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Covering symmetric skew-supermodular functions with hyperedges

Attila Bernáth and Tamás Király

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Attila Bernáth[⋆] and Tamás Király[⋆⋆]

Abstract

In this paper we give results related to a theorem of Szigeti that concerns the covering of symmetric skew-supermodular set functions with hyperedges of minimum total size. In particular, we show the following generalization using a variation of Schrijver's supermodular colouring theorem: if p_1 and p_2 are skew-supermodular functions whose maximum value is the same, then it is possible to find in polynomial time a hypergraph of minimum total size that covers both of them. Note that without the assumption on the maximum values this problem is NP-hard. The result has applications concerning the local edge-connectivity augmentation problem of hypergraphs and the global edge-connectivity augmentation problem of mixed hypergraphs. We also present some results on the case when the hypergraph must be obtained by merging given hyperedges.

1 Introduction

In this paper we give generalizations of a theorem of Szigeti (Theorem 2.1 below) that concerns covering symmetric skew-supermodular set functions with hyperedges. Szigeti's main motivation was local edge-connectivity augmentation of hypergraphs; we will show other applications, too.

Given a finite ground set V, a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *skew-supermodular* if at least one of the following two inequalities holds for every $X, Y \subseteq V$:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y), \tag{$\cap\cup$}$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{$-$}$$

A set function is *symmetric* if $p(X) = p(V - X)$ for every $X \subseteq V$. For an arbitrary set function p we define **the symmetrized of** p by $p^s(X) = \max\{p(X), p(\overline{X})\}$ for

[⋆]Dept. of Operations Research, Eötvös University, Pázmány P. s. 1/C, Budapest, Hungary H-1117. The author is a member of the Egerváry Research Group (EGRES). E-mail: `bernath@cs.elte.hu`

[⋆⋆]MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117. e-mail: `tkiraly@cs.elte.hu`. Supported by the Hungarian National Foundation for Scientific Research, OTKA K60802, and by European MCRTN Adonet, Contract Grant No. 504438.

any $X \subseteq V$. Observe that the symmetrized of a skew-supermodular function is skew-supermodular. Two sets $X, Y \subseteq V$ are *crossing* if $X \cap Y, V - X \cup Y, X - Y$ and $Y - X$ are all nonempty. A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *crossing supermodular* (*crossing negamodular*, resp.) if it satisfies $(\cap\cup)$ $((-)$, resp.) whenever X and Y are crossing. One can check that the symmetrized of a crossing supermodular or a crossing negamodular function is skew-supermodular.

For a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ we introduce the polyhedron

$$C(p) = \{x \in \mathbb{R}^V : x(Z) \geq p(Z) \ \forall Z \subseteq V, x \geq 0\}.$$

It is known that for a skew-supermodular function p this is an (integer) contrapolymatroid (for details see [1]).

An undirected hypergraph (or shortly hypergraph) $H = (V, \mathcal{E})$ is a pair of a finite set V and a family $\mathcal{E}$ of subsets of V (repetitions are allowed). The set V is called the *node set* of the hypergraph, the family $\mathcal{E}$ is called the *edge set* of the hypergraph. An element of $\mathcal{E}$ will be called a *hyperedge*.

In a hypergraph H, a *path* between nodes s and t is an alternating sequence of distinct nodes and hyperedges $s = v_0, e_1, v_1, e_2, \ldots, e_k, v_k = t$, such that $v_{i-1}, v_i \in e_i$ for all i between 1 and k. H is connected if there is a path between any two distinct nodes. A hyperedge e enters a set X if $e \cap X \neq \emptyset$ and $e \cap (V - X) \neq \emptyset$. For a set X we define $d_H(X) = |\{e \in \mathcal{E} : e \text{ enters } X\}|$ (the *degree* of X in H). This is a symmetric submodular function.

Definition 1.1. *Given a hypergraph $H = (V, \mathcal{E})$ and sets $X, Y \subseteq V$, let $\lambda_H(X, Y)$ denote the maximum number of edge-disjoint paths starting at a vertex of X and ending at a vertex of Y (we say that $\lambda_H(X, Y) = \infty$ if $X \cap Y \neq \emptyset$). The subscript H may be omitted if no confusion can arise.*

It is well known that Menger's theorem can be generalized for hypergraphs:

Theorem 1.2. *Let $H = (V, \mathcal{E})$ be a hypergraph, and $s, t \in V$ distinct nodes. Then*

$$\lambda_H(x, y) = \min\{d_H(X) : X \subseteq V, s \notin X, t \in X\}.$$

The hypergraph H is said to *cover* the function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ if $d_H(X) \geq p(X)$ for every $X \subseteq V$. The *total size* of a hypergraph is the sum of the sizes of its hyperedges. A hypergraph is said to be *nearly uniform* if the sizes of its hyperedges differ by at most one. A set $\{v\}$ containing exactly one element will also be called a *singleton* and we will sometimes write $d(v)$ instead of $d(\{v\})$ for a set function d.

Our results are presented in the following two sections. In Subsection 2.1 we extend the results of Szigeti [9] to the problem where a hypergraph covering a given skew-supermodular set function should be constructed by merging some hyperedges of a given hypergraph (Theorem 2.2).

Szigeti's result easily implied a special case of the supermodular colouring theorem of Schrijver [7]. We show in Subsection 2.2 an inverse implication: a variation of the supermodular colouring theorem implies a strengthening of the result of Szigeti which states that there is a solution where the hypergraph is nearly uniform, i.e. the sizes of the hyperedges differ by at most one (Theorem 2.10).

In Section 3 we present applications for the results given: besides local edge-connectivity augmentation of hypergraphs we introduce the node-to-area connectivity augmentation problem in hypergraphs and the global arc-connectivity augmentation problem of mixed hypergraphs.

The real strength of the supermodular colouring theorem is the fact that two requirement functions can be satisfied simultaneously. The main result presented in Subsection 3.4 is that we can generalize Theorem 2.10 to the problem of covering two symmetric skew-supermodular functions simultaneously, provided that the maximum values of the two functions are the same (Corollary 3.5). It turns out that without this last assumption the problem is NP-complete. As an example, we show that the local edge-connectivity augmentation problem for hypergraphs can be solved simultaneously for two hypergraphs if the maximum deficiencies are the same in the two instances.

2 Results

In this paper we consider the problem of covering a symmetric skew-supermodular set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ by a hypergraph. We distinguish two versions of this problem. In the *degree bounded version* we are also given a *degree bound* $m : V \to \mathbb{Z}_+$ and the question is whether a hypergraph H covering p exists with $d_H(v) \leq m(v)$ for every $v \in V$. In the *minimum version* we simply want to find a hypergraph covering p that has minimum total size. Possibly the latter problem seems more interesting and natural, however by the properties of a contrapolymatroid, a polynomial algorithm to the degree bounded covering problem will give rise to a solution to the minimum version of the problem, and to more general versions, too. As an example, the *minimum node-cost version* of the problem is the following: find a hypergraph H covering p that minimizes $\sum_{v \in V} c(v) d_H(v)$, where $c : V \to \mathbb{R}$ is a nonnegative cost function. Therefore we will mainly speak about the degree bounded version of the problem. For more details we refer to [1] and [4].

In [9], Szigeti proved the following result, which is fundamental for solving local edge-connectivity augmentation problems in hypergraphs.

Theorem 2.1 ([9]). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supermodular set function, and $m : V \to \mathbb{Z}_+$ a degree bound. There exists a hypergraph H s.t. $d_H(v) \leq m(v)$ for every $v \in V$ and $d_H(X) \geq p(X)$ for every $X \subseteq V$ if and only if*

$$\sum_{v \in X} m(v) \geq p(X) \quad \text{for every } X \subseteq V. \tag{1}$$

Furthermore, if $m(v) \leq k = \max\{p(X) : X \subseteq V\}$ for any $v \in V$, then H can be chosen so that it consists of exactly k hyperedges.

In this paper we give several generalizations of this theorem. As an application Szigeti considers local edge-connectivity augmentation of hypergraphs with hyperedges of minimum total size: we will give other applications, too.

Let us mention an algorithmic aspect. We think of the symmetric skew-supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ as a function that is given with a function-evaluation oracle. It will be clear that the proof below can be converted to a polynomial time algorithm to find the hypergraphs covering p if we can maximize the function $p - d_{H'} - b_{H''}$ in polynomial time for arbitrary hypergraphs H' and H'' (where these hypergraphs are given with their adjacency matrices, $b_{H''}$ will be defined later). However, there is no hope to maximize such a function in general, since the question "Is $p \le B$?" (where $B \in \mathbb{Z}$) is not even in NP, as can be checked easily. However, if p is finite then it is an open problem at the moment whether we can maximize p in polynomial time. In many applications however we can clearly do this: in what follows we assume that we can maximize such functions, since this will hold in the applications detailed below.

2.1 Merging hyperedges

Part of the results presented in this subsection has already appeared in [6]. Let $H = (V, \mathcal{E})$ be a hypergraph. By *merging* two disjoint hyperedges of H we mean the operation of replacing them in H by their union. "Merging some hyperedges of H" means repeating this operation a few times. Let us define the set function

$$b_H(X) := |\{e \in \mathcal{E} : \ e \cap X \neq \emptyset\}|.$$

It is easy to see that b_H is fully submodular, monotone, and

$$b_H(X) + b_H(Y) \ge b_H(X - Y) + b_H(Y - X) + |\{e \in \mathcal{E} : \ \emptyset \neq e \cap Y \subseteq X \cap Y\}|.$$

Theorem 2.2. *Let $H = (V, \mathcal{E})$ be a hypergraph, and let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supermodular set function with $k = \max\{p(X) : X \subseteq V\} \ge 0$, for which*

$$b_H(X) \ge p(X) \quad \text{for every } X \subseteq V. \tag{2}$$

(i) *Then by merging some hyperedges of H we can obtain a hypergraph $H_* = (V, \mathcal{E}_*)$ that covers p.*

(ii) *Furthermore, if there are k hyperedges $f^1, f^2, \ldots, f^k$ in H such that every hyperedge in $H - \{f^1, \ldots, f^k\}$ is a singleton and $b_H(v) \le k$ for any $v \in V$, then the merging operations can be organized in a way that $H_* = (V, \{f^1_*, f^2_* \ldots, f^k_*\})$ where $f^i \subseteq f^i_*$ for $i = 1, \ldots, k$.*

Proof. We prove (i) by induction on the number of hyperedges of H (it is clearly true if $\mathcal{E} = \emptyset$). A set $X \subseteq V$ is called *tight* if $b_H(X) = p(X)$. By the properties of b_H and p, if X and Y are tight, then either $X \cap Y$ and $X \cup Y$ are tight, or $X - Y$ and $Y - X$ are tight. Furthermore, if X and Y are tight and there is a hyperedge e such that $\emptyset \neq e \cap Y \subseteq X \cap Y$, then $X \cap Y$ and $X \cup Y$ are tight.

Let e_0 be an arbitrary hyperedge of H. If there is no tight set X such that $e_0 \subseteq X$, then let $H' := H - e_0$ and $p' = p - d_{H_0}$ where $H_0 = (V, \{e_0\})$. The set function p' is symmetric and skew-supermodular, and $b_{H'}(X) \ge p'(X)$ for every $X \subseteq V$, so by induction there is a hypergraph H'_*, obtained by merging some hyperedges of H', such that $d_{H'_*}(X) \geq p'(X)$ for every $X \subseteq V$. It follows that $H_* := H'_* + e_0$ covers p. We can thus assume that there is a tight set X_0 such that $e_0 \subseteq X_0$: let X_0 be a maximal tight set containing e_0.

Suppose that there is no hyperedge $e \in \mathcal{E}$ such that $e \cap X_0 = \emptyset$. Then $p(V - X_0) = p(X_0) = b_H(X_0) > b_H(V - X_0)$ since $e_0 \subseteq X_0$, contradicting (2). Thus there is a hyperedge $e_1 \in \mathcal{E}$ such that $e_1 \cap X_0 = \emptyset$. Consider the hypergraph $H' := (V, \mathcal{E} - \{e_0, e_1\} + (e_0 \cup e_1))$, i.e. the hypergraph obtained by merging e_0 and e_1. If $b_{H'}(Y_0) < p(Y_0)$ for some $Y_0 \subseteq V$, then $e_0 \cap Y_0 \neq \emptyset$, $e_1 \cap Y_0 \neq \emptyset$, and Y_0 was tight. Since $\emptyset \neq e_0 \cap Y_0 \subseteq X_0 \cap Y_0$, $X_0 \cup Y_0$ is also tight, which contradicts the maximality of X_0.

We proved that H' and p satisfy (2), so by induction there is a hypergraph H_*, obtained by merging some hyperedges of H' (hence obtained by merging some hyperedges of H), that covers p.

The proof of (ii) is similar to the proof of Theorem 2.1 by Szigeti. We will use the following observation. Let $X, Y \subseteq V$ such that X is tight and $p(Y) = k$. If $(\cap\cup)$ applies for X and Y then $p(X \cup Y) \leq p(Y) = k$ implies that $p(X \cap Y) \geq p(X) = b_H(X) \geq b_H(X \cap Y) \geq p(X \cap Y)$, so every inequality is satisfied with equality here (including $p(X \cup Y) = k$). On the other hand, if $(-)$ applies for X and Y then $p(Y - X) \leq p(Y) = k$ implies that $p(X - Y) \geq p(X) = b_H(X) \geq b_H(X - Y) \geq p(X - Y)$, so every inequality is satisfied with equality here (including $p(Y - X) = k$).

We will prove the statement indirectly: suppose that H, p and $f^1, \ldots, f^k$ form a counterexample with k as small as possible and, subject to that, $|V - f^k|$ as small as possible. Trivially, $k > 0$. Suppose that there is a set Y with $p(Y) = k$ that is disjoint from f^k. Since $b_H(Y) \geq p(Y) = k$ there must be a hyperedge $e \in \mathcal{E} - \{f^1, \ldots, f^{k-1}, f^k\}$ that intersects Y: since these hyperedges are singletons, in fact $e \subseteq Y$. Let H' be obtained from H by merging f^k and e into a hyperedge f'^k. We claim that H' does not violate (2): if it does then there was a tight set X such that $f^k \cap X \neq \emptyset$, $e \cap X \neq \emptyset$ ($e \subseteq X$ in fact). But $b_H(X) > b(X \cap Y)$ because of the edge f^k, implying that $(\cap\cup)$ cannot apply for X and Y. On the other hand, $b_H(X) > b_H(X - Y)$ because of the hyperedge e, so $(-)$ cannot apply for X and Y either, a contradiction. By the minimal choice of H, the statement is true for H', but then also for H, a contradiction. So in our minimal counterexample f^k intersects every set Y with $p(Y) = k$.

Similarly we claim that in this minimal counterexample f^k must cover every vertex v with $b_H(v) = k$. Assume that this is not the case and v is such a vertex not covered by f^k. Then there must be a hyperedge $e \in \mathcal{E} - \{f^1, \ldots, f^{k-1}, f^k\}$ that covers v and if we merge f^k with e then the hypergraph H' obtained will not violate (2), since every set Y that intersects both e and f^k has $b_H(Y) \geq k + 1$, so it cannot be tight. Therefore the statement is true for H', and then also for H, a contradiction.

We claim that there is no tight set X satisfying $f^k \subseteq X$. Assume that this is not true and let X be such a tight set. Let Y be an arbitrary set with $p(Y) = k$ (such a set exists by the definition of k). If $(\cap\cup)$ applies for X and Y then $p(X \cup Y) = k$, but then $p(V - (X \cup Y)) = k$, too, but this set is not covered by f^k. However, $(-)$ cannot apply for X and Y either, because then $Y - X$ would be a set with $p(Y - X) = k$ not covered by f^k. So we really obtained that there is no tight set containing f^k.

Let $H' = H - f^k$ and $p' = p - d_{H^k}$, where $H^k = (V, \{f^k\})$. Then $\max\{p'(X) :$

$X \subseteq V\} = k-1$, $b_{H'}(v) \leq k-1$ for every $v \in V$, and $b_{H'} \geq p'$, thus H', p' and $f^1, \dots, f^{k-1}$ must satisfy the statement to be proved (otherwise H was not a minimal counterexample), so there exists a hypergraph $H'_* = (V, \{f^1_*, \dots, f^{k-1}_*\})$ that covers p' and satisfies $f^i \subseteq f^i_*$ for any i between 1 and $k-1$. But then one can easily check that $H_* = H'_* + f^k$ satisfies our requirements, so H was not a counterexample. □

Theorem 2.1 corresponds to the case when H consists of hyperedges of size 1, and $m(v)$ is the multiplicity of $\{v\}$ in H.

2.2 Skew-supermodular colourings

Szigeti showed that Theorem 2.1 implies a special case of the supermodular colouring theorem of Schrijver [7]. We have to mention that the supermodular colouring theorem is true in a more general form, namely it is true for skew-supermodular functions instead of intersecting supermodular functions. This was observed by the second author in [5]: let us call this the "skew-supermodular colouring theorem", a proof of it will be given below. With similar methods to those of Szigeti we can prove the following theorem related to skew-supermodular colourings. Let us first introduce some terminology. A *k-colouring* is a partition $X_1, \dots, X_k$ of V (where $X_i \neq \emptyset$ for any $i = 1, \dots, k$). If a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is also given then a k-colouring is *good* (for p) if $|\{i : X_i \cap X \neq \emptyset\}| \geq p(X)$ for any $X \subseteq V$. More generally we will say that a hypergraph H *weakly covers* the set function p if $b_H(X) \geq p(X)$ for any $X \subseteq V$.

Theorem 2.3. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a skew-supermodular function with $k = \max\{p(X) : X \subseteq V\} > 0$. Let furthermore $X_1, \dots, X_k$ be a subpartition of V. Then there is a good k-colouring $X'_1, X'_2, \dots X'_k$ of V satisfying $X_i \subseteq X'_i$ for any $i = 1, \dots, k$ if and only if*

$$p(X) \leq |\{i : X_i \cap X \neq \emptyset\}| + |X - \cup_i X_i| \text{ for any } X \subseteq V. \tag{3}$$

Proof. The necessity of (3) is clear so let us prove its sufficiency. Let $x \notin V$ be a new vertex and $V' = V + x$. Define $p' : 2^{V'} \to \mathbb{Z} \cup \{-\infty\}$ the following way: let $p'(X) = p(X)$ if $X \subseteq V$ and $p'(X) = p(V - X)$ if $x \in X$. It is easy to check that p' is symmetric and skew-supermodular. Let $H = (V', \mathcal{E}')$ be the following: it contains hyperedges $f_i = X_i$ for $i = 1, \dots, k$ and singleton hyperedges $\{v\}$ for any $v \in V - \cup_i X_i$ with multiplicity one and the hyperedge $\{x\}$ with multiplicity k. Observe that H and p' satisfy (2): for a set $X \subseteq V$ this translates to (3), and for a set $X \subseteq V'$ with $x \in X$ we have $p'(X) = p(V - X) \leq k \leq b_H(X)$. So by Theorem 2.2 (ii) there exists $H_* = (V', \{f^1_*, \dots, f^k_*\})$ with $f_i \subseteq f^i_*$ for $i = 1, \dots, k$ that covers p': this hypergraph defines a good k-colouring of V by $X'_i = f^i_* - x$ for all $i = 1, 2, \dots, k$. □

Now we give a generalized version of the skew-supermodular colouring theorem that will have an interesting consequence concerning simultaneous augmentation of hypergraphs. Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be an arbitrary skew-supermodular function. Then it is well known that the polyhedron

$$C = C(p) = \{y \in \mathbb{R}^V : y(Z) \geq p(Z) \ \forall Z \subseteq V, y \geq 0\}$$

is a contrapolymatroid. Assume that $C \neq \emptyset$ (equivalently, $p(\emptyset) \leq 0$) and let $y \in C \cap \mathbb{Z}^V$ be an arbitrary integer vector in C. Let $k \geq \max\{p(X) : X \subseteq V\}$ (note that the vector $(k, k, \ldots, k)$ is in C). Then we can define the polyhedron

$$Q = Q(p,k,y) = \{x \in \mathbb{R}^V : x(Z) \geq 1 \text{ if } p(Z) = k, \\ x(Z) \leq y(Z) - p(Z) + 1 \ \forall Z \subseteq V,\ 0 \leq x \leq y\}.$$

One can simply prove that $y/k \in Q$, so Q is not empty. The proof of the following lemma essentially follows the line of the proof of the supermodular colouring theorem that appears in Schrijver's book [8].

Lemma 2.4. *Q is a (nonempty, integer) g-polymatroid.*

Proof. We will show that $Q' = y - Q$ is a g-polymatroid, from which the statement follows. One can see that

$$Q' = \{0 \leq x \leq y : x(Z) \geq p(Z) - 1 \ \forall Z \subseteq V,\ x(T) \leq y(T) - 1 \text{ if } p(T) = k\}.$$

Let

$$\mathcal{D} = \{T \subseteq V : p(T) = k \text{ but } p(Y) < k \text{ for any } Y \subsetneq T\},$$
$$\mathcal{C} = \{X \subseteq V : X \subseteq T \text{ for some } T \in \mathcal{D} \text{ or } X \cap T = \emptyset \ \forall T \in \mathcal{D}\}.$$

It is easy to check that D is a subpartition. It is also clear that

$$Q' = \{0 \leq x \leq y :\ x(Z) \geq p(Z) - 1 \ \forall Z \subseteq V,\ x(T) \leq y(T) - 1 \ \forall T \in \mathcal{D}\}.$$

We claim that Q' is actually equal to

$$Q'' = \{0 \leq x \leq y :\ x(Z) \geq p(Z) - 1 \ \forall Z \in \mathcal{C},\ x(T) \leq y(T) - 1 \ \forall T \in \mathcal{D}\}.$$

We only need to show that $Q'' \subseteq Q'$. Let $x \in Q''$ and $Z \subseteq V$ arbitrary: we have to show that $x(Z) \geq p(Z) - 1$. To prove this assume that Z intersects $t > 0$ members of $\mathcal{D}$ and prove by induction on t. Let $T \in \mathcal{D}$ be one of the t members of $\mathcal{D}$ intersected by Z.

Assume T and Z satisfy $(\cap\cup)$. This implies $p(Z) \leq p(T \cap Z)$ (since $p(T)$ is maximum). Then $x(Z) \geq x(Z \cap T) \geq p(Z \cap T) - 1 \geq p(Z) - 1$. Otherwise T and Z satisfy $(-)$, implying $p(Z) \leq p(Z - T)$. Then $x(Z) \geq x(Z - T) \geq p(Z - T) - 1 \geq p(Z) - 1$, since $Z - T$ intersects $t - 1$ members of $\mathcal{D}$, so we can use induction.

Let $f(X) = \max\{\sum_{i=1}^{t}(p(X_i) - 1) : X_1, \ldots, X_t \text{ is a subpartition of } X\}$ for any $X \in \mathcal{C}$ and $-\infty$ otherwise, $g(X) = y(X) - 1$ for any $X \in \mathcal{D}$ and ∞ otherwise. Then it is easy to check that f and g form a weak pair and Q' is the g-polymatroid determined by them. □

This lemma implies the following "skew-supermodular colouring theorem" which is an extension of Schrijver's supermodular colouring theorem.

Theorem 2.5 ([5]). *Let $p_1, p_2 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be two skew-supermodular functions and $k \geq 1$ an integer. Then there is a k-colouring that is good for both p_1 and p_2 if and only if*

$$p_i(X) \leq \min\{k, |X|\} \text{ for any } X \subseteq V \text{ and } i = 1, 2.$$

We want to relate the integer points of Q with hyperedges of a hypergraph. To this end we would need to generalize the notion of hypergraphs to allow multiplicities in hyperedges. Note however that we do not really need this: if $x \in Q$ then $\min(x, 1) \in Q$, too. From known results we have the following.

Corollary 2.6. *$Q_1 = Q \cap \{x \in \mathbb{R}^V : 0 \le x \le 1\}$ and $Q_2 = Q_1 \cap \{x \in \mathbb{R}^V : x(v) = 1$ if $y(v) = k$, $\lfloor \frac{y(V)}{k} \rfloor \le x(V) \le \lceil \frac{y(V)}{k} \rceil\}$ are g-polymatroids (Q_1 is not empty and if $y \le k$ then Q_2 is not empty, either).*

Q_1 corresponds to hypergraphs without multiplicities in the hyperedges and Q_2 corresponds to nearly uniform hypergraphs.

Theorem 2.7. *An integer vector $x \in \mathbb{Z}^V$ is in Q_1 if and only if it is the characteristic vector of a hyperedge of a hypergraph $H = (V, E)$ containing at most k hyperedges which weakly covers p and satisfies $d_H(v) \le y(v)$ for any $v \in V$.*

Proof. If $H = (V, E)$ is a hypergraph containing at most k hyperedges that satisfies $d_H(v) \le y(v)$ for every $v \in V$ and $b_H \ge p$ then clearly $\chi_e \in Q_1 \cap \mathbb{Z}^V$ for any $e \in E$.

Let $x \in Q_1 \cap \mathbb{Z}^V$ (possibly $x = 0$). We need to prove that there is a hypergraph H with the desired properties. We prove by induction on k, the $k = 0$ case being trivial. Let $H_k = (V, \{e_k\})$ where $\chi_{e_k} = x$, $p^* = p' - b_{H_k}$, $y^* = y - x$. By the assumptions made above $\max\{p^*(X) : X \subseteq V\} \le k - 1$ and $y^* \in C(p^*)$, so by induction (with an arbitrary choice of an $x^* \in Q(p^*, k-1, y^*) \cap \{0,1\}^V$) there is a hypergraph H^* with at most $k - 1$ hyperedges satisfying $b_{H^*} \ge p^*$ and $d_{H^*}(v) \le y^*(v)$ for every $v \in V$, and then $H^* + H_k$ is the hypergraph we were looking for. □

An analogous theorem can be proved for Q_2, the proof is omitted.

Theorem 2.8. *Assume $y \le k$. Then $x \in \mathbb{Z}^V \cap Q_2$ if and only if it is the characteristic vector of a hyperedge of a nearly uniform hypergraph $H = (V, E)$ containing at most k hyperedges which weakly covers p and satisfies $d_H(v) \le b_H(v) = y(v)$ for any $v \in V$.*

These theorems imply the existence of a (nearly uniform) hypergraph of at most k hyperedges satisfying the degree bound y that **weakly covers** p. However usually we are more interested in hypergraphs **covering** a skew-supermodular function. The following lemma shows that for symmetric set functions the two problems are closely related.

Lemma 2.9. *If $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is a* symmetric *skew-supermodular function, $k = \max\{p(X) : X \subseteq V\}$, and H is a hypergraph containing exactly k hyperedges, then $b_H \ge p$ implies that H* covers *p.*

Proof. The simplest way of proving this at this point is just to say that it obviously follows from Theorem 2.2 (i). However we give a direct proof, too. Suppose that H does not cover p, so there is a set X with $b_H(X) \ge p(X) > d_H(X) = b_H(X) - i_H(X)$, where $i_H(X)$ denotes the number of hyperedges of H *induced* by X. By the assumptions there is a set T with $p(T) = k$. Since H contains exactly k hyperedges, $p(X \cup T) = p(V - (X \cup T)) \le k - i_H(X)$ and $p(T - X) \le k - i_H(X)$ also follows. If

$(\cap\cup)$ applies for X and T then $p(X\cap T) \geq p(X) + i_H(X) > b_H(X) \geq b_H(X\cap T)$, and if $(-)$ applies for X and T then $p(X-T) \geq p(X) + i_H(X) > b_H(X) \geq b_H(X-T)$, either of which contradicts our assumptions. □

A consequence of these results is that the hypergraph H in Theorem 2.1 can be chosen to be nearly uniform.

Theorem 2.10. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supermodular set function, and $m : V \to \mathbb{Z}_+$ a degree specification, such that $m(v) \leq k = \max\{p(X) : X \subseteq V\}$ for any $v \in V$, and*

$$\sum_{v\in X} m(v) \geq p(X) \quad \textit{for every } X \subseteq V.$$

Then there exists a nearly uniform hypergraph H of exactly k hyperedges s.t. $d_H(v) = m(v)$ for every $v \in V$ and $d_H(X) \geq p(X)$ for every $X \subseteq V$.

Proof. Theorem 2.8 gives a hypergraph H of exactly k hyperedges which satisfies $d_H(v) = m(v)$ for every $v \in V$ and $b_H \geq p$ (observe that H cannot have less hyperedges and the strict inequality $d_H(v) < m(v)$ cannot hold for a node v, either). By Lemma 2.9, the hypergraph H covers p. □

3 Applications

In this section we present applications of our results. Besides the local edge-connectivity augmentation of hypergraphs discussed by Szigeti in [9], we show two other applications. For simplicity, we present the minimum version of the problems in the following three subsections: by the remarks above one can solve other variants, too.

3.1 Local edge-connectivity augmentation of hypergraphs

The **local edge-connectivity augmentation of hypergraphs with hyperedges of minimum total size** (solved by Szigeti in [9]) is the following. Given a hypergraph $H = (V, \mathcal{E})$ and a symmetric edge-connectivity requirement $r : V \times V \to \mathbb{Z}_+$, find a hypergraph H' of minimum total size such that $H + H'$ is r-edge-connected, meaning that

$$\lambda_{H+H'}(u, v) \geq r(u, v) \text{ for every } u, v \in V. \tag{4}$$

Let us define the set function R as $R(\emptyset) = R(V) = 0$ and

$$R(X) = \max_{u\in X, v\notin X} r(u, v) \qquad (\emptyset \neq X \subsetneq V). \tag{5}$$

One can simply check that a hypergraph H' satisfies (4) if and only if H' covers $p = R - d_H$. Since R is a skew supermodular function, applying Theorem 2.10 gives the following extension of Szigeti's result.

Theorem 3.1. *The optimal solution of the local edge-connectivity augmentation of hypergraphs with hyperedges of minimum total size can be chosen to be nearly uniform.*

3.2 The node-to-area connectivity augmentation problem in hypergraphs

Given a hypergraph $H = (V, \mathcal{E})$, a collection of subsets $\mathcal{W}$ of V and a function $r : \mathcal{W} \to \mathbb{Z}_+$ satisfying $r \geq 2$, find a hypergraph H' of minimum total size such that

$$\lambda_{H+H'}(x, W) \geq r(W) \text{ for any } W \in \mathcal{W} \text{ and } x \in V. \tag{6}$$

We will call this problem the **node-to-area connectivity augmentation problem in hypergraphs**. Define

$$R(X) = \max\{r(W) : W \in \mathcal{W},\ W \cap X = \emptyset\} \text{ for any } \emptyset \neq X \subseteq V \text{ and } R(\emptyset) = 0. \tag{7}$$

This is a crossing negamodular function and H' satisfies (6) if and only if it covers $p = R^s - d_H$. We mention that one can test membership in $C(p)$ for this special function p in polynomial time.

Theorem 3.2. *The optimal solution of the node-to-area connectivity augmentation problem in hypergraphs can be found in polynomial time and it can be chosen to be nearly uniform.*

3.3 Augmenting the global edge-connectivity of mixed hypergraphs

A **mixed hypergraph** $M = (V, \mathcal{A})$ is a pair of a finite set V and a family A of subsets of V (repetitions are allowed). For an $a \in \mathcal{A}$ every $v \in a$ can be either a **head node**, a **tail node** or even both (**head-tail node**), such that every hyperarc contains at least one head and one tail. More formally we could say that $\mathcal{A}$ contains nonempty ordered set-pairs (T, H) (T being the set of tails, H being the set of heads, possibly $H \cap T \neq \emptyset$). An undirected hypergraph can be considered (for our purposes) as a special mixed hypergraph where every node in a hyperarc is a head-tail node of this hyperarc. The set V is called the **node set** of the mixed hypergraph, the family $\mathcal{A}$ is called the **hyperarc set** (or sometimes shortly the **arc set**) of the mixed hypergraph. **Reversing a hyperarc in** $\mathcal{A}$ means switching the roles of the nodes in it, i.e. head nodes become tail nodes and vice versa (so head-tail nodes remain like that). When we say that v is a tail node of a hyperarc a then we also allow that it is a head-tail node (and similarly for head nodes).

In a mixed hypergraph M, a **path** between nodes s and t is an alternating sequence of distinct nodes and hyperarcs $s = v_0, a_1, v_1, a_2, \ldots, a_k, v_k = t$, such that v_{i-1} is a tail node of a_i and v_i is a head node of a_i for all i between 1 and k. A hyperarc a **enters a set** X if there is a head node of a in X and there is a tail node of a in $V - X$. A hyperarc **leaves a set** if it enters the complement of this set. For a set X we define $\varrho_M(X) = |\{a \in \mathcal{A} : a \text{ enters } X\}|$ (the **in-degree** of X) and $\delta_M(X) = \varrho_M(V - X)$ (the **out-degree** of X). It is easy to check that the functions ϱ and δ are submodular functions. Given a mixed hypergraph $M = (V, \mathcal{A})$ and sets $S, T \subseteq V$, let $\lambda_M(S, T)$

denote the maximum number of arc-disjoint paths starting in S and ending in T (we say that $\lambda_M(S,T)=\infty$ if $S\cap T\neq\emptyset$). By Menger's theorem:

$$\lambda_M(S,T)=\min\{\varrho_M(X): T\subseteq X\subseteq V-S\}.$$

If $M=(V,\mathcal{A})$ is a mixed hypergraph, $r\in V$ is a designated root node and k,l are nonnegative integers, then we say that M **is** (k,l)**-arc-connected from** r if $\lambda_M(r,v)\geq k$ and $\lambda_M(v,r)\geq l$ for any $v\in V$. Let us define the set function $q=q_{M,r,k,l}$ by $q(\emptyset)=q(V)=0$, $q(X)=k-\varrho_M(X)$ for any nonempty $X\subseteq V-r$ and $q(X)=l-\varrho_M(X)$ for any $X\subsetneq V$ with $r\in X$. Then one can check that q is crossing supermodular. For a hypergraph H one can prove that $M+H$ is (k,l)-arc-connected from r if and only if d_H covers q (or equivalently q^s). Theorem 2.10 gives the following.

Theorem 3.3. *If $M=(V,\mathcal{A})$ is a mixed hypergraph, $r\in V$ is a designated root node and k,l are nonnegative integers and $m\in\mathbb{Z}^V$ is a degree specification then there exists a hypergraph H such that $M+H$ is (k,l)-arc-connected from r and $d_H(v)\leq m(v)$ for any $v\in V$ if and only if $m\in C(p)$ with $p=q^s_{M,r,k,l}$. Furthermore, H can be chosen to be nearly uniform.*

3.4 Simultaneous augmentation of two hypergraphs

We can generalize these results one step further using the fact that the intersection of two integer g-polymatroids is an integer polyhedron. Let $p_1,p_2:2^V\to\mathbb{Z}\cup\{-\infty\}$ be two skew-supermodular functions with $\max\{p_1(X):X\subseteq V\}=\max\{p_2(X):X\subseteq V\}=k$.

Let $y\in C(p_1)\cap C(p_2)\cap\mathbb{Z}^V$ such that $y(v)\leq k$ for every $v\in V$ (such a vector clearly exists; in fact, for an arbitrary cost function on the nodes, we can find such a vector of minimum cost if we can "handle" the functions p_1 and p_2).

Let $R=R(p_1,p_2,k,y)=Q(p_1,k,y)\cap Q(p_2,k,y)\cap\{x\in[0,1]^V: x(v)=1 \text{ if } y(v)=k,\ \lfloor\frac{y(V)}{k}\rfloor\leq x(V)\leq\lceil\frac{y(V)}{k}\rceil\}$. Then $y/k\in R$, and R is the intersection of two integer g-polymatroids, so R is a non-empty integer polyhedron.

Theorem 3.4. *An integer vector $x\in\mathbb{Z}^V$ is in R if and only if it is the characteristic vector of a hyperedge of a nearly uniform hypergraph $H=(V,E)$ containing k hyperedges which satisfies $b_H\geq\max\{p_1,p_2\}$ and $b_H(v)=y(v)$ for every $v\in V$.*

Proof. The proof is similar to that of Theorem 2.7. If a nearly uniform hypergraph $H=(V,E)$ containing k hyperedges satisfies $b_H\geq\max\{p_1,p_2\}$ and $b_H(v)=y(v)$ for every $v\in V$, then clearly $\chi_e\in R\cap\mathbb{Z}^V$ for any $e\in E$.

Let $x\in R\cap\mathbb{Z}^V$. We need to prove that there is a hypergraph H with the desired properties. We prove by induction on k, the $k=0$ case being trivial. Let $H_k=(V,\{e_k\})$ where $\chi_{e_k}=x$, $p_1^*=p_1-b_{H_k}$, $p_2^*=p_2-b_{H_k}$, and $y^*=y-x$. By the assumptions made above, $\max\{p_1^*(X):X\subseteq V\}=\max\{p_2^*(X):X\subseteq V\}=k-1$, $y^*\in C(p_1^*)\cap C(p_2^*)\cap\mathbb{Z}^V$, and $y^*(v)\leq k-1$ for every $v\in V$. Thus, by induction (with an arbitrary choice of an $x^*\in R(p_1^*,p_2^*,k-1,y^*)\cap\mathbb{Z}^V$), there is a nearly uniform hypergraph H^* with $k-1$ hyperedges that satisfies $b_{H^*}\geq\max\{p_1^*,p_2^*\}$, and $b_{H^*}(v)=y^*(v)\ \forall v\in V$. This means that the hypergraph $H=H^*+H_k$ has the required properties. $\square$

Using Lemma 2.9 we obtain the following corollary.

Corollary 3.5. *Let $p_1 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ and $p_2 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be two symmetric skew-supermodular set functions such that*

$$\max\{p_1(X) : X \subseteq V\} = \max\{p_2(X) : X \subseteq V\} = k, \tag{8}$$

and let $m : V \to \mathbb{Z}_+$ be a degree specification such that $m(v) \leq k$ for any $v \in V$ and

$$\sum_{v \in X} m(v) \geq \max\{p_1(X), p_2(X)\} \quad \text{for every } X \subseteq V.$$

Then there exists a nearly uniform hypergraph H of exactly k hyperedges s.t. $d_H(v) = m(v)$ for every $v \in V$ and $d_H(X) \geq \max\{p_1(X), p_2(X)\}$ for every $X \subseteq V$.

Using this corollary we obtain, that under certain circumstances we can solve an arbitrary combination of two problems from the above classes simultaneously. For example we can optimally augment the local edge-connectivity of a hypergraph and solve a node-to-area connectivity augmentation problem in another hypergraph simultaneously, if the fairly artificial condition (8) on the maximum deficiencies holds. In what follows we detail this argument for a special case and we show that without the assumption (8) we obtain NP-complete problems.

Consider the following problem, the *simultaneous local edge-connectivity augmentation problem of two hypergraphs*. Given two hypergraphs H_1, H_2 on the same ground set V, two symmetric requirement functions $r_1, r_2 : V \times V \to \mathbb{Z}$, and a nonnegative cost function $c : V \to \mathbb{R}$, find a hypergraph H of minimum total cost such that $H_i + H$ is r_i-edge-connected for $i = 1, 2$ (the cost of H is $\sum_{v \in V} c(v) d_H(v)$). Our results imply that if we assume that $\max\{r_1(u,v) - \lambda_{H_1}(u,v) : u,v \in V\} = \max\{r_2(u,v) - \lambda_{H_2}(u,v) : u,v \in V\}$, then we can solve the problem optimally in polynomial time. Furthermore, we can even achieve that the hypergraph H is nearly uniform. The following theorem shows that without the assumption on the maximum deficiencies the problem becomes NP-complete: the reduction is similar to that of [3].

Theorem 3.6. *The simultaneous local edge-connectivity augmentation problem of two hypergraphs is in general NP-complete, even if the cost function is constant.*

Proof. The problem is clearly in NP. To show its completeness consider the *Special Bin-Packing Problem (SBP)*. An instance of this problem consists of a set of positive integers $W = \{w_1, w_2, \dots, w_n\}$ (*weights*), a set of positive integers $B = \{b_1, b_2, \dots, b_m\}$ (*bins*) such that $\gamma = \sum_{w \in W} w = \sum_{b \in B} b$. The SBP problem asks whether there exists a partition $W_1, W_2 \dots, W_m$ of W such that $\sum_{w \in W_j} w = b_j$ for every $j = 1, 2, \dots, m$. This problem is shown to be strongly NP-complete in [2], i.e. it remains NP-complete even if the weights and bins are unary encoded. We will reduce the unary-encoded SBP problem to our problem. For each weight $w_i \in W$ consider a set X_i such that $|X_i| = w_i$ and similarly, for each bin $b_j \in B$ let Y_j be such that $|Y_j| = b_j$. The sets X_i $(i = 1, 2, \dots, n)$ and Y_j $(j = 1, 2, \dots, m)$ are assumed to be pairwise disjoint. Let $X = \cup_{i=1}^n X_i$ and $Y = \cup_{j=1}^m Y_j$. The ground set of the two hypergraphs is $V = X \cup Y$.

The edge-set of H_1 consists of a $\gamma - 1$ regular and $\gamma - 1$-edge-connected graph on X and a similar graph on Y: one can check that such a graph exists. The requirement function r_1 is uniformly γ. The edge-set of H_2 consists of hyperedges Y_j for every $j = 1, 2, \ldots, m$ and the requirement is $r_2(u, v) = 1$ if $u, v \in X_i$ for some i, and 0 otherwise. One can check that there is a hypergraph H of total size at most γ such that $H_i + H$ is r_i-edge-connected for $i = 1, 2$ if and only if there is a solution of the SBP problem. The details are left to the reader. $\square$

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