The main question we ask in approximation theory concerns the possibility of approximation .Is the given family of functions from which we plan to approximate dense in the set of functions we wish to approximate? The first significant density results were those of Weierstrass who proved in1885 the density of algebraic polynomials in the class of continuous real-valued functions on a compact interval. The Weierstrass approximation theorems spawned numerous generalizations which were applied to other families of functions. They also led to the development of general methods for determining density. One of these methods is the Stone-Weierstrass theorem generalizing the Weierstrass theorem to subalgebras of $C(X)$; X a compact space. In particular the Stone-
Weierstrass theorem yields that multivariate polynomials are also dense in the space of continuous functions on compact subsets of the *d*-dimensional Euclidean Space. It is considerably harder to verify density for those families of functions which do not satisfy the subalgebra condition. A classical example of such a problem is the Müntz problem of density of lacunary polynomials. (Lacunary polynomials are not closed relative to multiplication and thus the subalgebra condition fails.) Another density problem which has been widely investigated in the past 20 years concerns the density of weighted polynomials of the form $w_n p_n$, where p_n is an algebraic polynomial of degree at most n. The union of such polynomials is not a linear space hence the Stone-Weierstrass theorem is again not applicable.
An important family of multivariate polynomials arising frequently in various areas of analysis is the space of homogeneous polynomials of exact degree n. These polynomials appear in problems related to approximation by neural networks, ridge functions, approximation of curves and
surfaces by algebraic surfaces, etc. Thus it is natural to ask if the Weierstrass theorem holds in some form for homogeneous polynomials? Note that the set of *homogeneous polynomials* is not a linear space, i.e., the Stone-Weierstrass
theorem again does not apply here. Due to the algebraic structure of
homogeneous polynomials the natural domain for this problem is a star-like 0-symmetric surface, in particular the boundary of a 0-symmetric convex body. Also since any homogeneous polynomial is either even or odd, in order
to be able to approximate an arbitrary continuous function we clearly need at least 2 polynomials!

In his plenary talk at the 5th International Conference on Functional Analysis and Approximation Theory held in Maratea (Italy) in 2004 the Principle Investigator proposed the following conjecture:

CONJECTURE. *Let K be an arbitrary* 0-*symmetric convex body in d-space. Then any function continuous on the boundary of K can be approximated arbitrarily close in the uniform norm by sums of two homogeneous polynomials.*

Let us mention that in general the convexity of K is necessary for the density, one can give examples of non convex star-like domains for which density fails.
The above conjecture inspired considerable attention and substantial progress was made towards its resolution. Namely it was verified in case when

d=2, in case when K is a convex polytope, and also in case when the boundary of K is sufficiently smooth.

In the framework of the present project we developed a new technique of using partitions of unity to smooth functions with non compact support and verified the Main Conjecture in two new important cases:

1) The conjecture was proved in uniform norm for an arbitrary **regular convex body,** that is a convex body possessing a unique tangent plane at each point of its boundary.

2) The conjecture was verified in Lp norm in its **full generality** for arbitrary convex bodies.

The second fundamental question in Approximation Theory is that of the rate of approximation. In case of approximation by multivariate polynomials of total degree the well known Jackson-type theorems give the rate of best approximation in terms of the modulus of continuity of functions considered. On the other hand the rate of approximation by homogeneous polynomials depends not only on the modulus of continuity of the approximated functions but also on the smoothness of the boundary of the underlying domain. This connection between analytic properties of functions and geometric properties of the domain make the problem especially intriguing.
We explored this connection and obtained sharp Jackson-type estimates for the rate of best approximation by homogeneous polynomials in uniform norm.

The proof of sharpness of Jackson-type estimates for best approximation is usually related to the so-called Markov-type bounds for the derivatives of polynomials, when the norm of the gradients of polynomials is estimated under the assumption that the norm of polynomials is given. In the past
20 years the study of these inequalities was extended to the multivariate case, however these investigations where concentrated solely on uniform norm estimates.

In the framework of the present project we obtained new multivariate Markov-type inequalities in the case of arbitrary *Lp* norms. There is a principal difference between studying these inequalities in the uniform versus
Lp norms. In order to estimate the uniform norm of derivatives the standard approach usually consists in inscribing into the domain a proper curve passing through the given point and thus reducing the question to the case of one variable. In case of Lp norms such a straightforward linearization technique does not work, a more complex geometric approach was needed.

Weighted approximation is a classical topic in approximation theory.

Generalizing some classical results by Bernstein and Szegő we obtained an asymptotic representation for weighted Chebyshev polynomials with arbitrary weights both in uniform and Lp norms on the interval [-1, 1].

We also considered polynomial approximation problems on the real line with generalized Freud weights. The generalization means multiplying these weights by so-called generalized polynomials which have real roots only. Analogues of classical polynomial inequalities, as well as direct and converse approximation theorems were proved.

We proved that for Jacobi and Pollaczek weights the weighted error of approximation by generalized Bernstein polynomials introduced earlier is equivalent to the modulus of smoothness of the function. This result is analogous to a well-known theorem of Ditzian and Ivanov for the classical Bernstein polynomials.

We gave error estimates for the weighted approximation of functions on the semiaxis by some modifications of the Szász-Mirakyan operators. To do so we defined a new weighted modulus of smoothness and proved its equivalence to the weighted K-functional. Also, the class of functions for which the modified Szász--Mirakyan operator can be defined was extended to a much wider set than the original operator.

We also proved that the weighted error of approximation by these operators is equivalent to the modulus of smoothness of function. This result is analogous to the results for Bernstein-type operators mentioned above.

We established sufficient conditions on the parameter defining the Cesaro means of Fourier--Jacobi series in spaces of locally continuous functions in order to have bounded weighted norm. For some values of the parameters a Stechkin type error estimate for the order of convergence was also given.

We gave error estimates for the weighted approximation of functions with Freud-type weights, by entire functions interpolating at finitely or infinitely many points on the real line. The error estimates involve weighted moduli of continuity corresponding to general Freud-type weights for which the density of polynomials is not always guaranteed.