

# WIGNER FUNCTIONS AND SPIN TOMOGRAMS FOR QUBIT STATES

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## Abstract

We establish the relation of the spin tomogram to the Wigner function on a discrete phase space of qubits. We use the quantizers and dequantizers of the spin tomographic star-product scheme for qubits to derive the expression for the kernel connecting Wigner symbols on the discrete phase space with the tomographic symbols.

**Keywords:** discrete Wigner function, spin tomography, star product, quantizer, dequantizer.

## 1 Introduction

The Wigner function is a powerful tool for representing quantum states and treating quantum-mechanical problems. It is a quasiprobability distribution and it has also been generalized for discrete quantum systems [1–5]. Properties of quasidistributions in a finite Hilbert space have been studied in the literature [6–8]. The quasidistributions can be associated with mutually unbiased bases (MUB) [9, 10]. Also there exists the construction of tomographic-probability distributions (spin tomography) [11–14] and unitary-matrix tomography [15] describing the quantum states. The star product of functions [16, 17] is the usual framework to consider the Wigner function [18] and associative product of functions in the phase space. The star-product approach was generalized [19] for considering different schemes. The approach is based on the existence of the so-called dequantizer  $\hat{Q}(x)$  and quantizer  $\hat{D}(x)$  acting in a Hilbert space and depending on a collective coordinate  $x$  of a point in a manifold.

There exists the geometrical description of quantum states based on the discrete phase space. The discrete phase space for a quantum system characterized by a dimension equal to the power of the prime  $d = p^n$  was constructed in [2, 4] with the help of  $d^2$  points  $(x_1, x_2)$ , where  $x_1$  runs along the horizontal axis and  $x_2$ , along the vertical one. A line is described by a subset of  $d$  points. A given set of  $d$  parallel lines defines a striation [2, 4]. Two striations are called mutually orthogonal if each line of the first striation has exactly one intersecting point with each line of the second striation [20]. There are  $d + 1$  mutually unbiased striations.

The correspondence between a line  $\lambda$  and a quantum state is determined by the function  $Q$  [4], namely,  $Q(\lambda)$  is the projection operator of the pure state. The discrete Wigner function was introduced in [4] and is based on a special family of Hermitian operators  $\mathcal{A}_\alpha$ , which depend on a point in the discrete phase space. If  $\alpha$  is a point in the discrete phase space, the phase-space point operators are defined as

$$\mathcal{A}_\alpha = \sum_{\lambda \ni \alpha} Q(\lambda) - I, \quad (1)$$

where the sum is taken over all lines  $\lambda$  that contain the point  $\alpha$ . Here  $I$  is the identity operator. These operators satisfy  $\text{Tr } \mathcal{A}_\alpha = 1$ . The discrete Wigner function of a quantum state  $\rho$  is defined as [4]

$$W_\alpha = \text{Tr}(\rho \mathcal{A}_\alpha)/d. \quad (2)$$

The set of Hermitian operators  $\mathcal{A}_\alpha$  is not unique; it depends on the complete set of mutually orthogonal striations constructed with the help of mutually unbiased bases. It turns out that the MUBs are determined by the bases associated with each striation. Starting from this geometrical description, more results were obtained for different systems: two qubits [5, 21], three qubits [22, 23], and  $n$  qubits [24]. An interesting analogy was made between the mutually orthogonal striations and Latin squares [20, 25, 26] and a more general concept called supersquares [27]. A recent detailed review presents different constructions of MUBs [28].

An algorithm for constructing the discrete Wigner function in the case of composed systems, whose dimension can be factorized into prime factors,  $d = d_1 \dots d_p$ , was proposed in [29]. In this case, the phase-space point operators can be written as tensor products of the phase-space point operators of each subsystem. The discrete Wigner function of two qubits was used for evaluating the entanglement in [30]. The entanglement was analyzed with the help of the partial transposition criterion and the local uncertainty relations, which were reformulated in terms of the discrete Wigner function.

The aim of this work is to associate the discrete Wigner function construction of [4] with the star-product quantizer–dequantizer scheme and find an explicit formula connecting the tomographic probabilities and the quasidistributions on the discrete phase spaces, using the elaborated framework of the star-product schemes [19, 31]. In this paper, we present the one-qubit case. We study in detail a concrete example of the qubit state, using explicit forms of the quantizer and dequantizer determining the tomographic probability distribution given in [32].

This paper is organized as follows.

In Sec. 2, we review a general scheme of the star-product quantization. In Sec. 3, we give the construction of the Wigner function for one qubit within the framework of the star-product scheme. We study the relation of the Wigner functions to qubit tomograms in Sec. 4. It is worth noting that some aspects of the problem of connection of quasidistributions and tomographic-probability distributions of spin states were discussed in [33]. We present our conclusions and perspectives in Sec. 5.

## 2 General Description of the Star-Product Scheme

Following [19], we present a general scheme of the star-product construction. Given a Hilbert space  $\mathcal{H}$  and operators called the dequantizers  $\hat{Q}(x)$  and quantizers  $\hat{D}(x)$  acting on the Hilbert space, these operators satisfy the condition that, for an arbitrary operator  $\hat{A}$ , one has

$$\int dx' \text{Tr}(\hat{Q}(x)\hat{D}(x'))\text{Tr}(\hat{A}\hat{Q}(x')) = \text{Tr}(\hat{A}\hat{Q}(x)). \quad (3)$$

In some cases, such an equality can be rewritten as

$$\text{Tr}(\hat{Q}(x)\hat{D}(x')) = \delta(x - x'). \quad (4)$$

We consider the manifold point coordinate  $x$  as  $(q, p)$  for the standard phase space of an oscillator. Also, in other cases, this coordinate can contain discrete components. So far we are dealing with the spin- $j$  tomography  $x = (m, \vec{n})$ , where  $m$  is the spin projection taking values  $-j, -j + 1, \dots, j$ , and  $\vec{n} = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$  is a vector determining the direction, in which we obtain the spin projection  $m$ .

We introduce the function

$$f_A(x) = \text{Tr}(\hat{A}\hat{Q}(x)), \quad (5)$$

called the symbol of the operator  $\hat{A}$ . Relation (3) provides the possibility to reconstruct the operator  $\hat{A}$  from its symbol

$$\hat{A} = \int f_A(x)\hat{D}(x) dx. \quad (6)$$

There exists a dual symbol (see also [30]) of the operator  $\hat{A}$  given as

$$f_A^d(x) = \text{Tr}(\hat{A}\hat{D}(x)). \quad (7)$$

The reconstruction relation reads

$$\hat{A} = \int f_A^d(x)\hat{Q}(x) dx. \quad (8)$$

The mean value of the observable  $\hat{A}$  in the state characterized by the density operator  $\hat{\rho}$  is

$$\langle \hat{A} \rangle = \text{Tr}(\hat{\rho}\hat{A}) = \int w_\rho(x)f_A^d(x) dx, \quad w_\rho(x) = \text{Tr}(\hat{\rho}\hat{Q}(x)). \quad (9)$$

We assume that there exists another pair of operators  $\hat{\hat{Q}}(y)$  and  $\hat{\hat{D}}(y)$  with the properties of dequantizers and quantizers. Then a new symbol of the operator  $\hat{A}$

$$F_A(y) = \text{Tr}(\hat{\hat{Q}}(y)\hat{A}) \quad (10)$$

can be related to the symbol  $f_A(x)$  by means of the kernel

$$F_A(y) = \int K(y, x)f_A(x) dx, \quad K(y, x) = \text{Tr}(\hat{\hat{Q}}(y)\hat{D}(x)). \quad (11)$$

Analogously

$$f_A(x) = \int \mathcal{K}(x, y)F_A(y) dy, \quad \mathcal{K}(x, y) = \text{Tr}(\hat{Q}(x)\hat{\hat{D}}(y)). \quad (12)$$

The associative product of two symbols, called the star product, is defined as

$$(f_A * f_B)(x) = f_{AB}(x), \quad (13)$$

and it is determined by the integral kernel

$$(f_A * f_B)(x) = \int f_A(x') f_B(x'') K(x', x'', x) dx' dx'', \quad K(x', x'', x) = \text{Tr}(\hat{D}(x') \hat{D}(x'') \hat{Q}(x)). \quad (14)$$

### 3 The Oscillator Phase Space

For the standard Wigner function  $W(q, p)$  of an oscillator state, one has the quantizer

$$\hat{Q}(x) \equiv \hat{Q}(q, p) = 2 \{ \exp [2\alpha \hat{a}^\dagger - 2\alpha^* \hat{a}] \} \hat{I}, \quad (15)$$

where  $\hat{a}$  and  $\hat{a}^\dagger$  are the creation and annihilation operators  $\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}$  and  $\hat{a}^\dagger = (\hat{q} - i\hat{p})/\sqrt{2}$ , respectively,  $\hat{I}$  is the parity operator, i.e.,  $\hat{I}\psi(x) = \psi(-x)$ , and the complex number  $\alpha = (q + ip)/\sqrt{2}$ .

In the Weyl–Wigner star-product scheme, the quantizer reads

$$\hat{D}(x) \equiv \hat{D}(q, p) = \hat{Q}(q, p)/2\pi. \quad (16)$$

The scheme is self-dual due to (16).

The Grönewald star-product kernel is given by Eq. (14); it is

$$K(q_1, p_1, q_2, p_2, q_3, p_3) = (2\pi)^{-2} \exp \{ 2i[q_1 p_2 - q_2 p_1 + q_2 p_3 - q_3 p_2 + q_3 p_1 - q_1 p_3] \}. \quad (17)$$

If the symbols of two observables  $\hat{A}$  and  $\hat{B}$  are given as functionals  $A(q, p)$  and  $B(q, p)$  in the oscillator phase space, the Weyl symbol  $f_{AB}(q, p)$  of the product  $AB$  is given by the integral

$$f_{AB}(q, p) = \int (2\pi)^{-2} f_A(q_1, p_1) f_B(q_2, p_2) \exp \{ 2i[q_1 p_2 - q_2 p_1 + q_2 p_3 - q_3 p_2 + q_3 p_1 - q_1 p_3] \}. \quad (18)$$

### 4 Example of the Spin 1/2

For spin equal to 1/2, the dequantizer  $\hat{Q}(x) \equiv \hat{Q}(m, \vec{n})$  has the form [12]

$$\hat{Q}(m, \vec{n}) = U^\dagger |m\rangle \langle m| U, \quad (19)$$

where  $m = \pm 1/2$ , and the unitary matrix reads

$$U = \begin{pmatrix} \cos \vartheta/2 e^{i(\varphi+\psi)/2} & \sin \vartheta/2 e^{i(\varphi-\psi)/2} \\ -\sin \vartheta/2 e^{i(-\varphi+\psi)/2} & \cos \vartheta/2 e^{-i(\varphi+\psi)/2} \end{pmatrix}. \quad (20)$$

The dequantizer can be presented in the form of a 2×2 matrix as follows [32]:

$$\hat{Q}(m, \vec{n}) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}. \quad (21)$$

The quantizer reads

$$\hat{D}(m, \vec{n}) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3m \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}. \quad (22)$$

For any qubit state with the density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix}, \quad (23)$$

the tomogram reads

$$w(m, \vec{n}) = \text{Tr} [\rho \hat{Q}(m, \vec{n})]. \quad (24)$$

It is the standard probability distribution of the spin projection  $m$  onto the quantization axes  $\vec{n}$ , i.e., it is nonnegative,  $w(m, \vec{n}) \geq 0$ , and the normalization condition  $\sum_{m=-1/2}^{1/2} w(m, \vec{n}) = 1$  holds.

The construction of spin tomograms can be generalized for multiqubit systems.

For the state of two qubits with the density matrix  $\rho(1, 2)$ , one has the tomogram

$$w(m_1, m_2, \vec{n}_1, \vec{n}_2) = \text{Tr} [\rho(1, 2) \hat{Q}(m_1, m_2, \vec{n}_1, \vec{n}_2)], \quad (25)$$

where

$$\hat{Q}(m_1, m_2, \vec{n}_1, \vec{n}_2) = \hat{Q}_1(m_1, \vec{n}_1) \otimes \hat{Q}_2(m_2, \vec{n}_2), \quad (26)$$

and  $\hat{Q}_1$  and  $\hat{Q}_2$  are given by (21). It is the joint probability distribution of two spin projections  $m_1$  and  $m_2$  onto the quantization axes  $\vec{n}_1$  and  $\vec{n}_2$ , respectively.

Also the quantizer is the tensor product

$$\hat{D}(m_1, m_2, \vec{n}_1, \vec{n}_2) = \hat{D}_1(m_1, \vec{n}_1) \otimes \hat{D}_2(m_2, \vec{n}_2). \quad (27)$$

## 5 Wigner Function of the One-Qubit State

To demonstrate our general construction, we consider the one-qubit state. The density matrix of this state can be presented in two forms: either as

$$\hat{\rho} = \begin{pmatrix} a & ce^{i\xi} \\ ce^{-i\xi} & b \end{pmatrix}, \quad a + b = 1, \quad (28)$$

or as

$$\hat{\rho} = \frac{1}{2} \begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix}, \quad (29)$$

corresponding to the two bases:  $A$  and  $B$ . The explicit forms of basis  $A$  and basis  $B$  are given below.

We introduce four matrices  $\hat{\mathcal{A}}_\alpha$  for one qubit, where the collective index  $\alpha(j, k)$  takes the values  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ . The four matrices  $\hat{\mathcal{A}}_\alpha$  read

$$\begin{aligned} \hat{\mathcal{A}}_{0,0} &= \begin{pmatrix} 1 & (1-i)/2 \\ (1+i)/2 & 0 \end{pmatrix}, & \hat{\mathcal{A}}_{0,1} &= \begin{pmatrix} 1 & (-1+i)/2 \\ (-1-i)/2 & 0 \end{pmatrix}, \\ \hat{\mathcal{A}}_{1,0} &= \begin{pmatrix} 0 & (1+i)/2 \\ (1-i)/2 & 1 \end{pmatrix}, & \hat{\mathcal{A}}_{1,1} &= \begin{pmatrix} 0 & (-1-i)/2 \\ (-1+i)/2 & 1 \end{pmatrix}. \end{aligned}$$

The four matrices  $\hat{\mathcal{B}}_\alpha$  for one qubit are given by the transposed matrices  $\hat{\mathcal{A}}_\alpha$ ; they are

$$\begin{aligned}\hat{\mathcal{B}}_{0,0} &= \hat{\mathcal{A}}_{0,0}^T = \begin{pmatrix} 1 & (1+i)/2 \\ (1-i)/2 & 0 \end{pmatrix}, & \hat{\mathcal{B}}_{0,1} &= \hat{\mathcal{A}}_{0,1}^T = \begin{pmatrix} 1 & (-1-i)/2 \\ (-1+i)/2 & 0 \end{pmatrix}, \\ \hat{\mathcal{B}}_{1,0} &= \hat{\mathcal{A}}_{1,0}^T = \begin{pmatrix} 0 & (1-i)/2 \\ (1+i)/2 & 1 \end{pmatrix}, & \hat{\mathcal{B}}_{1,1} &= \hat{\mathcal{A}}_{1,1}^T = \begin{pmatrix} 0 & (-1+i)/2 \\ (-1-i)/2 & 1 \end{pmatrix}.\end{aligned}$$

We are looking for the Wigner function of the one-qubit state in basis  $A$ , where its components read

$$\begin{aligned}W^A(j, k) &= \text{Tr}(\hat{\rho} \hat{\mathcal{A}}_{j,k})/2, & j, k &= 0, 1, \\ W^A(0, 0) &= (1 + z + x - y)/4, & W^A(0, 1) &= (1 + z - x + y)/4, \\ W^A(1, 0) &= (1 - z + x + y)/4, & W^A(1, 1) &= (1 - z - x - y)/4,\end{aligned}\tag{30}$$

while in basis  $B$  they are

$$\begin{aligned}W^B(j, k) &= \text{Tr}(\hat{\rho} \hat{\mathcal{B}}_{j,k})/2 & j, k &= 0, 1, \\ W^B(0, 0) &= ((1 + z + x + y))/4, & W^B(0, 1) &= (1 + z - x - y)/4, \\ W^B(1, 0) &= ((1 - z + x - y))/4, & W^B(1, 1) &= (1 - z - x + y)/4.\end{aligned}\tag{31}$$

We have two sorts of Wigner functions – the first one is determined by dequantizer (30), and the second one is determined by dequantizer (31).

One can easily check that the following reconstruction formulas are valid:

$$\hat{\rho} = \sum_{j,k=0}^1 W^A(j, k) \hat{\mathcal{A}}_{j,k}, \quad \hat{\rho} = \sum_{j,k=0}^1 W^B(j, k) \hat{\mathcal{B}}_{j,k}.\tag{32}$$

The components of the Wigner function in basis  $A$  and basis  $B$  are related as follows:

$$W^A(i, j) = \frac{1}{2} \sum_{l,k=0}^1 W^B(l, k) \text{Tr}(\hat{\mathcal{A}}_{i,j} \hat{\mathcal{B}}_{l,k}).\tag{33}$$

The qubit-state tomogram is determined by various quantizer–dequantizer pairs.

The tomographic dequantizer operator (19), with unitary matrix (20) and a unit vector  $\vec{n}$  with components  $(\sin \vartheta \sin \psi, \sin \vartheta \cos \psi, \cos \vartheta)$ , has the explicit matrix form

$$\begin{aligned}\hat{Q}(1/2, \vartheta, \psi) &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \cos \vartheta & \sin \vartheta e^{-i\psi} \\ \sin \vartheta e^{i\psi} & -\cos \vartheta \end{pmatrix}, \\ \hat{Q}(-1/2, \vartheta, \psi) &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \cos \vartheta & \sin \vartheta e^{-i\psi} \\ \sin \vartheta e^{i\psi} & -\cos \vartheta \end{pmatrix}.\end{aligned}\tag{34}$$

The tomographic quantizer operator  $\hat{D}$  has the matrix form with matrix elements depending on the coordinates of the unit vector  $\vec{n}$ , namely,

$$\begin{aligned}\hat{D}(1/2, \vartheta, \psi) &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} \cos \vartheta & \sin \vartheta e^{-i\psi} \\ \sin \vartheta e^{i\psi} & -\cos \vartheta \end{pmatrix}, \\ \hat{D}(-1/2, \vartheta, \psi) &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} \cos \vartheta & \sin \vartheta e^{-i\psi} \\ \sin \vartheta e^{i\psi} & -\cos \vartheta \end{pmatrix}.\end{aligned}\tag{35}$$

In view of formulas (11) and (12), we obtain the kernels connecting tomograms and Wigner functions.  
 After some algebra, we obtain the components of the kernel  $\text{Ker}^A(m, \vec{n}; j, k) = \text{Tr}(\hat{Q}(m, \vec{n})\hat{\mathcal{A}}_{j,k})$  as follows:

$$\begin{aligned}
 \text{Ker}^A(1/2, \vartheta, \psi; 0, 0) &= (1 + \cos \vartheta + \sin \vartheta(\cos \psi + \sin \psi))/2, \\
 \text{Ker}^A(-1/2, \vartheta, \psi; 0, 0) &= (1 - \cos \vartheta - \sin \vartheta(\cos \psi + \sin \psi))/2, \\
 \text{Ker}^A(1/2, \vartheta, \psi; 0, 1) &= (1 + \cos \vartheta - \sin \vartheta(\cos \psi + \sin \psi))/2 \\
 \text{Ker}^A(-1/2, \vartheta, \psi; 0, 1) &= (1 - \cos \vartheta + \sin \vartheta(\cos \psi + \sin \psi))/2, \\
 \text{Ker}^A(1/2, \vartheta, \psi; 1, 0) &= (1 - \cos \vartheta + \sin \vartheta(\cos \psi - \sin \psi))/2, \\
 \text{Ker}^A(-1/2, \vartheta, \psi; 1, 0) &= (1 + \cos \vartheta - \sin \vartheta(\cos \psi - \sin \psi)), \\
 \text{Ker}^A(1/2, \vartheta, \psi; 1, 1) &= (1 - \cos \vartheta - \sin \vartheta(\cos \psi - \sin \psi))/2, \\
 \text{Ker}^A(-1/2, \vartheta, \psi; 1, 1) &= (1 + \cos \vartheta + \sin \vartheta(\cos \psi - \sin \psi))/2.
 \end{aligned} \tag{36}$$

In addition, we find the components of the kernel  $\widetilde{\text{Ker}}^A(m, \vec{n}; j, k) = \text{Tr}(\hat{D}(m, \vec{n})\hat{\mathcal{A}}_{j,k})/2$  connecting the Wigner functions with the dual tomograms; they read

$$\begin{aligned}
 \widetilde{\text{Ker}}^A(1/2, \vartheta, \psi; 0, 0) &= (1 + 3 \cos \vartheta + 3 \sin \vartheta(\cos \psi + \sin \psi))/4, \\
 \widetilde{\text{Ker}}^A(-1/2, \vartheta, \psi; 0, 0) &= (1 - 3 \cos \vartheta - 3 \sin \vartheta(\cos \psi + \sin \psi))/4, \\
 \widetilde{\text{Ker}}^A(1/2, \vartheta, \psi; 0, 1) &= (1 + 3 \cos \vartheta - 3 \sin \vartheta(\cos \psi + \sin \psi))/4, \\
 \widetilde{\text{Ker}}^A(-1/2, \vartheta, \psi; 0, 1) &= (1 - 3 \cos \vartheta + 3 \sin \vartheta(\cos \psi + \sin \psi))/4, \\
 \widetilde{\text{Ker}}^A(1/2, \vartheta, \psi; 1, 0) &= (1 - 3 \cos \vartheta + 3 \sin \vartheta(\cos \psi - \sin \psi))/4, \\
 \widetilde{\text{Ker}}^A(-1/2, \vartheta, \psi; 1, 0) &= (1 + 3 \cos \vartheta - 3 \sin \vartheta(\cos \psi - \sin \psi))/4, \\
 \widetilde{\text{Ker}}^A(1/2, \vartheta, \psi; 1, 1) &= (1 - 3 \cos \vartheta - 3 \sin \vartheta(\cos \psi - \sin \psi))/4, \\
 \widetilde{\text{Ker}}^A(-1/2, \vartheta, \psi; 1, 1) &= (1 + 3 \cos \vartheta + 3 \sin \vartheta(\cos \psi - \sin \psi))/4.
 \end{aligned} \tag{37}$$

Now we show the components of the kernalers  $\text{Ker}^B(m, \vec{n}; j, k) = \text{Tr}(\hat{Q}(m, \vec{n})\hat{\mathcal{B}}_{j,k})$  and  $\widetilde{\text{Ker}}^B(m, \vec{n}; j, k) = \text{Tr}(\hat{D}(m, \vec{n})\hat{\mathcal{B}}_{j,k})/2$  constructed with the help of operators  $\hat{\mathcal{B}}_{j,k}$ ; they are

$$\begin{aligned}
 \text{Ker}^B(1/2, \vartheta, \psi; 0, 0) &= (1 + \cos \vartheta + \sin \vartheta(\cos \psi - \sin \psi))/2, \\
 \text{Ker}^B(-1/2, \vartheta, \psi; 0, 0) &= (1 - \cos \vartheta - \sin \vartheta(\cos \psi - \sin \psi))/2, \\
 \text{Ker}^B(1/2, \vartheta, \psi; 0, 1) &= (1 + \cos \vartheta - \sin \vartheta(\cos \psi - \sin \psi))/2, \\
 \text{Ker}^B(-1/2, \vartheta, \psi; 0, 1) &= (1 - \cos \vartheta + \sin \vartheta(\cos \psi - \sin \psi))/2, \\
 \text{Ker}^B(1/2, \vartheta, \psi; 1, 0) &= (1 - \cos \vartheta + \sin \vartheta(\cos \psi + \sin \psi))/2, \\
 \text{Ker}^B(-1/2, \vartheta, \psi; 1, 0) &= (1 + \cos \vartheta - \sin \vartheta(\cos \psi + \sin \psi))/2, \\
 \text{Ker}^B(1/2, \vartheta, \psi; 1, 1) &= (1 - \cos \vartheta - \sin \vartheta(\cos \psi + \sin \psi))/2, \\
 \text{Ker}^B(-1/2, \vartheta, \psi; 1, 1) &= (1 + \cos \vartheta + \sin \vartheta(\cos \psi + \sin \psi))/2,
 \end{aligned} \tag{38}$$

and

$$\begin{aligned}
\widetilde{\text{Ker}}^B(1/2, \vartheta, \psi; 0, 0) &= (1 + 3 \cos \vartheta + 3 \sin \vartheta (\cos \psi - \sin \psi))/4, \\
\widetilde{\text{Ker}}^B(-1/2, \vartheta, \psi; 0, 0) &= (1 - 3 \cos \vartheta - 3 \sin \vartheta (\cos \psi - \sin \psi))/4, \\
\widetilde{\text{Ker}}^B(1/2, \vartheta, \psi; 0, 1) &= (1 + 3 \cos \vartheta - 3 \sin \vartheta (\cos \psi - \sin \psi))/4, \\
\widetilde{\text{Ker}}^B(-1/2, \vartheta, \psi; 0, 1) &= (1 - 3 \cos \vartheta + 3 \sin \vartheta (\cos \psi - \sin \psi))/4, \\
\widetilde{\text{Ker}}^B(1/2, \vartheta, \psi; 1, 0) &= (1 - 3 \cos \vartheta + 3 \sin \vartheta (\cos \psi + \sin \psi))/4, \\
\widetilde{\text{Ker}}^B(-1/2, \vartheta, \psi; 1, 0) &= (1 + 3 \cos \vartheta - 3 \sin \vartheta (\cos \psi + \sin \psi))/4, \\
\widetilde{\text{Ker}}^B(1/2, \vartheta, \psi; 1, 1) &= (1 - 3 \cos \vartheta - 3 \sin \vartheta (\cos \psi + \sin \psi))/4, \\
\widetilde{\text{Ker}}^B(-1/2, \vartheta, \psi; 1, 1) &= (1 + 3 \cos \vartheta + 3 \sin \vartheta (\cos \psi + \sin \psi))/4.
\end{aligned} \tag{39}$$

## 6 Tomograms of the One-Qubit State

The tomograms of the one-qubit state (28) are

$$\begin{aligned}
w_1 &= \frac{1}{2} + \frac{(a-b)}{2} \cos \vartheta + c \sin \vartheta \cos(\psi + \xi) = \frac{1}{2}(1 + z \cos \vartheta + x \sin \vartheta \cos \psi - y \sin \vartheta \sin \psi), \\
w_2 &= \frac{1}{2} - \frac{(a-b)}{2} \cos \vartheta - c \sin \vartheta \cos(\psi + \xi) = \frac{1}{2}(1 - z \cos \vartheta - x \sin \vartheta \cos \psi + y \sin \vartheta \sin \psi).
\end{aligned} \tag{40}$$

We can reconstruct these tomograms, in view of the kernel  $\text{Ker}^A(m, \vec{n}; j, k) = \text{Tr}(\hat{Q}(m, \vec{n})\hat{\mathcal{A}}_{j,k})$ , through the Wigner functions (30) and (31). After some algebra, we obtain the following relationships:

$$\begin{aligned}
w_1 &= w(1/2, \vartheta, \psi) = \sum_{j,k=0}^1 \text{Ker}^A(1/2, \vartheta, \psi; j, k) W^A(j, k) = \sum_{j,k=0}^1 \text{Ker}^B(1/2, \vartheta, \psi; j, k) W^B(j, k), \\
w_2 &= w(-1/2, \vartheta, \psi) = \sum_{j,k=0}^1 \text{Ker}^A(-1/2, \vartheta, \psi; j, k) W^A(j, k) = \sum_{j,k=0}^1 \text{Ker}^B(-1/2, \vartheta, \psi; j, k) W^B(j, k).
\end{aligned} \tag{41}$$

We can also reconstruct tomograms (40), in view of the kernels  $\text{Ker}^B(m, \vec{n}; j, k) = \text{Tr}(\hat{Q}(m, \vec{n})\hat{\mathcal{B}}_{j,k})$  and  $\widetilde{\text{Ker}}^B(m, \vec{n}; j, k) = \text{Tr}(\hat{D}(m, \vec{n})\hat{\mathcal{B}}_{j,k})/2$ , through the Wigner functions (30) and (31). After some algebra, we obtain the following relationships:

$$\begin{aligned}
W^A(j, k) &= \frac{1}{4\pi} \sum_{m=-1/2}^{1/2} \int_0^\pi \int_0^{2\pi} w(m, \vartheta, \psi) \widetilde{\text{Ker}}^A(m, \vartheta, \psi; j, k) \sin \vartheta d\vartheta d\psi, \\
W^B(j, k) &= \frac{1}{4\pi} \sum_{m=-1/2}^{1/2} \int_0^\pi \int_0^{2\pi} w(m, \vartheta, \psi) \widetilde{\text{Ker}}^B(m, \vartheta, \psi; j, k) \sin \vartheta d\vartheta d\psi.
\end{aligned} \tag{42}$$



## 7 Conclusions

In conclusion, we list the main results of our study.

We constructed the Wigner functions of the one-qubit state using the framework of the star-product quantization scheme and explicit forms of the quantizer and dequantizer operators. We also constructed the probability distributions for the one-qubit state applying the star-product scheme and the pair of quantizer and dequantizer operators determining the state tomogram. We elaborated the procedure of finding the relation of the tomograms of the qubit state to the explicit form of kernels providing the map of the qubit state tomograms onto the Wigner functions and vice versa.

In our approach, the known formulas determining the Wigner function of qubit state in terms of operators  $\hat{A}_\alpha$  [4] are reformulated as formulas used in the star-product quantization schemes, where the quantizer–dequantizer operator pair provides an invertible map of operators onto their symbols. Also the qubit-state tomograms were mapped onto the Wigner functions, in view of the procedure based on the tomographic quantizer–dequantizer pair. We calculated the interwinning kernels connecting the different sorts of Wigner functions and the qubit tomograms given in terms of quantizer–dequantizer pairs. The application of the elaborated scheme to the two-qubit and multiqubit states, using an analogous approach, will be considered in a future publication.

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## References

- [1] J. Schwinger, *Proc. Natl. Acad. Sci. U.S.A.*, **46**, 570 (1960).
- [2] W. K. Wootters, *IBM J. Res. Dev.*, **48**, 99 (2004).
- [3] A. Vourdas, *Acta Appl. Math.*, **93**, 197 (2006).
- [4] K. S. Gibbons, M. J. Hoffman, and W. K. Wootters, *Phys. Rev. A* **70**, 062101 (2004).
- [5] A. B. Klimov, J. L. Romero, G. Björk, and L. L. Sánchez-Soto, *J. Phys. A: Math. Theor.*, **40**, 3987 (2007).
- [6] P. Stóvicék and J. Tolar, *Rep. Math. Phys.*, **20**, 157 (1984).

- [7] S. Chaturvedi, E. Ercolessi, G. Marmo, et al., *J. Phys. A: Math. Gen.*, **39**, 1405 (2006).
- [8] S. N. Filippov and V. I. Man'ko, *Phys. Scr.*, **T143**, 014010 (2011).
- [9] I. D. Ivanović, *J. Phys. A: Math. Gen.*, **14**, 3241 (1981).
- [10] W. K. Wootters, *Ann. Phys. (N.Y.)*, **176**, 1 (1987).
- [11] V. V. Dodonov and V. I. Man'ko, *Phys. Lett. A*, **229** 335 (1997).
- [12] V. I. Man'ko and O. V. Man'ko, *J. Exp. Theor. Phys.*, **85**, 430 (1997).
- [13] V. A. Andreev and V. I. Man'ko, *J. Exp. Theor. Phys.*, **87**, 239 (1998).
- [14] V. A. Andreev, O. V. Man'ko, V. I. Man'ko, and S. S. Safonov, *J. Russ. Laser Res.*, **19**, 340 (1998).
- [15] V. I. Man'ko, G. Marmo, A. Simoni, and F. Ventriglia, *Phys. Lett. A*, **372**, 6490 (2004).
- [16] S. L. Stratonovich, *Sov. Phys. JETP*, **4**, 891 (1957).
- [17] F. Bayen, M. Flato, C. Fronsdal, et al., *Lett. Math. Phys.*, **1**, 521 (1977).
- [18] E. Wigner, *Phys. Rev.*, **40**, 749 (1932).
- [19] O. V. Man'ko, V. I. Man'ko, and G. Marmo, *J. Phys. A: Math. Gen.*, **35**, 699 (2002).
- [20] W. K. Wootters, *Found. Phys.*, **36**, 112 (2006).
- [21] I. Ghiu, *J. Phys.: Conf. Ser.*, **338**, 012008 (2012).
- [22] G. Björk, J. L. Romero, A. B. Klimov, and L. L. Sánchez-Soto, *J. Opt. Soc. Am. B*, **24**, 371 (2007).
- [23] I. Ghiu, *Phys. Scr.*, **T153**, 014027 (2013).
- [24] A. B. Klimov, J. L. Romero, G. Björk, and L. L. Sánchez-Soto, *Ann. Phys. (N.Y.)*, **324**, 53 (2009).
- [25] T. Paterek, B. Dakic, and C. Brukner, *Phys. Rev. A*, **83**, 036102 (2011).
- [26] T. Paterek, M. Pawłowski, M. Grassl, and C. Brukner, *Phys. Scr.*, **T140**, 014031 (2010).
- [27] C. Ghiu and I. Ghiu, *Cent. Eur. J. Math.*, **12**, 337 (2014).
- [28] T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski, *Int. J. Quantum Inform.*, **8**, 535 (2010).
- [29] R. Asplund and G. Björk, *Phys. Rev. A*, **64**, 012106 (2001).
- [30] R. Franco and V. Penna, *J. Phys. A: Math. Gen.*, **39**, 5907 (2006).
- [31] A. Ibort, V. I. Man'ko, G. Marmo, et al., *Phys. Scr.*, **79**, 065013 (2009).
- [32] S. N. Filippov and V. I. Man'ko, *J. Russ. Laser Res.*, **31**, 32 (2010).
- [33] M. O. Terra Cunha, V. I. Man'ko, and M. O. Scully, *Found. Phys. Lett.*, **14**, 103 (2001).