Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
34 (2023), 65–77.
www.emis.de/journals
ISSN 1786-0091

PUTNAM'S INEQUALITY FOR QUASI-$*$-CLASS A OPERATORS

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Abstract. An operator $T \in \mathscr{B}(\mathscr{H})$ is called quasi-$*$-class (A, k) (abbreviation, $T \in \mathcal{Q}^*(\mathcal{A}, k)$) if $T^{*k}(|T^2| - |T^*|^2)T^k \geq 0$ for a positive integer k, which is a generalization of $*$-class A. In this paper, firstly we consider some spectral properties of quasi-$*$-class (A, k) operators; it is shown that if $T \in \mathcal{Q}^*(\mathcal{A}, k)$, then the nonzero points of its point spectrum and the joint point spectrum are identical, the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal and the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical. Also, we consider the Putnam's inequality for quasi-$*$-class (A, k) operators. Moreover, we prove that two quasisimilar quasi-$*$-class (A, k) operators have equal essential spectra.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be separable complex Hilbert spaces, and let $\mathscr{B}(\mathscr{H}, \mathscr{K})$ denote the algebra of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$. When $\mathscr{H} = \mathscr{K}$, we write $\mathscr{B}(\mathscr{H})$ for $\mathscr{B}(\mathscr{H}, \mathscr{H})$. An operator $T \in \mathscr{B}(\mathscr{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and U is partial isometry satisfying $\ker(U) = \ker(T) = \ker(|T|)$ and $\ker(U) = \ker(T^*)$. Recall [2, 4, 11] that an operator T is p-hyponormal if $|T|^{2p} \geq |T^*|^{2p}$ for $p \in (0, 1]$, T is called paranormal if $\|T^2x\| \geq \|Tx\|^2$ for all unit vector $x \in \mathscr{H}$, T is called normaloid if $\|T\| = r(T)$, the spectral radius of T. Following [15], we say that $T \in \mathscr{B}(\mathscr{H})$ belongs to class A if $|T^2| \geq |T|^2$. According to [12], we say that $T \in \mathscr{B}(\mathscr{H})$ is a $*$-class A (abbreviation, $T \in \mathcal{A}^*$) if $|T^2| \geq |T^*|^2$ and T is said to be $*$-paranormal if $\|T^*x\|^2 \leq \|T^2x\|$ for every unit vector $x \in \mathscr{H}$. Following [18], we say that $T \in \mathscr{B}(\mathscr{H})$ is a quasi-class A if $T^*|T^2|T \geq T^*|T|^2T$. We introduce a new class of operators:

2020 *Mathematics Subject Classification.* 47A55, 47A10, 47A11.

Key words and phrases. Riesz idempotent, tensor product, $*$-class A, quasi-$*$-class A.

Definition 1. We say that an operator $T \in \mathscr{B}(\mathscr{H})$ is a quasi-$*$-class (A, k) (abbreviate $\mathcal{Q}^*(\mathcal{A}, k)$) if

$$T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k,$$

where k is a positive integer.

Throughout this paper, we shall denote the spectrum, the point spectrum and the isolated points of the spectrum of $T \in \mathscr{B}(\mathscr{H})$ by $\sigma(T), \sigma_p(T)$ and iso$\sigma(T)$, respectively. The range and the kernel of $T \in \mathscr{B}(\mathscr{H})$ will be denoted by $\Re(T)$ and $\ker(T)$, respectively. We shall denote the set of all complex numbers and the complex conjugate of a complex number λ by $\mathbb{C}$ and $\overline{\lambda}$, respectively. The numerical range of an operator S will be denoted by $W(S)$. The closure of a set S will be denoted by $\overline{S}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$.

In this paper, firstly we consider some spectral properties of quasi-$*$-class (A, k) operators; it is shown that if T is a quasi-$*$-class (A, k) operator for a positive integer k, then the nonzero points of its point spectrum and joint point spectrum are identical; furthermore, the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal; the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical. Secondly, we show that Putnam's theorems hold for quasi-$*$-class (A, k) operator.

2. Some properties of quasi-$*$-class A operators

We recall the following result which summarizes some basic properties of quasi-$*$-class (A, k) operators.

Theorem 1. [34] *Let $T \in \mathcal{Q}^*(\mathcal{A}, k)$ such that T does not have a dense range. Then*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathscr{H} = \overline{\Re(T^k)} \oplus \ker(T^{*k}),$$

where $T_1 = T|_{\overline{\Re(T^k)}}$ is the restriction of T to $\overline{\Re(T^k)}$, and $T_1 \in \mathcal{A}^*$. Moreover, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in \mathscr{H}$ such that $(T - \lambda)x = 0$. If in addition, $(T^* - \overline{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. Clearly, $\sigma_p(T) \subseteq \sigma_{jp}(T)$. In general, $\sigma_p(T) \neq \sigma_{jp}(T)$.

In [44], Xia showed that if T is a semi-hyponormal operator, then $\sigma_p(T) = \sigma_{jp}(T)$; Tanahashi extended this result to log-hyponormal operators in [39]. Aluthge [3] showed that if T is w-hyponormal, then the nonzero points of $\sigma_p(T)$ and $\sigma_{jp}(T)$ are identical; Uchiyama extended this result to class A operators in [41]. In the following, we will point out that if T is a quasi-$*$-class (A, k) operator for a positive integer k, then the nonzero points of $\sigma_{jp}(T)$ and $\sigma_p(T)$ are also identical and the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal.

Theorem 2. *Let $T \in \mathscr{B}(\mathscr{H})$ be a $\mathcal{Q}^*(\mathcal{A},k)$ operator and $\mathscr{M}$ be its invariant subspace. Then the restriction $T|_{\mathscr{M}}$ of T to $\mathscr{M}$ is also a $\mathcal{Q}^*(\mathcal{A},k)$ operator.*

Proof. Decompose

$$T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix} \text{ on } \mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}.$$

Let Q be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. Then

$$|S^*|^2 \leq (Q|T^*|^2 Q)|_{\mathscr{M}}$$

and

$$|S^2| = (Q|T^2|^2 Q)^{\frac{1}{2}}|_{\mathscr{M}} \geq (Q|T^2|Q)|_{\mathscr{M}}.$$

Let $x \in \mathscr{M}$. Then

$$\begin{aligned} \left\langle S^{*k}|S^*|^2 S^k x, x\right\rangle &\leq \left\langle S^{*k}(Q|T^*|^2 Q)|_{\mathscr{M}} S^k x, x\right\rangle \\ &= \left\langle |T^*|^2 T^k x, T^k x\right\rangle \leq \left\langle |T^2| T^k x, T^k x\right\rangle \\ &= \left\langle S^{*k}(Q|T^2|Q)|_{\mathscr{M}} S^k x, x\right\rangle \leq \left\langle S^{*k}|S^2|S^k x, x\right\rangle. \end{aligned}$$

□

Lemma 1. *Let $T \in \mathscr{B}(\mathscr{H})$ be a $*$-class A. Let $\lambda \in \mathbb{C}$. Assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$.*

Proof. We consider two cases:

Case I. ($\lambda = 0$): Since T is a $*$-class A, T is normaloid. Therefore $T = 0$.

Case II. ($\lambda \neq 0$): Here T is invertible, and since T is a $*$-class A, we see that T^{-1} is also a $*$-class A. Therefore T^{-1} is normaloid. On the other hand, $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. It follows that T is convexoid, so the numerical range $W(T) = \{\lambda\}$. Therefore $T = \lambda$. □

Lemma 2. *Let $T \in \mathcal{Q}^*(\mathcal{A},k)$ and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$ if $\lambda \neq 0$ and $T^{k+1} = 0$ if $\lambda = 0$.*

Proof. If T^k has dense range, then T is a $*$-class A. So, the result follows from Lemma 1. If the range of T^k is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathscr{H} = \overline{\Re(T^k)} \oplus \ker(T^{*k}),$$

where $T_1 = T|_{\overline{\Re(T^k)}}$ is a $*$-class A, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$. In this case $\lambda = 0$. Hence $T_1 = 0$ by Lemma 1. Thus

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2 T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

□

Theorem 3. *Let $T \in \mathcal{Q}^*(\mathcal{A},k)$. Then the following assertions hold.*

(a) *If $\mathscr{M}$ is an invariant subspace of T and $T|_{\mathscr{M}}$ is an injective normal operator, then $\mathscr{M}$ reduces T.*

(b) *If* $(T-\lambda)x=0$ *and* $\lambda\neq 0$, *then* $(T-\lambda)^*x=0$.

Proof. (a) Decompose T into $T=\begin{pmatrix} S & A \\ 0 & B \end{pmatrix}$ on $\mathscr{H}=\mathscr{M}\oplus\mathscr{M}^\perp$ and let $S=T|_{\mathscr{M}}$ be an injective normal operator. Let Q be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. Since $\ker(S)=\ker(S^*)=\{0\}$, we have

$$\mathscr{M}=\overline{\Re(S)}=\overline{\Re(S^k)}\subset\overline{\Re(T^k)}.$$

Then

$$\begin{pmatrix} |S^*|^2 & 0 \\ 0 & 0 \end{pmatrix}\leq Q|T^*|^2Q\leq Q|T^2|Q\leq (Q|T^2|^2Q)^{\frac{1}{2}}=\begin{pmatrix} |S^2| & 0 \\ 0 & 0 \end{pmatrix}$$

by Hansen's inequality. Since S is normal, we can write

$$|T^2|=\begin{pmatrix} |S|^2 & C \\ C^* & D \end{pmatrix}.$$

Then

$$\begin{pmatrix} |S|^4 & 0 \\ 0 & 0 \end{pmatrix}=QT^*T^*TTQ=Q|T^2||T^2|Q=\begin{pmatrix} |S|^4+CC^* & 0 \\ 0 & 0 \end{pmatrix}$$

and hence $C=0$. Thus

$$\begin{aligned}\begin{pmatrix} |S|^4 & 0 \\ 0 & D \end{pmatrix}&=|T^2|^2=T^*T^*TT\\ &=\begin{pmatrix} S^*S^*SS & S^*S^*(SA+AB) \\ (A^*S^*+B^*A^*)SS & (A^*S^*+B^*A^*)(SA+AB)+B^*B^*BB \end{pmatrix}.\end{aligned}$$

Since S is an injective normal operator, $SA+AB=0$ and $D=|B^2|$. If $k\geq 1$, then

$$\begin{aligned}0&\leq T^{*k}(|T^2|-|T^*|^2)T^k\\ &=\begin{pmatrix} -S^{*k}|A^*|^2S^k & Y \\ Y^* & X+B^{*k}(|B^2|-|B^*|^2)B^k \end{pmatrix}.\end{aligned}$$

Thus $A=0$.

(b) Let $\mathscr{M}=span\,\{x\}$. Then $T|_{\mathscr{M}}=\lambda$ and $T|_{\mathscr{M}}$ is an injective normal operator. Hence $\mathscr{M}$ reduces T and $T=\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ on $\mathscr{H}=\mathscr{M}\oplus\mathscr{M}^\perp$. Thus $(T-\lambda)^*x=0$. □

Theorem 4. *Let* $T\in\mathscr{B}(\mathscr{H})$ *be a quasi-$*$-class* (A,k) *operator for a positive integer* k. *Then the following assertions hold.*

(a) $\sigma_{jp}(T)\setminus\{0\}=\sigma_p(T)\setminus\{0\}$.

(b) *If* $Tx=\lambda x$, $Ty=\mu y$ *and* $\lambda\neq\mu$, *then* $\langle x,y\rangle=0$.

Proof. (a) Clearly by Theorem 3.

(b) Without loss of generality, we assume $\mu\neq 0$. Then we have $(T-\mu)^*y=0$ by Theorem 3. Thus we get $\mu\langle x,y\rangle=\langle x,T^*y\rangle=\langle Tx,y\rangle=\lambda\langle x,y\rangle$. Since $\lambda\neq\mu$, we have $\langle x,y\rangle=0$. □

A complex number λ is said to be in the approximate point spectrum $\sigma_a(T)$ of T if there is a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$ such that $(T-\lambda)x_n \longrightarrow 0$. If in addition, $(T-\lambda)^*x_n \longrightarrow 0$, then λ is said to be in the joint approximate point spectrum $\sigma_{ja}(T)$ of T. Clearly, $\sigma_{ja}(T) \subseteq \sigma_a(T)$. In general, $\sigma_{ja}(T) \neq \sigma_a(T)$. In [44], Xia showed that if T is a semi-hyponormal operator, then $\sigma_{ja}(T) = \sigma_a(T)$; Aluthge and Wang [3] showed that if T is w-hyponormal, then the nonzero points of $\sigma_{ja}(T)$ and $\sigma_a(T)$ are identical. In the following, we will show that if T is a quasi-$*$-class (A,k) operator for a positive integer k, then the nonzero points of $\sigma_{ja}(T)$ and $\sigma_a(T)$ are also identical.

Theorem 5. *Let $T \in \mathscr{B}(\mathscr{H})$ be a quasi-$*$-class (A,k) operator for a positive integer k. Then $\sigma_{ja}(T)\setminus\{0\} = \sigma_a(T)\setminus\{0\}$.*

To prove Theorem 5, we need the following auxiliary results.

Theorem 6. [6] *Let $\mathscr{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathscr{K}$ such that $\mathscr{H} \subset \mathscr{K}$ and a map $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ such that*

(a) *ϕ is a faithful $*$-representation of the algebra $\mathscr{B}(\mathscr{H})$ on $\mathscr{K}$;*
(b) *$\phi(A) \geq 0$ for any $A \geq 0$ in $\mathscr{B}(\mathscr{H})$;*
(c) *$\sigma_a(T) = \sigma_a(\phi(T)) = \sigma_p(\phi(T))$ for every $T \in \mathscr{B}(\mathscr{H})$.*

Lemma 3. [44] *Let $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ be Berberian's faithful $*$-representation. Then $\sigma_{ja}(T) = \sigma_{jp}(\phi(T))$.*

Proof of Theorem 5. Let $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ be Berberian's faithful $*$-representation of Theorem 6. In the following, we shall show that $\phi(T)$ is also a quasi-$*$-class (A,k) operator for a positive integer k. In fact, since T is a quasi-$*$-class (A,k) operator, we have

$$(\phi(T))^{*k}(|(\phi(T))^2| - |\phi(T^*)|^2)(\phi(T))^k = \phi\left(T^{*k}(|T^2| - |T^*|^2)T^k\right) \geq 0.$$

Hence, we have

$$\begin{aligned}\sigma_a(T)\setminus\{0\} = \sigma_a(\phi(T))\setminus\{0\} &= \sigma_p(\phi(T))\setminus\{0\} \\ &= \sigma_{jp}(\phi(T))\setminus\{0\} = \sigma_{ja}(T)\setminus\{0\}.\end{aligned}$$

□

Theorem 7. *Let $T \in \mathscr{B}(\mathscr{H})$ be a quasi-$*$-class (A,k) operator for a positive integer k. Then*

$$\sigma(T)\setminus\{0\} = (\sigma_a(T^*)\setminus\{0\})^* = \{\lambda : \bar{\lambda} \in \sigma_a(T^*)\setminus\{0\}\}.$$

Proof. It suffices to prove $\sigma(T)\setminus\{0\} \subset (\sigma_a(T^*)\setminus\{0\})^* = \{\lambda : \bar{\lambda} \in \sigma_a(T^*)\setminus\{0\}\}$ for every $T \in \mathscr{B}(\mathscr{H})$. Hence we have

$$\sigma_a(T)\setminus\{0\} = \sigma_{ja}(T)\setminus\{0\} \subset (\sigma_a(T^*)\setminus\{0\})^*$$

by Theorem 5. This achieves the proof. □

Putnam [29] proved some theorems concerning spectral properties of hyponormal operators. These theorems were generalized to p-hyponormal operators by Chō et al. in [8, 9]. In the following, we extend these theorems to quasi-$*$-class (A, k) operators.

We show the first generalization concerning points in the approximate point spectrum of a quasi-$*$-class (A, k) operator for a positive integer k as follows.

Theorem 8. *Let $T \in \mathscr{B}(\mathscr{H})$ be a quasi-$*$-class (A, k) for a positive integer k. If $\lambda \neq 0$ such that $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.*

To prove Theorem 8, we need the following auxiliary results.

Theorem 9. *Let $T = U|T|$ be the polar decomposition of T, $\lambda \neq 0$, and $\{x_n\}$ be a sequence of vectors. Then the following assertions are equivalent.*

(a) $(T - \lambda)x_n \to 0$ *and* $(T^* - \bar{\lambda})x_n \to 0$,
(b) $(|T| - |\lambda|)x_n \to 0$ *and* $(U - e^{i\theta})x_n \to 0$,
(c) $(|T^*| - |\lambda|)x_n \to 0$ *and* $(U^* - e^{-i\theta})x_n \to 0$.

Proof of Theorem 8. If $\lambda \neq 0$ and $\lambda \in \sigma_a(T)$, a sequence of unit vectors exists such that $(T - \lambda)x_n \to 0$ and $(T^* - \bar{\lambda})x_n \to 0$ by Theorem 5. Hence the result holds by Theorem 9. □

Corollary 1. *Let $T \in \mathscr{B}(\mathscr{H})$ be a quasi-$*$-class A operator. If $\lambda \neq 0$ such that $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.*

Definition 2. [7] An operator T is said to have *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if for every open neighborhood G of λ, the function $f_n \in \mathrm{Hol}(G)$ with $(T - \lambda)f_n(\mu) \to 0$ uniformly on every compact subset of G implies that $f_n(\mu) \to 0$ uniformly on every compact subset of G, where $\mathrm{Hol}(G)$ means the space of all analytic functions on G. When T has Bishop's property (β) at each $\lambda \in \mathbb{C}$, simply say that T has property (β).

Lemma 4. [24] *Let G be an open subset of the complex plane $\mathbb{C}$ and let $f_n \in \mathrm{Hol}(G)$ be functions such that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of G. Then $f_n(\mu) \to 0$ uniformly on every compact subset of G.*

Lemma 5. *Let $T \in \mathcal{Q}^*(\mathcal{A}, k)$. Then T has Bishop's property (β).*

Proof. If T^k has a dense range, then T is a $*$-class A, so the result follows from Proposition 2.4 of [12] (T is $*$-paranormal). Assume that T^k does not have a dense range. Then T has the matrix representation

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathscr{H} = \overline{\Re(T^k)} \oplus \ker(T^{*k}),$$

where $T_1 = T|_{\overline{\Re(T^k)}}$ is a $*$-class A, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Let $f_n(z)$ be analytic on D. Let $(T - z)f_n(z) \to 0$ uniformly on each compact subset of D. Then

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_{n_1}(z) \\ f_{n_2}(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_{n_1}(z) + T_2 f_{n_2}(z) \\ (T_3 - z)f_{n_2}(z) \end{pmatrix} \to 0$$

since $T_3^k = 0$, T_3 has Bishop's property (β) and $f_{n_2}(z) \to 0$. If an operator T_1 is a $*$-class A, then T_1 has Bishop's property (β). Thus $f_{n_1}(z) \to 0$. □

The quasinilpotent part of $T - \lambda$ is defined as

$$H_0(T-\lambda) = \left\{ x \in \mathscr{H} : \lim_{n\to\infty} \|(T-\lambda)^n\|^{1/n} = 0 \right\}.$$

In general, $\ker(T-\lambda) \subset H_0(T-\lambda)$ and $H_0(T-\lambda)$ is not closed. Let $F \subset \mathbb{C}$ be a closed set. Then the global spectral subspace is defined by

$$\chi_T(F) = \{x \in \mathscr{H} \mid \exists \text{ analytic } f(z) : (T-\lambda)f(z) = x \text{ on } \mathbb{C} \setminus F\}.$$

Theorem 10. *Let $T \in \mathcal{Q}^*(\mathcal{A}, k)$. Then*

$$H_0(T-\lambda) = \begin{cases} \ker(T-\lambda), & \text{if } \lambda \neq 0; \\ \ker(T^{m+1}), & \text{if } \lambda = 0. \end{cases}$$

Moreover, if $0 \neq \lambda$, then $H_0(T-\lambda) = \ker(T-\lambda) \subset \ker(T-\lambda)^*$.

Proof. Since T has Bishop's property (β) by Lemma 5 and $H_0(T-\lambda) = \chi_T(\{\lambda\})$ by Theorem 2.20 of [1], $H_0(T-\lambda)$ is closed and $\sigma(T|_{H_0(T-\lambda)}) \subset \{\lambda\}$ by Proposition 1.2.19 of [26]. Let $S = T|_{H_0(T-\lambda)}$. Then S is a $\mathcal{Q}^*(\mathcal{A}, k)$ operator by Theorem 2. Hence, we divide the proof into 3 cases:
Case I. If $\sigma(S) = \sigma(T|_{H_0(T-\lambda)}) = \emptyset$, then $H_0(T-\lambda) = \{0\}$, and so $\ker(T-\lambda) = \{0\}$.
Case II. If $\sigma(S) = \{\lambda\}$ and $\lambda \neq 0$, then $S = \lambda$ by Lemma 2, and $H_0(T-\lambda) = \ker(S-\lambda) \subset \ker(T-\lambda)$.
Case III. If $\sigma(S) = \{0\}$, then $S^{m+1} = 0$ by Lemma 2, and $H_0(T) = \ker(S^{m+1}) \subset \ker(T^{m+1})$.

Moreover, let $\lambda \neq 0$. In this case, $S = \lambda$. Hence S is normal and invertible, so $H_0(T-\lambda)$ reduces T by Theorem 3. Thus $H_0(T-\lambda) = \ker(T-\lambda) \subset \ker(T-\lambda)^*$. □

Theorem 11. *The eigenvalues of a $*$-class A operator are normal (i.e., the corresponding eigenspaces are reducing).*

Proof. If $T \in \mathscr{B}(\mathscr{H})$ is $*$-class A, $\lambda \in \sigma_p(T)$ and $Tx = \lambda x$ for some nontrivial $x \in \mathscr{H}$, $\|x\| = 1$, then

$$\begin{aligned} \left\|(T^* - \bar{\lambda})x\right\|^2 &= \|T^*x\|^2 - \lambda \langle T^*x, x\rangle - \bar{\lambda}\langle x, T^*x\rangle + \|\lambda\|^2 \\ &\leq \left\|T^2x\right\| \|x\| - \lambda\langle x, Tx\rangle - \bar{\lambda}\langle Tx, x\rangle + \|\lambda\|^2 = 0. \end{aligned}$$

□

3. Putnam's inequality of quasi-∗-class A operators

In general, by the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$ we cannot get that T is normal. For instance, [35], if $T = SB$, where S is positive and invertible, B is self-adjoint, and S and B do not commute, then $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, but T is not normal.

I. H. Sheth showed that if T is a hyponormal operator and $S^{-1}TS = T^*$ satisfying $0 \notin \overline{W(S)}$, then T is self-adjoint. I. H. Kim [22] extended the result of Sheth to the class of p-hyponormal operators. In the following, we shall show that if T or T^* is $*$-class A operator, the result of Sheth also holds.

Theorem 12. *Let T or T^* be a $*$-class A and S be an operator satisfying $0 \notin \overline{W(S)}$ such that $ST = T^*S$. Then T is self-adjoint.*

To prove Theorem 12 the following lemmas are needed.

Lemma 6. [35] *Let $T \in \mathscr{B}(\mathscr{H})$ be an operator such that $S^{-1}TS = T^*$, where S is an operator satisfying $0 \notin \overline{W(S)}$. Then $\sigma(T) \subset \mathbb{R}$.*

Lemma 7. [33] *Let $T \in \mathscr{B}(\mathscr{H})$ be a $*$-class A operator, then the following inequality holds*

$$\left\||T^2| - |T^*|^2\right\| \leq \left\||\tilde{T}_{1,1}| - |\tilde{T}^*_{1,1}|\right\| \leq \frac{1}{\pi}\text{meas}\,\sigma(T),$$

where $T = U|T|$ is the polar decomposition of T, $\tilde{T}_{1,1} = |T|U|T|$ and $\text{meas}\,\sigma(T)$ *is the planar Lebesgue measure of the spectrum of T. Moreover, if* $\text{meas}\,\sigma(T) = 0$, *then T is normal.*

Proof of Theorem 12. Assume that T or T^* is a $*$-class A operator. Since $0 \notin \overline{W(S)}$ and $\sigma(T) \subset \overline{W(S)}$, we have S is invertible and $0 \notin \overline{W(S^{-1})}$. Hence $(S^{-1})^{-1}TS^{-1} = T^*$ holds by $ST = T^*S$. Hence we have $\sigma(T) \subset \mathbb{R}$ by applying Lemma 6. Thus $\sigma(T^*) = \overline{\sigma(T)} \subset \mathbb{R}$. So we have that meas $\sigma(T) =$ meas $\sigma(T^*) = 0$ for the planar Lebesgue measure, whence we get that T or T^* is normal by Lemma 7. Hence T is self-adjoint since $\sigma(T) = \sigma(T^*) \subset \mathbb{R}$. □

It is well known that a class A operator with real spectrum is self-adjoint. More generally, from the proof of Theorem 12 we have the following.

Corollary 2. *Let $T \in \mathscr{B}(\mathscr{H})$ be a $*$-class A operator, and $\sigma(T) \subset \mathbb{R}$, then T is self-adjoint.*

The following theorem is about Putnam's inequality for $\mathcal{Q}^*(\mathcal{A}, k)$ operators.

Theorem 13. *Let $T \in \mathcal{Q}^*(\mathcal{A}, k)$ be an operator for a positive integer k. Then*

$$\left\|P(|T^2| - |T^*|^2|)P\right\| \leq \frac{1}{\pi}\text{meas}\ \sigma(T),$$

where P is the orthogonal projection of $\mathscr{H}$ onto $\overline{\Re(T^k)}$ and meas $\sigma(T)$ *is the planar Lebesgue mesure of the spectrum of T.*

Proof. Consider the matrix representation of T with respect to the decomposition $\mathscr{H} = \overline{\Re(T^k)} \oplus \ker(T^{*k})$, $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$. Let P be the orthogonal projection of $\mathscr{H}$ onto $\overline{\Re(T^k)}$. Then $T_1 = TP = PTP$. Since $T \in \mathcal{Q}^*(\mathcal{A}, k)$, we have

$$P(|T^2| - |T^*|^2)P \geq 0.$$

Then

$$|T_1^2| = (PT^*PT^*TPTP)^{\frac{1}{2}} = (PT^*T^*TTP)^{\frac{1}{2}} = (P|T^2|^2P)^{\frac{1}{2}} \geq P|T^2|P$$

by Hansen's inequality [28]. On the other hand

$$|T_1^*|^2 = T_1T_1^* = PTPP^*T^*P^* = P|T^*|^2P \leq P|T^2|P.$$

So we have

$$|T_1^*|^2 = P|T_1^*|^2P \leq P|T^2|P \leq |T_1^2|.$$

Hence

$$0 \leq P(|T^2| - |T^*|^2)P \leq |T_1^2| - |T_1^*|^2.$$

Since T_1 is a $*$-class A operator by Theorem 1, we have

$$\left\| P(|T^2| - |T^*|^2)P \right\| \leq \left\| |T_1^2| - |T_1^*|^2 \right\| \leq \frac{1}{\pi}\text{meas } \sigma(T_1) = \frac{1}{\pi}\text{meas } \sigma(T),$$

by Lemma 7 and Theorem 1. This achieves the proof. □

Theorem 14. *Let $T \in \mathscr{B}(\mathscr{H})$ be an injective quasi-$*$-class $(\mathcal{A}, k)$ operator for a positive integer k and S be a positive operator satisfying $0 \notin \overline{W(S)}$ such that $ST = T^*S$. Then T is a direct sum of a self-adjoint and a nilpotent operator.*

Proof. Since $T \in \mathcal{Q}^*(\mathcal{A}, k)$, we have the following matrix representation by Theorem 1

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathscr{H} = \overline{\Re(T^k)} \oplus \ker(T^{*k}),$$

where T_1 is a $*$-class A operator on $\overline{\Re(T^k)}$ and $T_3^k = 0$. Since $ST = T^*S$ and $0 \notin \overline{W(S)}$, we have $\sigma(T) \subset \mathbb{R}$ by Lemma 6. Hence $\sigma(T_1) \subset \mathbb{R}$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$. So, we have that T_1 is self-adjoint by Corollary 2 since T_1 is a $*$-class A operator on $\overline{\Re(T^k)}$. Let P be the orthogonal projection of $\mathscr{H}$ onto $\overline{\Re(T^k)}$. By Hansen's inequality [28], we have

$$\begin{pmatrix} |T_1^2| & 0 \\ 0 & 0 \end{pmatrix} = (P|T^2|^2P)^{\frac{1}{2}} \geq P|T^2|P \geq P|T^*|^2P = PTT^*P = \begin{pmatrix} |T_1^*|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since T_1 is self-adjoint, hence we can write

$$|T^2| = \begin{pmatrix} T_1^2 & A \\ A^* & B \end{pmatrix}.$$

So, we have

$$\begin{aligned} \begin{pmatrix} T_1^4 & 0 \\ 0 & 0 \end{pmatrix} &= P|T^2||T^2|P \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1^2 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} T_1^2 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T_1^4 + AA^* & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This implies that $A = 0$ and $|T^2|^2 = \begin{pmatrix} T_1^4 & 0 \\ 0 & B^2 \end{pmatrix}$. On the other hand,

$$\begin{aligned} |T^2|^2 &= T^*T^*TT \\ &= \begin{pmatrix} T_1 & 0 \\ T_2^* & T_3^* \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ T_2^* & T_3^* \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \\ &= \begin{pmatrix} T_1^4 & T_1^2(T_1T_2 + T_2T_3) \\ (T_1T_2 + T_2T_3)^*T_1^2 & |T_1T_2 + T_2T_3|^2 + |T_3^2|^2 \end{pmatrix}. \end{aligned}$$

Since T is injective and $\ker(T_1) \subseteq \ker(T)$, we have that T_1 is injective. Hence $T_1T_2 + T_2T_3 = 0$ and $B = |T_3^2|$. Since $T \in \mathcal{Q}^*(\mathcal{A}, k)$, by simple calculation, we have

$$\begin{aligned} 0 &\leq T^{*k}(|T^2| - |T^*|^2)T^k \\ &= \begin{pmatrix} -T_1^k|T_2^*|^2T_1^k & Y \\ Y^* & X + T_3^{*k}(|T_3^2| - |T_3^*|^2)T_3^k \end{pmatrix}. \end{aligned}$$

Recall that $\begin{pmatrix} A & B \\ B^* & Z \end{pmatrix} \geq 0$ if and only if $A, Z \geq 0$ and $Y = A^{\frac{1}{2}}WZ^{\frac{1}{2}}$ for some contraction W. Thus we have $T_2 = 0$. This achieves the proof. □

4. Quasisimilarity

For two bounded linear operators S and T on the Hilbert spaces, S and T are said to be quasisimilar if there are two injective operators with dense ranges, X and Y such that $XS = TX$ and $SY = YT$. Though quasi-similarity is a weaker equivalence relation for operators, it is an interesting equivalence relation for the seminormal operators since quasi-similarity preserves spectrum and essential spectrum [27] as well as some other properties for $*$-class A operators.

Recall that a subspace $\mathscr{M}$ of $\mathscr{H}$ is called spectral maximal space for T if $\mathscr{M}$ contains every invariant subspace $\mathcal{C}$ of T for which $\sigma(T|_{\mathcal{C}}) \subset \sigma(T|_{\mathscr{M}})$.

Definition 3. [1] An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be decomposable if for any finite open covering $\{U_1, U_2, \ldots, U_n\}$ of spectrum of T, there exist spectral maximal subspaces $\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_n$ of T such that

(i) $\mathscr{H} = \mathscr{M}_1 + \mathscr{M}_2 + \cdots + \mathscr{M}_n$ and
(ii) $\sigma(T|_{\mathscr{M}_i}) \subset U_i$, for $i = 1, 2, \ldots, n$.

We say that an operator T is subdecomposable operator if it is the restriction of a decomposable operator to its invariant space (see [1]). It is well known that T is decomposable if and only if T has Bishop property (β). The following result of Yang is crucial to our purpose.

Proposition 1. [45] *Let $T \in \mathscr{B}(\mathscr{H})$ and $S \in \mathscr{B}(\mathscr{K})$ be two quasisimilar subdecomposable operators. Then $\sigma(T) = \sigma(S)$.*

Theorem 15. *If quasi-$*$-class (A,k) operators $T,S\in\mathscr{B}(\mathscr{H})$ are quasisimilar, then they have equal spectrum.*

Proof. Let $T,S\in\mathscr{B}(\mathscr{H})$ be quasi-$*$-class (A,k) operators. From Theorem 5, T and S satisfy Bishop property (β) and hence T and S are subdecomposable operators. Then by Proposition 1, it follows that the spectrum of T and S are equal. □

Two operators $T\in\mathscr{B}(\mathscr{H})$ and $S\in\mathscr{B}(\mathscr{K})$ are densely similar if there exist $X\in\mathscr{B}(\mathscr{H},\mathscr{K})$ and $Y\in\mathscr{B}(\mathscr{K},\mathscr{H})$ such that they have dense ranges and $XT=SX$ and $YS=TY$.

Theorem 16. *If quasi-$*$-class (A,k) operators $T,S\in\mathscr{B}(\mathscr{H})$ are densely similar, then they have equal essential spectrum.*

Proof. Since T and S are quasi-$*$-class (A,k) operators, both T and S satisfies Bishop property (β). Then by applying [26, Theorem 3.7.13], it follows that they have equal essential spectrum. □

Proposition 2. [33] *Let $T\in\mathscr{B}(\mathscr{H})$ and $S\in\mathscr{B}(\mathscr{K})$ be two quasisimilar $*$-class A operators. Then they have the same essential spectrum.*

Let $M_Q=\begin{pmatrix} S & Q\\ 0 & T\end{pmatrix}$ be an 2×2 upper-triangular operator matrix acting on the Hilbert space $\mathscr{H}\oplus\mathscr{K}$ and let $\sigma_e(T)$ denote the essential spectrum of $T\in\mathscr{B}(\mathscr{H})$.

Proposition 3. [20] *Assume that $\sigma_e(T)\cap\sigma_e(S)$ has no interior points. Then, for every $Q\in\mathscr{B}(\mathscr{K},\mathscr{H})$,*

$$\sigma_e(M_Q)=\sigma_e(T)\cup\sigma_e(S).$$

Now we prove that two quasisimilar quasi-$*$-class (A,k) operators have equal essential spectrum.

Theorem 17. *If quasi-$*$-class (A,k) operators $T,S\in\mathscr{B}(\mathscr{H})$ are quasisimilar, then they have equal essential spectrum.*

Proof. Let $T,S\in\mathscr{B}(\mathscr{H})$ be quasisimilar quasi-$*$-class (A,k) operators. Then there exist quasi-affinities X and Y such that $XT=SX$ and $YS=TY$. By Theorem 1, decompose T and S a as follows:

$$T=\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}\ \text{on}\ \mathscr{H}=\overline{\mathcal{R}(T^k)}\oplus\ker(T^{*k}),$$
$$S=\begin{pmatrix} S_1 & S_2\\ 0 & S_3\end{pmatrix}\ \text{on}\ \mathscr{K}=\overline{\mathcal{R}(S^k)}\oplus\ker(S^{*k}),$$

where $T_1=T|_{\overline{\mathcal{R}(T^k)}}$, $S_1=S|_{\overline{\mathcal{R}(S^k)}}$ are $*$-class A operators, $\sigma(T)=\sigma(T_1)\cup\{0\}$ and $\sigma(S)=\sigma(S_1)\cup\{0\}$. Since quasisimilar $*$-class A operators have the same essential spectrum by Proposition 2, in view of Theorem 1 and Proposition 3, it is enough to show that the domain of T_3 is $\{0\}$ if and only if the domain of S_3 is $\{0\}$. Since $XT = SX$, $XT^k = S^kX$. Let $0 \neq x \in \mathscr{H}$ such that $T^{*k}x = 0$. Then by the equality $XT^k = S^kX$, we have $S^{*k}Y^* = 0$. Since Y^* is one to one, we have that the domain of S_3 is $\{0\}$ implies that the domain of T_3 is $\{0\}$. By a similar argument as above using the equality $YS = TY$ we obtain that the domain of T_3 is $\{0\}$ and hence the domain of S_3 is $\{0\}$. □

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Received April 09, 2017.

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