} |\varphi(x)| < \infty, \quad 0 < \Delta_0 < \infty, \quad 0 < M_0 < \infty.$$

If $y(x)$ — oscillating solution of equation (1), the kernel $r(x,s)$ not decreasing on the variable s for each fixed x and $(-1)^\alpha = -1$, then on any interval of length Δ_0 it at least once changes its sign.

Theorem 1. *Let the kernel $r(x,s)$ is a non-decreasing function on s for each fixed x, $\alpha > 1$, $\Delta_0 M_0 \Phi_0^{\alpha-1} \leqslant 1$ $y(x)$ — the oscillating solution of the equation (1). Then*

$$|y(x)| \leqslant \left(\frac{\Delta_0 M_0}{2}\right)^{\frac{1}{1-\alpha}} \left(\left(\frac{\Delta_0 M_0}{2}\right)^{\frac{1}{\alpha-1}} \Phi_0\right)^{\alpha^{\frac{x-A-\Delta_0}{2\Delta_0}}} \quad (A \leqslant x < \infty). \tag{2}$$

Proof. Define $A_0 = A - \Delta_0$, $A_k = A + (2k-1)\Delta_0$, $l_k = [A_{k-1}, A_k]$ $(k = 1, 2, \ldots)$. Pre-prove that $\max\limits_{[A,A+\Delta_0]} |y(x)| \leqslant \Phi_0$.

If $\varphi(A) = 0$, then

$$\max_{[A,A+\Delta_0]} |y(x)| \leqslant \frac{\Delta_0 M_0}{2} \Phi_0^\alpha < \Phi_0.$$

Indeed, let $[a,b]$, $(a \geqslant A)$ — the first half-cycle of the solution $y(x)$, located to the right of point A, that is $y(a) = y(b) = 0$, $y(x) \neq 0$ $(a < x < b)$.

Then $b - a < \Delta_0$. For definiteness, we assume it is negative, that is $y(x) < 0$ $(a < x < b)$. Move the start point to the point a and denote by T any point of (a,b), in which $y(x)$ takes the least value on $[a,b]$.

From equation (1) we have

$$y(T) = y(T) - y(b) = (T-b)y'(x_1) = (T-b)\int_0^\infty y^\alpha(x_1 - s)\,dr(x_1,s) < 0 \quad (T < x_1 < b),$$

from which

$$\int_0^\infty y^\alpha(x_1 - s)\,dr(x_1,s) > 0.$$

As $[a,b]$ $(a \geqslant A)$ — the first half-cycle of the solution $y(x)$, then $y(x) \equiv 0$ $(A \leqslant x \leqslant a)$. And as $y(x) < 0$ $(a < x < b)$, then $|y(T)| \leqslant (b-T)\Phi_0^\alpha M_0$. So

$$b - T \geqslant \frac{|y(T)|}{\Phi_0^\alpha M_0}.$$

Similarly get

$$T - a \geqslant \frac{|y(T)|}{\Phi_0^\alpha M_0}.$$

In the end we have

$$\Delta_0 \geqslant b-a=(b-T)+(T-a) \geqslant \frac{2|y(T)|}{\Phi_0^\alpha M_0},$$

that is

$$|y(T)| < \frac{\Delta_0 M_0}{2}\Phi_0^\alpha < \Phi_0.$$

Let $\varphi(A) \neq 0$. For definiteness, put $\varphi(A) < 0$. Because on the interval $[A, A+\Delta_0]$ there is at least one zero of solution $y(x)$, let $y(x)$ for the first time becomes zero at $[A, A+\Delta_0]$ at the point b. Then $y(x) < 0$ on the interval $[A, b)$.

Denote by T_1 any point of $[A, b)$, in which $y(x)$ on $[A, b]$ reaches the smallest value. Then

$$|y(T_1)| \leqslant (b-T_1)\Phi_0^\alpha M_0 \leqslant \Delta_0 M_0 \Phi_0^\alpha \leqslant \Phi_0.$$

Moving the start point to the point b, we have

$$\max_{[A,A+\Delta_0]}|y(x)| \leqslant \Phi_0.$$

It follows that $\max\limits_{l_1} |y(x)| \leqslant \Phi_0$,

$$\max_{l_{k+1}} |y(x)| \leqslant \frac{\Delta_0 M_0}{2} \max_{l_k} |y^\alpha(x)|, \quad (k=1,\, 2\dots).$$

The case $\varphi(A) > 0$ analyzed similarly.

For $k = 1,\, 2, \dots$ we have

$$\max_{l_{k+1}} |y(x)| \leqslant \left(\frac{\Delta_0 M_0}{2}\right)^{1+\alpha+\dots+\alpha^{k-1}} \Phi_0^{\alpha^k} = \left(\frac{\Delta_0 M_0}{2}\right)^{\frac{\alpha^k-1}{\alpha-1}} \Phi_0^{\alpha^k} =$$

$$= \left(\frac{\Delta_0 M_0}{2}\right)^{\frac{1}{1-\alpha}} \left(\left(\frac{\Delta_0 M_0}{2}\right)^{\frac{1}{\alpha-1}} \Phi_0\right)^{\alpha^k}.$$

From $A+(2k-1)\Delta_0 \leqslant x \leqslant A+(2k+1)\Delta_0$ we find $k \geqslant \frac{x-A-\Delta_0}{2\Delta_0}$. And as

$$\left(\frac{\Delta_0 M_0}{2}\right)^{\frac{1}{\alpha-1}} \Phi_0 < 1,$$

then

$$|y(x)| \leqslant \left(\frac{\Delta_0 M_0}{2}\right)^{\frac{1}{1-\alpha}} \left(\left(\frac{\Delta_0 M_0}{2}\right)^{\frac{1}{\alpha-1}} \Phi_0\right)^{\alpha^{\frac{x-A-\Delta_0}{2\Delta_0}}}$$

for $A \leqslant x < \infty$. □

Thus, the oscillating solutions dampted, if $\alpha > 1$, $\Delta_0 M_0 \Phi_0^{\alpha-1} \leqslant 1$.

Now let $0 < \alpha < 1$. In this case $\Delta_0 M_0 < 2$ it is possible to prove the boundedness of the solution $y(x)$ of equation (1) at $[A, \infty)$, if it changes its sign on any interval of length Δ_0. It should be emphasized that this property holds for non-monotonic kernel $r(x,s)$, satisfying the conditions imposed on the kernel in [1], § 1.

Theorem 2. *Let* $0 < \alpha < 1$, $\varphi(A) = 0$,

$$\Delta_0 M_0 < 2 \tag{3}$$

and the solution $y(x)$ $(A \leqslant x < B)$ *changes its sign on any interval* $[a, a+\Delta_0]$, $A \leqslant a \leqslant a+\Delta_0$, $B - A > 2\Delta_0$. *Then*

1) $\max\limits_{[A,A+\Delta_0]} |y(x)| \leqslant \frac{\Delta_0 M_0}{2} \Phi_0^\alpha < \Phi_0$ *when* $\Phi_0 \geqslant 1$,

2) $\max\limits_{[A,A+\Delta_0]} |y(x)| < \frac{\Delta_0 M_0}{2}$ *when* $\Phi_0 < 1$.

Proof. If $y(x) \equiv 0$ $(A \leqslant x \leqslant A+\Delta_0)$, the theorem is obvious.

Let $x = T$ $(A < T \leqslant A+\Delta_0)$ — any point of the interval $[A, A+\Delta_0]$, in which $|y(x)|$ reaches the highest value at $[A, A+\Delta_0]$. Let $B_1 < T$ and $B_2 > T$ — coming to the point T zeros of solution $y(x)$. Such zeroes exist by the condition of the theorem, and $B_1 \geqslant A$, $T < B_2 \leqslant B_1 + \Delta_0$.

From equation (1) we find that

$$|y(T)| \leqslant (T - B_1) \max_{[A-\Delta_0, B_2]} |y^\alpha(x)|\, M_0.$$

Similarly

$$|y(T)| \leqslant (B_2 - T) \max_{[A-\Delta_0, B_2]} |y^\alpha(x)|\, M_0.$$

Since $B_2 - B_1 \leqslant \Delta_0$, then

$$\Delta_0 \geqslant (B_2 - T) + (T - B_1) \geqslant \frac{2|y(T)|}{M_0 \max\limits_{[A-\Delta_0, B_2]} |y^\alpha(x)|}. \tag{4}$$

Here two cases are possible.

1) $\max\limits_{[A-\Delta_0, B_2]} |y(x)| \geqslant 1$. Then $\max\limits_{[A-\Delta_0, B_2]} |y^\alpha(x)| \leqslant \max\limits_{[A-\Delta_0, B_2]} |y(x)|$ and from (4) we find

$$\Delta_0 \geqslant \frac{2|y(T)|}{M_0 \max\limits_{[A-\Delta_0, B_2]} |y^\alpha(x)|} \geqslant \frac{2|y(T)|}{M_0 \max\limits_{[A-\Delta_0, B_2]} |y(x)|}.$$

Hence

$$|y(T)| \leqslant \frac{\Delta_0 M_0}{2} \max_{[A-\Delta_0, B_2]} |y(x)| < \max_{[A-\Delta_0, B_2]} |y(x)|.$$

At the point T the function $|y(x)|$ reaches the highest value at $[A, A+\Delta_0]$. Since $\max\limits_{[A-\Delta_0, A]} |y(x)| = \Phi_0$, $|y(T)| < \max\limits_{[A-\Delta_0, B_2]} |y(x)|$, then $\max\limits_{[A-\Delta_0, B_2]} |y(x)| = \Phi_0$.

Substituting this in (4), we obtain the first assertion of the theorem.

2) $\max\limits_{[A-\Delta_0, B_2]} |y(x)| < 1$. From (4) we immediately have

$$|y(T)| \leqslant \frac{\Delta_0 M_0}{2} \max_{[A-\Delta_0, B_2]} |y^\alpha(x)| < \frac{\Delta_0 M_0}{2}.$$

□

Theorem 2 allows to get the following conclusions. If a — zero of solution $y(x)$ and $\max\limits_{[a-\Delta_0,a]}|y(x)| \geqslant 1$, then realization of the conditions of the theorem 2 the solution $y(x)$ satisfies the inequality

$$\max_{[a,a+\Delta_0]}|y(x)| < \max_{[a-\Delta_0,a]}|y(x)|.$$

This inequality can be disrupted only when $\max\limits_{[a-\Delta_0,a]}|y(x)| < 1$, but then the solution $y(x)$ never comes out of the band, limited straight lines $y = \pm\frac{\Delta_0 M_0}{2}$.

If the kernel $r(x,s)$ is a monotonic function of s for each fixed $x \in [A,\infty)$, the condition (3) is it possible to allow the equal sign, the assertion 1) of theorem 2 is strict.

In theorems 1 and 2 we not consider the case $\alpha = 1$. The corresponding equation was investigated in detail in [1]. In particular it is proved that if $\Delta_0 M_0 < 2$ and the solution $y(x)$ changes its sign on any interval of length Δ_0, when $\alpha = 1$ a true evaluation

$$|y(x)| \leqslant \left(\frac{2}{\Delta_0 M_0}\right)^{\frac{1}{2}} \sup_{[-\infty,A+\Delta_0]}|y(x)| \left(\frac{\Delta_0 M_0}{2}\right)^{\frac{x-A}{2\Delta_0}} \quad (A \leqslant x < \infty).$$

Note that equation (1) with monotone kernel was considered in [3], and with non-monotone kernel — in [4].

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