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Combinatorial Conditions for the Unique Completability of Low Rank Matrices

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Abstract

We consider the problems of completing a low-rank positive semidefinite square matrix M or a low-rank rectangular matrix N from a given subset of their entries. Following the approach initiated by Singer and Cucuringu [20] we study the local and global uniqueness of such completions by analysing the structure of the graphs determined by the positions of the known entries of M or N.

We present combinatorial characterizations of local and global (unique) completability for special families of graphs. We characterize local and global completability in all dimensions for cluster graphs, i.e. graphs which can be obtained from disjoint complete graphs by adding a set of independent edges. These results correspond to theorems for body-bar frameworks in rigidity theory. We also provide a characterization of two-dimensional local completability of planar bipartite graphs, which leads to a characterization of two-dimensional local completability in the rectangular matrix model when the underlying bipartite graph is planar. These results are based on new observations that certain graph operations preserve local or global completability, as well as on a further connection between rigidity and completability.

We also prove that a rank condition on the completability stress matrix of a graph is a sufficient condition for global completability. This verifies a conjecture of Singer and Cucuringu [20].

1 Introduction

In matrix completion problems a partially filled matrix is given and the goal is to determine the missing entries so that the resulting matrix belongs to a certain class of matrices. Such problems arise in several practical problems where one has to deal

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with incomplete data sets. Here we consider the completion of low-rank positive semidefinite matrices.

A square matrix M of size n is called a *Gram matrix* if $M = P^\top P$ for some $d \times n$ matrix P with real entries. Thus a Gram matrix is symmetric and positive semidefinite with rank at most d. Conversely, any symmetric positive semidefinite matrix can be expressed in the form $P^\top P$ for some matrix P.

A related problem is to decide whether such a completion is unique. Singer and Cucuringu [20] investigated the uniqueness of the Gram matrix completion problem and pointed out that several concepts and techniques from rigidity theory can be adapted to the matrix completion setting. In this paper we explore these connections further to obtain new results on uniquely completable matrices.

1.1 Local and global completability

The (two levels of) the uniqueness of the completion can be defined by using notions which are similar to (the two levels of) the rigidity of bar-and-joint frameworks.

For a given Gram matrix $P^\top P$, each column of P may be regarded as a point in $\mathbb{R}^d$, and hence an $n \times n$ Gram matrix can be defined by specifying n points in $\mathbb{R}^d$. Let $V = \{1, \ldots, n\}$. Then $\boldsymbol{p} : i \in V \mapsto p_i \in \mathbb{R}^d$ determines a Gram matrix $M = P(\boldsymbol{p})^\top P(\boldsymbol{p})$, where $P(\boldsymbol{p})$ is the matrix whose i'th column is p_i. Note that the entry $M[i, j]$ is equal to the scalar product $\langle p_i, p_j \rangle$.

Suppose that we are given a subset of the entries of some Gram matrix $M = P(\boldsymbol{p})^\top P(\boldsymbol{p})$. The given entries define an undirected graph $G = (V, E)$ on V in which two vertices i, j are adjacent if and only if $M[i, j]$ is given. Note that G is *semi-simple*, that is, it contains no parallel edges but it may contain loops. A *d-dimensional framework* (or simply a *framework*) is a pair $(G, \boldsymbol{p})$, where $G = (V, E)$ is a semi-simple graph and $\boldsymbol{p} : V \to \mathbb{R}^d$ is a map. Thus each partially filled Gram matrix has an underlying framework and each framework defines a partially filled Gram matrix M in which $M[i, j] = \langle p_i, p_j \rangle$. We consider the situation where we only have an incomplete Gram matrix in our hand and do not have $\boldsymbol{p}$ and investigate under which circumstances we can decide if it has a unique completion.

We say that $(G, \boldsymbol{q})$ is *equivalent* to $(G, \boldsymbol{p})$ if

$$\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \qquad (ij \in E) \tag{1}$$

and they are *congruent* if (1) holds for every pair i, j in V (including i, j with $i = j$). This is equivalent to saying that $q_i = Ap_i$ for all $i \in V$ for some fixed orthogonal matrix A.

We say that a d-dimensional framework $(G, \boldsymbol{p})$ is *globally completable* in $\mathbb{R}^d$ if for every d-dimensional framework $(G, \boldsymbol{q})$ which is equivalent to $(G, \boldsymbol{p})$ we have that $(G, \boldsymbol{q})$ and $(G, \boldsymbol{p})$ are congruent. Similarly, $(G, \boldsymbol{p})$ is called *locally completable* in $\mathbb{R}^d$ if there exists an open neighborhood $N(\boldsymbol{p})$ of $\boldsymbol{p}$ in $\mathbb{R}^{d|V|}$ such that for any $\boldsymbol{q} \in N(\boldsymbol{p})$ the equivalence of $(G, \boldsymbol{q})$ to $(G, \boldsymbol{p})$ implies that the two frameworks are congruent[1].

[1]It can be seen that, as in the case of rigidity, the local completability of $(G, \boldsymbol{p})$ is equivalent to the fact that every continuous motion of the vertices of $(G, \boldsymbol{p})$ in $\mathbb{R}^d$ which preserves equivalence must also preserve congruence.

One may also define the infinitesimal version of local completability. A map $\dot{\boldsymbol{p}} : V \to \mathbb{R}^d$ is called an *infinitesimal c-motion* of $(G, \boldsymbol{p})$ if

$$\langle p_i, \dot{p}_j \rangle + \langle p_j, \dot{p}_i \rangle = 0 \qquad (ij \in E) \tag{2}$$

The $|E| \times d|V|$-matrix representing this system of linear equations with variables $\dot{\boldsymbol{p}}$ is the *completability matrix* of $(G, \boldsymbol{p})$, denoted by $C(G, \boldsymbol{p})$. (Thus the entries of $C(G, \boldsymbol{p})$ in the d-tuples of positions i and j of row $e = ij$ are p_j and p_i, respectively, and all other entries are zeros.) For example, if G is a graph with $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{11, 12, 23, 24\}$, the completability matrix becomes as follows

$$\begin{array}{c} \\ 11 \\ 12 \\ 23 \\ 24 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 2p_1 & 0 & 0 & 0 \\ p_2 & p_1 & 0 & 0 \\ 0 & p_3 & p_2 & 0 \\ 0 & p_4 & 0 & p_2 \end{pmatrix} \end{array}.$$

As observed by Singer and Cucuringu [20], for any $d \times d$ skew-symmetric matrix S, the map $\dot{\boldsymbol{p}} : V \to \mathbb{R}^d$ defined by $\dot{p}_i = Sp_i$ for $i \in V$ is an infinitesimal c-motion. (The infinitesimal c-motions of this kind are called *trivial*.) Therefore, if $|V| \geq d$, then

$$\operatorname{rank} C(G, \boldsymbol{p}) \leq dn - \binom{d}{2}. \tag{3}$$

Clearly the rank of $C(G, \boldsymbol{p})$ is also bounded above by the number of edges in the complete semi-simple graph on n vertices. A framework $(G, \boldsymbol{p})$ is said to be *infinitesimally completable* if $\operatorname{rank} C(G, \boldsymbol{p}) = dn - \binom{d}{2}$ when $n \geq d$ or $\operatorname{rank} C(G, \boldsymbol{p}) = \binom{n+1}{2}$ when $n \leq d$. It is *c-independent* if $\operatorname{rank} C(G, \boldsymbol{p}) = |E|$. A map $\boldsymbol{p} : V \to \mathbb{R}^d$ is called *generic* if the set of coordinates of $\boldsymbol{p}$ is algebraically independent over the rational field. Thus the rank of $C(G, \boldsymbol{p})$ will be the same for all generic realizations of G. Singer and Cucuringu [20] showed that infinitesimally completability is a sufficient condition for local completability, and that the two properties are equivalent when $(G, \boldsymbol{p})$ is generic. Hence we say that the graph G is *locally completable* or *c-independent* in $\mathbb{R}^d$ if some (or equivalently, every) generic realization of G in $\mathbb{R}^d$ is locally completable or c-independent. It follows that in the generic case, the local uniqueness of a completion of a partial Gram matrix depends only on the underlying graph G, which is determined by the positions of the known entries. This is similar to the well-studied property of generic rigidity of bar-and-joint frameworks, where the rigidity of a framework depends only on the underlying graph if the positions of the joints are generic. Unlike in the case of global rigidity, it is not yet known whether the global uniqueness of a completion of a partial rank d Gram matrix depends only on the positions of its known entries when $d \geq 2$. We say that a graph G is *globally completable* in $\mathbb{R}^d$ if *every* generic realization of G in $\mathbb{R}^d$ is globally completable.

The d-dimensional *completability matroid* $\mathcal{C}_d(G)$ of G is the matroid on E in which a set of edges is independent if and only if the corresponding set of rows in $C(G, \boldsymbol{p})$ is linearly independent, for some generic $\boldsymbol{p} : V \to \mathbb{R}^d$. We shall see that the complete

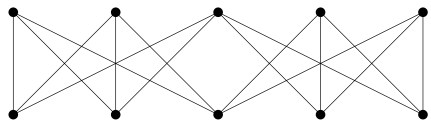

Figure 1: This graph G satisfies the necessary conditions of Lemma 1 for $d = 2$ but it is not c-independent in $\mathbb{R}^2$. To see this consider the graph $G + uv$ obtained by adding an edge between the vertices u, v in the unique 2-vertex cut of G. Let B be a base of $\mathcal{C}_2(G + uv)$ which contains uv. The fact that $K_{3,3}$ is dependent implies that $|B| \le 1 + 7 + 7 = 15$. Hence the rank of $\mathcal{C}_2(G)$ is at most 15 which is less than $|E|$.

semi-simple graph K_n° on n vertices is locally completable in $\mathbb{R}^d$. (For a loopless graph G we use G° to denote the graph obtained from G by adding a loop incident with each vertex.) Hence it has rank $dn - \binom{d}{2}$, when $n \ge d$ (resp. $\binom{n+1}{2}$, when $n \le d$) and its bases are the (edge sets of the) minimally locally completable graphs on n vertices.

As mentioned above the map $\dot{\boldsymbol{p}}$ defined by $\dot{p}(i) = Sp(i)$ for some skew-symmetric matrix S is an infinitesimal c-motion of $(G, \boldsymbol{p})$. Singer and Cucuringu [20] also noticed that if G is bipartite with vertex bipartition $\{V_1, V_2\}$, then the map $\dot{\boldsymbol{p}}$ defined by $\dot{p}(i) = Ap(i)$ for $i \in V_1$ and $\dot{p}(j) = -A^T p(j)$ for $j \in V_2$ is also an infinitesimal c-motion of $(G, \boldsymbol{p})$ for any $d \times d$ matrix A. Therefore,

$$\operatorname{rank} C(G, p) \le dn - d^2 \tag{4}$$

if G is bipartite with $|V_i| \ge d$, $i = 1, 2$. The inequalities (3) and (4) imply the following necessary condition for c-independence. We shall use $i(X)$ to denote the number of edges induced by some vertex set $X \subseteq V$ in a graph $G = (V, E)$.

Lemma 1. *Let $G = (V, E)$ be c-independent in $\mathbb{R}^d$. Then*
(i) $i(X) \le d|X| - \binom{d}{2}$ for all $X \subseteq V$ with $|X| \ge d$, and
(ii) for each bipartite subgraph $H = (V_1, V_2; F)$ on vertex set $X = V_1 \cup V_2$ with $|V_i| \ge d$, $i = 1, 2$ we have $i(X) \le d|X| - d^2$.

1.2 Previous work

Singer and Cucuringu [20, Proposition 5.3] showed that for $d = 1$ the pair of necessary conditions of c-independence in Lemma 1 is also sufficient. For $d = 2$ this is not the case (see Figure 1) and it remains a challanging open problem to characterize c-independence in $\mathbb{R}^d$, for $d \ge 2$.

They also characterized global 1-completability [20, Proposition 5.4] by showing that G is globally completable when $d = 1$ if and only if it contains a connected c-independent subgraph with $|V|$ edges. The characterization of global d-completability for $d \ge 2$ is also open.

Local and global completability of frameworks correspond to local and global rigidity of bar-and-joint frameworks, which are analogously defined by replacing the inner product in (1) with the Euclidean distance between the two points. Singer and Cucuringu [20] pointed out that rigidity and completability are equivalent when G contains a loop at every vertex (i.e., all the diagonal entries of the Gram matrix are known). We will return to this in section 2.3 below. Laurent and Varvitsiotis [14] also worked on the link between rigidity and completability, where they discussed the relation between universal completability and universal rigidity in terms of SDP formulations, again assuming that G contains a loop at every vertex.

1.3 The rectangular matrix model

Singer and Cucuringu [20] also considered the unique completability of *low rank rectangular matrices*, i.e. rectangular matrices of the form $P^\top Q$ for some $d \times n$ matrix P and $d \times m$ matrix Q. In this case the known entries of the rectangular matrix define a bipartite graph $G = (V, U; E)$ in which the colour classes are of size n and m, respectively, and an edge ij corresponds to the known scalar product of row i in $P^\top$ and column j in Q. The definition of local and global completability for partially filled rectangular matrices is analogous to that for Gram matrices.

The low rank rectangular matrix model may appear more general than the Gram matrix model, but it is actually equivalent to the Gram matrix completion model restricted to bipartite underlying graphs. As we saw above, $\dim\ker C(G,p) \geq d^2$ if G is bipartite. Hence we say that a bipartite graph G is *locally completable in the (rank d) rectangular matrix model* if rank $C(G,p) = d|V(G)| - d^2$.

We note that Király et al. [12] also considered the uniqueness of the matrix completion in the rectangular matrix model in the complex field. They discussed combinatorial characterizations of 1-dimensional completability and corank-1-dimensional completability, a sufficient condition for global completability, and completability of random graphs.

1.4 New results

In this paper we first present combinatorial characterizations of local and global completability of special families of graphs. We characterize local and global completability in all dimensions for cluster graphs, i.e. graphs which can be obtained from disjoint complete semi-simple graphs by adding a set of independent edges. These results correspond to theorems of Tay [21], and Connelly, Jordán, and Whiteley [3], for 'body-bar frameworks' in rigidity theory. We also provide a characterization of two-dimensional local completability of planar bipartite graphs, which leads to a characterization of two-dimensional local completability in the rectangular matrix model when the underlying bipartite graph is planar[2].

These results are based on new observations stating that some (old or new) graph operations preserve local or global completability, as well as on a further connection between rigidity and completability.

We also prove that a certain rank condition on the completability stress matrix (defined later) is a sufficient condition for global completability. This verifies a conjecture of Singer and Cucuringu [20].

[2]The latter result has been obtained independently by Kalai, Nevo, and Novak [11] in a different context.

2 Preliminaries

We have seen that the concepts of global and local completability correspond to global and local rigidity, respectively, in the theory of bar-and-joint frameworks. We first give a brief summary of the relevant results from rigidity theory. We then adapt an inductive technique for rigidity to completability. We close this section by describing the above mentioned equivalence of rigidity with completability of a graph with a loop at each vertex and use it to deduce necessary connectivity conditions for completability.

2.1 Rigidity of frameworks

Deciding whether a given framework is globally (or locally) rigid in $\mathbb{R}^d$ is NP-hard for $d \geq 1$ (resp. $d \geq 2$), see [19]. The first order approximation of local rigidity and the generic (local or global) rigidity behaviour of graphs is better understood.

Let $(G, \boldsymbol{p})$ be a d-dimensional framework, where G is a simple undirected graph (no parallel edges, no loops). We say that a map $\dot{\boldsymbol{p}} : i \in V \mapsto \dot{p}_i \in \mathbb{R}^d$ is an *infinitesimal motion* of $(G, \boldsymbol{p})$ if

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad \text{for all } ij \in E. \tag{5}$$

An infinitesimal motion $\dot{\boldsymbol{p}}$ is called *trivial* if $\dot{p}_i = Sp_i + t$ holds for all $i \in V$ for some skew-symmetric matrix S and some $t \in \mathbb{R}^d$. We say that $(G, \boldsymbol{p})$ is *infinitesimally rigid* if every infinitesimal motion of $(G, \boldsymbol{p})$ is trivial.

The *rigidity matrix* of $(G, \boldsymbol{p})$ is a matrix $R(G, \boldsymbol{p})$ of size $|E| \times d|V|$ representing the system of linear equations (5) with variables $\dot{\boldsymbol{p}}$. Hence the rows are indexed by E and sets of d consecutive columns are indexed by V, and the entries in the row of $e = ij$ and in the d columns of i and j contain the d coordinates of $p_i - p_j$ and $p_j - p_i$, respectively, and the remaining entries are zeros. By definition, $\dot{\boldsymbol{p}}$ is an infinitesimal motion of $(G, \boldsymbol{p})$ if and only if $\dot{\boldsymbol{p}}$ is in the kernel of $R(G, \boldsymbol{p})$. Since the set of trivial infinitesimal motions forms a $\binom{d+1}{2}$-dimensional linear space, $(G, \boldsymbol{p})$ is infinitesimally rigid if and only if either $|V| \leq d$ and rank $R(G, \boldsymbol{p}) = \binom{n}{2}$ or $|V| \geq d$ and rank $R(G, \boldsymbol{p}) = d|V| - \binom{d+1}{2}$.

Since the rank of $R(G, \boldsymbol{p})$ is maximized for all generic $\boldsymbol{p}$, $(G, \boldsymbol{p})$ is infinitesimally rigid for some generic $\boldsymbol{p}$ if and only if $(G, \boldsymbol{p})$ is infinitesimally rigid for all generic $\boldsymbol{p}$. Moreover, if $\boldsymbol{p}$ is generic then $(G, \boldsymbol{p})$ is infinitesimally rigid if and only if $(G, \boldsymbol{p})$ is rigid. We can define the d-dimensional *rigidity matroid* $\mathcal{R}_d(G)$ on the edge set of G by linear independence in $R(G, \boldsymbol{p})$, for some generic $\boldsymbol{p}$. Thus, assuming that $\boldsymbol{p}$ is generic, the rigidity of $(G, \boldsymbol{p})$ (or more generally, the rank of E in $\mathcal{R}_d(G)$) depends only on G. Motivated by this fact, we say that a graph G is *rigid* in $\mathbb{R}^d$ if $(G, \boldsymbol{p})$ is rigid for some generic $\boldsymbol{p} : V \to \mathbb{R}^d$, or equivalently, if either G is a complete graph on at most d vertices or $|V| \geq d$ and rank $R(G, \boldsymbol{p}) = d|V| - \binom{d+1}{2}$. It is not hard to see that $\mathcal{R}_1(G)$ is isomorphic to the circuit matroid of G and that G is rigid in $\mathbb{R}^1$ if and only if G is connected. In $\mathbb{R}^2$ the following celebrated result of Laman [13] characterizes independence in $\mathcal{R}_2(G)$ and hence also the rigidity of G. The corresponding characterization for $d \geq 3$ is a major open problem in combinatorial rigidity.

Theorem 2 (Laman)**.** *Let $G = (V, E)$ be a graph. Then E is independent in $\mathcal{R}_2(G)$ if and only if $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$.*

Theorem 3 below implies that global rigidity in $\mathbb{R}^d$ is also a *generic property* for all $d \geq 1$ i.e. the global rigidity of a generic framework depends only on its underlying graph.

We say that $\boldsymbol{\omega} : E \to \mathbb{R}$ is a *self-stress* of $(G, \boldsymbol{p})$ if $R(G, \boldsymbol{p})^\top \boldsymbol{\omega} = 0$. The *stress matrix* Ω associated with $\boldsymbol{\omega}$ is a $|V| \times |V|$ symmetric matrix whose columns and rows are associated with vertices $V = \{1, \dots, n\}$ and each entry is given by

$$\Omega[i, j] = \begin{cases} \sum_{k \in N_G(i)} \boldsymbol{\omega}(ik) & \text{if } i = j \\ -\boldsymbol{\omega}(ij) & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3 (Connelly [2], Gortler, Healy, and Thurston [5])**.** *Let $(G, \boldsymbol{p})$ be a d-dimensional generic framework. Then $(G, \boldsymbol{p})$ is globally rigid in $\mathbb{R}^d$ if and only if there is a self-stress $\boldsymbol{\omega}$ of $(G, \boldsymbol{p})$ for which the associated stress matrix Ω has rank $n - d - 1$.*

Thus we say that G is *globally rigid* in $\mathbb{R}^d$ if (G, p) is globally rigid for some (or equvalently, for all) generic $\boldsymbol{p} : V \to \mathbb{R}^d$. As in the case of rigidity, global rigidity is well-characterized up to dimension two. A graph is globally rigid in $\mathbb{R}^1$ if and only if it is 2-connected. By a theorem of Jackson and Jordán [7] a graph G is globally rigid in $\mathbb{R}^2$ if and only if it is 3-connected and $G - e$ is rigid for all $e \in E(G)$. The higher dimensional cases remain difficult open problems.

2.2 Inductive constructions

Inductive constructions of graphs provide a powerful technique for analyzing local and global rigidity [25]. In this section we recall some well-known operations that we shall further develop in subsequent sections and make some preliminary observations. Perhaps the best known operations are the so-called Henneberg operations (also called vertex addition and edge splitting). We shall call them *0-extension* and *1-extension*, which are also frequently used in the literature.

Let $G = (V, E)$ be a semi-simple graph. The (d-dimensional) *0-extension* operation adds a new vertex v to G and d new edges $vu_1, \dots, vu_d$ for distinct vertices $u_1, \dots, u_d \in V + v$. The *1-extension* operation removes an existing non-loop edge $u_1u_2 \in E$, adds a new vertex v to G and $d + 1$ new edges $vu_1, vu_2, vu_3, \dots vu_{d+1}$ for distinct vertices $u_3, \dots, u_{d+1} \in (V + v) \setminus \{u_1, u_2\}$. See Figure 2. Note that we allow one of the new edges to be a loop in both 0- and 1-extensions by taking $u_i = v$. If necessary, we will specify whether or not a loop is added by referring to the operation as a *looped extension* or a *simple extension*.

It is known that simple 0-extension and simple 1-extension both preserve the rigidity of graphs in $\mathbb{R}^d$. It is also known that simple 1-extension preserves global rigidity in $\mathbb{R}^d$. (Note that the simple 0-extension operation cannot preserve global rigidity since $(d + 1)$-connectivity is a necessary condition for global rigidity in $\mathbb{R}^d$.)

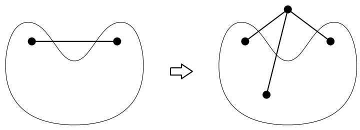

Figure 2: 1-extension in $\mathbb{R}^2$.

One can easily check that the 0-extension operation preserves c-independence (and hence local completability) in $\mathbb{R}^d$.

Lemma 4. *Let $(G, \boldsymbol{p})$ and $(G', \boldsymbol{p}')$ be d-dimensional frameworks and suppose that G is obtained from G' by a 0-extension operation which adds a new vertex v and new edges $vu_1, \ldots, vu_d$, and $\boldsymbol{p}'$ is the restriction of $\boldsymbol{p}$ to G'. Suppose further that $(G', \boldsymbol{p}')$ is c-independent and that the points $0, p(u_1), \ldots, p(u_d)$ are in general position in $\mathbb{R}^d$. Then $(G, \boldsymbol{p})$ is c-independent.*

Proof. It is easy to check that $\operatorname{rank} C(G, \boldsymbol{p}) = \operatorname{rank} C(G', \boldsymbol{p}') + d$. □

It is easy to see that the 'partial 0-extension operation' i.e. the operation which adds at most d new edges will also preserve c-independence. Since the complete semi-simple graph K_n° can be obtained from K_1° by a sequence of partial 0-extensions and edge additions, Lemma 4 implies that

Corollary 5. *Suppose $(K_n^\circ, \boldsymbol{p})$ is a d-dimensional framework and the points $\{0\} \cup p(V)$ are in general position in $\mathbb{R}^d$. Then $(K_n^\circ, \boldsymbol{p})$ is infinitesimally locally completable.*

We shall prove later (Theorem 26) that the simple 0-extension operation also preserves global completability in $\mathbb{R}^d$.

On the other hand, 1-extension does not always preserve local completability. This follows by observing that C_4 can be obtained from C_3 by a 1-extension in $\mathbb{R}^1$. We can deduce from this observation that 1-extension does not always preserve global completability either. Thus the behaviour of the extension operations in rigidity and completability are quite different.

We shall introduce a new operation, called double-1-extension, and prove that this operation preserves local completability. As an application of this result, we shall obtain a new proof of the combinatorial characterization of locally completable graphs in $\mathbb{R}^1$.

Another key operation used in rigidity theory is vertex-splitting. Let $G = (V, E)$ be a graph, let $v \in V$ be a vertex, and let $\{U_1, U^*, U_2\}$ be a partition of $N(v)$ for which $|U^*| = d - 1$ (and such that U_1 or U_2 may be empty). The *vertex-splitting* operation at v (with respect to $\{U_1, U^*, U_2\}$) in $\mathbb{R}^d$ removes v from G and inserts two new vertices v_1 and v_2 and new edges $v_i u$ for $u \in U^*$ for $i = 1, 2$, $v_i u$ for $u \in U_i$ for $i = 1, 2$, and $v_1 v_2$. Whiteley [24] showed that vertex-splitting preserves rigidity. The corresponding statement for global rigidity (in the case when U_1 and U_2 are both non-empty) is known to hold in $\mathbb{R}^2$ [10] and is conjectured to hold in higher dimensions [4].

We shall also introduce a more general version of the vertex-splitting operation mentioned above and prove that it preserves local completability. This result will be the key step in the proof of our result on bipartite planar graphs.

2.3 Completability, rigidity, and coning

We use $G * \{v\}$ to denote the *cone* graph of G, that is, the graph obtained by adding a new vertex v and connecting each vertex of G to v by a new edge. For a framework $(G, \boldsymbol{p})$, let $\boldsymbol{p}^*$ be the extension of $\boldsymbol{p}$ to $V(G) \cup \{v\}$ by $\boldsymbol{p}^*(v) = 0$. The following property was observed by Singer and Cucuringu [20].

Proposition 6. *Let $G = (V, E)$ be a simple graph and let $(G, \boldsymbol{p})$ be a d-dimensional framework with $\boldsymbol{p}(v) \neq 0$ for all $v \in V$. Then $(G^\circ, \boldsymbol{p})$ is infinitesimally (resp. globally) completable in $\mathbb{R}^d$ if and only if $(G * \{v\}, \boldsymbol{p}^*)$ is infinitesimally (resp. globally) rigid in $\mathbb{R}^d$.*

Whiteley [23] showed that a graph G is rigid in $\mathbb{R}^{d-1}$ if and only if $G * \{v\}$ is rigid in $\mathbb{R}^d$. This fact and Proposition 6 imply the following. (A direct proof will be given in Section 7.)

Corollary 7. *Let G be a simple graph. Then G° is locally completable in $\mathbb{R}^d$ if and only if G is rigid in $\mathbb{R}^{d-1}$.*

Connelly and Whiteley [4] proved that a graph G is globally rigid in $\mathbb{R}^{d-1}$ if and only if $G * \{v\}$ is globally rigid in $\mathbb{R}^d$. Thus we have:

Corollary 8. *Let G be a simple graph. Then G° is globally completable in $\mathbb{R}^d$ if and only if G is globally rigid in $\mathbb{R}^{d-1}$.*

This implies that global completability of frameworks whose underlying graph has a loop at each vertex is a generic property. It is not known whether this holds for all graphs.

Corollaries 7 and 8 also imply the following necessary conditions for completability.

Corollary 9. *(a) If G is locally completable in $\mathbb{R}^d$ then either G is $(d-1)$-connected or G is a complete semi-simple graph on at most $d-1$ vertices.*
(b) If G is globally completable in $\mathbb{R}^d$ then either G is d-connected or G is a complete semi-simple graph on at most d vertices.

This corollary can also be proved directly by considering a rotation about a $(d-2)$-dimensional subspace which contains the vertices in a $(d-2)$ cutset in (a) and a reflection in a $(d-1)$ dimensional subspace which contains the vertices in a $(d-1)$-cutset in (b).

We close this section by obtaining an analogue to Whiteley's above mentioned result which linked coning and rigidity.

Let $G = (V, E)$ be a semi-simple graph. The *looped cone extension* $G \circ v$ of G is obtained by adding a new vertex v and all edges uv for $u \in V + v$.

Lemma 10. *Let $G = (V, E)$ be a graph and $G \circ v$ be its looped cone extension. Then G is locally completable in $\mathbb{R}^d$ if and only if $G \circ v$ is locally completable in $\mathbb{R}^{d+1}$.*

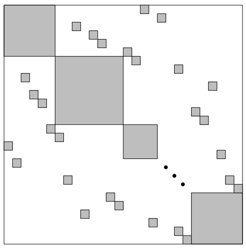

Figure 3: A partially filled matrix with a cluster structure.

Proof. We use the fact that a framework is infinitesimally locally completable if and only if every infinitesimal c-motion is a rotation which fixes the origin. Choose a generic framework $(G \circ v, \boldsymbol{p})$ in $\mathbb{R}^{d+1}$. By applying a suitable rotation, we may transform $(G \circ v, \boldsymbol{p})$ to a framework $(G \circ v, \boldsymbol{q})$ with $\boldsymbol{q}(v) = (t, 0, 0, \ldots, 0)$ for some $t \in \mathbb{R}$ and is such that $(G \circ v, \boldsymbol{p})$ is infinitesimally completable if and only if $(G \circ v, \boldsymbol{q})$ is infinitesimally completable. Let $\dot{\boldsymbol{q}}$ be an infinitesimal c-motion of $(G \circ v, \boldsymbol{q})$. Since $G \circ v$ has a loop at v, we may assume that $\dot{\boldsymbol{q}}(v) = (0, 0, \ldots, 0)$ (by composing $\dot{\boldsymbol{q}}$ with a suitable infinitesimal rotation of $\mathbb{R}^{d+1}$ which fixes the origin). The facts that $\dot{\boldsymbol{q}}$ is an infinitesimal c-motion of $(G \circ v, \boldsymbol{q})$ and uv is an edge of $G \circ v$ for all $u \in V$ now tell us that the first component of $\dot{\boldsymbol{q}}(u)$ is zero for all $u \in V$. This in turn implies that the projection of $\dot{\boldsymbol{q}}$ onto its last d coordinates is an infinitesimal c-motion of the framework $(G, \bar{\boldsymbol{q}})$ obtained from $(G, \boldsymbol{q})$ by projecting each $\boldsymbol{q}(u)$ onto its last d coordinates. Conversely any infinitesimal c-motion of $(G, \bar{\boldsymbol{q}})$ can be extended to an infinitesimal c-motion of $(G, \boldsymbol{q})$ which fixes v by putting the first component of the infinitesimal velocity of each vertex $u \in V$ equal to zero. This gives us a bijection between the infinitesimal c-motions of $(G, \boldsymbol{q})$ which fix v and the the infinitesimal c-motions of $(G, \bar{\boldsymbol{q}})$. ☐

3 Cluster graphs

Let $H = (V, E)$ be a loopless multigraph. The *cluster graph induced by H*, denoted by G_H°, is the graph obtained from H by replacing each vertex $v \in V$ by $K_{d(v)}^\circ$, that is a complete graph on $d(v)$ vertices in which a loop is added to each vertex (we call this subgraph the *cluster C_v* associated with v), and replacing each edge $st \in E$ by an edge between the clusters C_s and C_t in such a way that the edges of G_H° connecting distinct clusters are pairwise disjoint. A graph obtained in this manner is called a cluster graph. In this section we consider the characterization of local and global completability of cluster graphs. The pattern of known entries in a matrix corresponding to a cluster graph is illustrated in Figure 3.

3.1 Local completability

Tay [21] gave a combinatorial characterization for the rigidity of *body-bar graphs*, i.e. cluster graphs with all loops deleted. By combining Tay's theorem and Corollary 7 one can derive a combinatorial characterization of locally completable cluster graphs in each dimension. Here we shall present a simpler direct proof.

Let $H = (V, E)$ be a multigraph. For a partition $\mathcal{P}$ of V let $E_H(\mathcal{P})$ denote the set, and $e_H(\mathcal{P})$ the number of edges of H connecting distinct members of $\mathcal{P}$. We say that H is *m-tree-connected* if

$$e_H(\mathcal{P}) \geq m(t-1) \tag{6}$$

for all partitions $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of V. Note that a theorem of Nash-Williams [16] and Tutte [22] implies that H satisfies (6) if and only if H contains m edge-disjoint spanning trees.

Lemma 11. *Let $H = (V, E)$ be a multigraph and suppose that the cluster graph G_H° induced by H is locally completable in $\mathbb{R}^d$. Then H is $\binom{d}{2}$-tree-connected.*

Proof. If G_H° has less than d vertices then it must be a complete semi-simple graph. This implies that $|V| = 1$ and the lemma is trivially true. Hence we may assume that G_H° has at least d vertices.

For a contradiction suppose that $e_H(\mathcal{P}) \leq \binom{d}{2}(t-1) - 1$ for a partition $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of V with $t \geq 2$. Let $Y_i = \cup\{V(C_v) : v \in X_i\}$, for $1 \leq i \leq t$, and let $\mathcal{Q} = \{Y_1, Y_2, ..., Y_t\}$ be the corresponding partition of $V(G_H^\circ)$. Observe that $e_{G_H}(\mathcal{Q}) = e_H(\mathcal{P})$ holds.

Let $S \subseteq E(G_H^\circ)$ be a maximal set of independent edges in G_H°, i.e. a base in the d-dimensional generic completability matroid of G_H°. Since G_H° is locally completable and has at least d vertices, we have $|S| = d|V(G_H^\circ)| - \binom{d}{2}$. Thus, by using the fact that each vertex in a locally completable semi-simple graph on at least d vertices has degree at least d (which follows from Lemma 1) we obtain that each subset $Y \subseteq V(G_H^\circ)$ has $|Y| \geq d-1$ and hence it induces at most $d|Y| - \binom{d}{2}$ edges of S (again by Lemma 1). Thus we obtain

$$d|V(G_H^\circ)| - \binom{d}{2} = |S| \leq \sum_{1}^{t}(d|Y_i| - \binom{d}{2}) + e_{G_H^\circ}(\mathcal{Q}) =$$

$$d|V(G_H^\circ)| - \binom{d}{2}t + e_H(\mathcal{P}) \leq d|V(G_H^\circ)| - \binom{d}{2} - 1,$$

a contradiction. Thus H satisfies (6) with $m = \binom{d}{2}$ and the lemma follows. □

Before the proof of the next theorem recall the correspondence between the trivial infinitesimal c-motions of a d-dimensional framework $(G, \boldsymbol{p})$ with $n \geq d$ and the skew-symmetric matrices of size $d \times d$.

Theorem 12. *Let H be a multigraph. Then the cluster graph G_H° induced by H is locally completable in $\mathbb{R}^d$ if and only if H is $\binom{d}{2}$-tree-connected .*

Proof. Necessity follows from Lemma 11. We prove sufficiency by showing that G_H° has a d-dimensional realization $(G_H^\circ, \boldsymbol{p})$ which is infinitesimally completable.

We shall assign coordinates to the vertices of G_H° so that each vertex is located on some of the coordinate axes at unit distance from the origin. It will also follow that the points corresponding to the vertices of some cluster contain such points from at least $d-1$ different axes, and hence each cluster gives rise to an infinitesimally locally completable subframework (this follows from Corollary 5).

Let us fix $\binom{d}{2}$ edge-disjoint spanning trees $T_{i,j}$, $1 \leq i < j \leq d$ as well as a root vertex r in H and then orient the trees away from r, so that they become out-arborescences rooted at r.

Consider an edge st of tree $T_{i,j}$ (for some $1 \leq i < j \leq d$) and suppose that it is oriented from s to t. This edge corresponds to an edge uv of G_H° connecting the clusters C_s and C_t (with $u \in C_s, v \in C_t$, say). Define the location of vertices u, v by

$$p(u) = \mathbf{e}_i, \quad p(v) = \mathbf{e}_j,$$

where $\mathbf{e}_l$ is the unit vector $(0, 0, ..., 0, 1, 0, ..., 0)$ on the l'th coordinate axis. This completes the definition of $\boldsymbol{p}$.

Now consider an infinitesimal c-motion $\mathbf{m}$ of $(G_H^\circ, \boldsymbol{p})$. Since the clusters are infinitesimally locally completable, each cluster C_w has a $d \times d$ skew-symmetric matrix A_w for which $m(x) = A_w p(x)$ for all vertices x in $V(C_w)$. The fact that $\mathbf{m}$ is an infinitesimal c-motion, the definition of $\boldsymbol{p}$, and the skew-symmetry of the matrices imply that

$$0 = \langle m(u), p(v) \rangle + \langle m(v), p(u) \rangle = \langle A_s p(u), p(v) \rangle + \langle A_t p(v), p(u) \rangle =$$

$$= A_s[j, i] + A_t[i, j] = A_t[i, j] - A_s[i, j],$$

which gives $A_s[i, j] = A_t[i, j]$. This argument, applied to all edges of the trees, implies that $A_w = A_r$ for all $w \in V(H)$ and hence $m(x) = A_r p(x)$ for all vertices x of G_H°. Thus m is a trivial infinitesimal c-motion and $(G_H^\circ, \boldsymbol{p})$ is infinitesimally locally completable. It follows that G_H° is locally completable in $\mathbb{R}^d$, as claimed. □

3.2 Global completability

The charaterization of globally rigid generic body-bar frameworks is also known. Let $H = (V, E)$ be a multigraph. We say that H is *highly m-tree-connected* if $H - e$ is m-tree-connected for all $e \in E$.

Theorem 13. *[3] Let $H = (V, E)$ be a multigraph with $|V| \geq 2$ and $|E| \geq 2$ and let G_H be the body-bar graph induced by H. Let $d \geq 1$ be an integer. Then the following are equivalent:*
(a) G_H is generically globally rigid in $\mathbb{R}^d$,
(b) H is highly $\binom{d+1}{2}$-tree-connected.

Thus we can deduce, by using Corollary 8, that:

Theorem 14. *Let $H = (V, E)$ be a multigraph with $|V| \geq 2$ and $|E| \geq 2$ and let G_H° be the cluster graph induced by H. Let $d \geq 1$ be an integer. Then the following are equivalent:*
(a) G_H° is generically globally completable in $\mathbb{R}^d$,
(b) H is highly $\binom{d}{2}$-tree-connected.

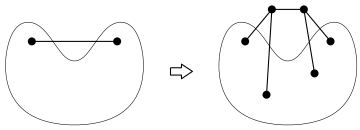

Figure 4: 2-dimensional double-1-extension.

Theorem 14 shows that global completability implies redundant local completability for cluster graphs. This implication does not hold in general (see Section 6.2 for more details).

4 Operations preserving c-independence

In this section we introduce some new graph operations and prove that they preserve c-independence in $\mathbb{R}^d$.

4.1 Double-1-extension

Let $G = (V, E)$ be a semi-simple graph. The (d-dimensional) *double 1-extension* operation removes an existing edge $e = ab$ from G and inserts two new vertices v_1 and v_2 with new edges av_1, v_1v_2, v_2b and $v_1u_1^1, v_1u_1^2, \ldots, v_1u_1^{d-1}$ and $v_2u_2^1, v_2u_2^2, \ldots, v_2u_2^{d-1}$, where $\{u_1^1, u_1^2, \ldots, u_1^{d-1}\}$ and $\{u_2^1, u_2^2, \ldots, u_2^{d-1}\}$ are $d-1$ distinct vertices in $(V + v_1) \setminus \{a\}$ and $(V + v_2) \setminus \{b\}$, respectively. We allow the possibility that e is a loop (in which case $a = b$). See Figure 4 for an example.

Lemma 15. *Let $G = (V, E)$ be a graph and $G' = (V', E')$ be the graph obtained from G by a double 1-extension. If G is c-independent in $\mathbb{R}^d$ then G' is also c-independent in $\mathbb{R}^d$.*

Proof. Suppose that the double 1-extension removes the edge $ab \in E$. We use the notation given above to denote the vertices involved in the operation.

Since G is c-independent, there is a generic $\boldsymbol{p} : V \to \mathbb{R}^d$ such that $C(G, \boldsymbol{p})$ is row independent. We define $\boldsymbol{p}' : V' \to \mathbb{R}^d$ by

$$\boldsymbol{p}'(u) = \begin{cases} \boldsymbol{p}(b) & (\text{if } u = v_1) \\ \boldsymbol{p}(a) & (\text{if } u = v_2) \\ \boldsymbol{p}(u) & (\text{otherwise}) \end{cases} \qquad (u \in V').$$

We show that $C(G', p')$ is row independent.

We first assume that $a \neq b$. Consider the rows of $C(G', \boldsymbol{p}')$ associated with av_1, v_1v_2, v_2b:

	v_1	v_2	a	b	$V' \setminus \{v_1, v_2, a, b\}$
av_1	p_a	0	p_b	0	0
v_2b	0	p_b	0	p_a	0
v_1v_2	p_a	p_b	0	0	0

The sum of the rows of av_1 and v_2b minus the row of v_1v_2 is equal to

v_1	v_2	a	b	$V' \setminus \{v_1, v_2, a, b\}$
0	0	p_b	p_a	0

which is equal to the row of ab in $R(G, \boldsymbol{q})$. Therefore, by using row operations, $C(G', \boldsymbol{p}')$ can be transformed to

$$
\begin{array}{r|c|c|c|}
 & v_1 & v_2 & \\
\hline
av_1 & p_a & 0 & * \\
\hline
v_1u_1^1 & p_{u_1^1} & & \\
\vdots & \vdots & 0 & * \\
v_1u_1^{d-1} & p_{u_1^{d-1}} & & \\
\hline
v_2b & 0 & p_b & * \\
\hline
v_2u_2^1 & & p_{u_2^1} & \\
\vdots & 0 & \vdots & * \\
v_2u_2^{d-1} & & p_{u_2^{d-1}} & \\
\hline
 & 0 & 0 & C(G, \boldsymbol{p}) \\
\hline
\end{array}
$$

This implies the row independence of $C(G', \boldsymbol{p}')$.

An almost identical argument holds when $a = b$. $\square$

Lemmas 4 and 15 imply the following characterization of local completability in $\mathbb{R}^1$. The equivalence of (i) and (ii) was verified by Singer and Cucuringu [20], using a different approach.

Theorem 16. *The following statements are equivalent for a graph $G = (V, E)$:*

(i) *G is minimally locally completable in $\mathbb{R}^1$;*

(ii) *Each connected component of G contains exactly one cycle, which is odd;*

(iii) *G can be constructed from a graph with one vertex with one loop by a sequence of one-dimensional 0-extensions and double-1-extensions.*

4.2 Vertex-splitting

Let $G = (V, E)$ be a semi-simple graph. The *d-dimensional vertex-splitting* (or simply *vertex-d-splitting*) operation (at vertex v with some fixed partition $\{U_1, U^*, U_2\}$ of $N(v)$ with $d - 1 \leq |U^*| \leq d$) consists of the following steps:

- It removes v and inserts two new vertices v_1 and v_2 with new edges v_iu for $u \in U_i \cup U^*$ for $i = 1, 2$.
- If v is incident with a loop in G, then it further adds a new edge v_1v_2.
- If $|U^*| = d - 1$, then it further adds a loop incident with v_1.

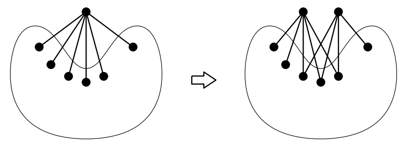

Figure 5: 3-dimensional vertex splitting (with $|U^*| = 3$).

See Figure 5 for an example.

Lemma 17. *Let $G = (V, E)$ be a graph and $G' = (V', E')$ be the graph obtained from G by a vertex-d-splitting at vertex v. If G is c-independent in $\mathbb{R}^d$ then G' is also c-independent in $\mathbb{R}^d$.*

Proof. Let $\{U_1, U^*, U_2\}$ be the partition of the neighbors of v used by the vertex-d-splitting operation.

(a) Let us first consider the case when $|U^*| = d$ and v is not incident to a loop.

Since G is c-independent, there is a generic $\boldsymbol{p} : V \to \mathbb{R}^d$ such that $C(G, \boldsymbol{p})$ is row independent. We define $\boldsymbol{p}' : V' \to \mathbb{R}^d$ by

$$\boldsymbol{p}'(u) = \begin{cases} \boldsymbol{p}(v) & (\text{if } u = v_1 \text{ or } u = v_2) \\ \boldsymbol{p}(u) & (\text{otherwise}) \end{cases} \qquad (u \in V').$$

We show that $C(G', \boldsymbol{p}')$ is row independent.

For $u \in U^*$, consider the rows of $C(G', \boldsymbol{p}')$ associated with v_1u and v_2u:

$$\begin{array}{c|c|c|c|c|} & v_1 & v_2 & u & V(G') \setminus \{u, v_1, v_2\} \\ \hline v_1u & p_u & 0 & p_v & 0 \\ \hline v_2u & 0 & p_u & p_v & 0 \\ \hline \end{array}$$

We first add the column of v_1 to that of v_2. Notice that if we then subtract the row of v_2u from that of v_1u, we obtain

$$\begin{array}{c|c|c|c|c|} & v_1 & v_2 & u & V' \setminus \{u, v_1, v_2\} \\ \hline v_1u & p_u & 0 & 0 & 0 \\ \hline v_2u & 0 & p_u & p_v & 0 \\ \hline \end{array}$$

where the resulting row of v_2u is equal to the row of vu in $C(G, \boldsymbol{p})$. It follows that, if we subtract the row of v_2u from that of v_1u for all $u \in U^*$, then, by identifying the columns of v_2 in $C(G', \boldsymbol{p}')$ with that of v in $C(G, \boldsymbol{p})$, and rearranging the rows, the resulting matrix $C(G', \boldsymbol{p}')$ can be written as

$$\begin{array}{c|c|c|} & v_1 & V' \setminus \{v_1\} \\ \hline v_1u_1 & p_{u_1} & \\ \vdots & \vdots & 0 \\ v_1u_d & p_{u_d} & \\ \hline E' \setminus \{v_1u_1, \ldots, v_1u_d\} & * & C(G, \boldsymbol{p}) \\ \hline \end{array}$$

where $U^* = \{u_1, u_2, \ldots, u_d\}$. Since the top-left $d \times d$-block is row independent (by the genericity of $\boldsymbol{p}$) and $C(G, \boldsymbol{p})$ is row independent, $C(G', \boldsymbol{p}')$ is row independent. This completes the proof for the case when $|U^*| = d$ and v is not incident to a loop.

(b) Next consider the case when $|U^*| = d$ and v is incident to a loop ℓ. Then ℓ is replaced by a new edge v_1v_2 in the vertex-d-splitting operation. By using the map $\boldsymbol{p}'$ defined above, the row of v_1v_2 in $C(G', \boldsymbol{p}')$ can be written as

$$\begin{array}{c|c|c|c|} & v_1 & v_2 & V' \setminus \{v_1, v_2\} \\ \cline{2-4} v_1v_2 & p_v & p_v & 0 \\ \cline{2-4} \end{array}$$

Notice that the restriction of this row to the columns associated with $V' \setminus \{v_1\}$ coincides with the row of ℓ in $C(G, \boldsymbol{p})$. This means that, by the same column and row operations, $C(G', \boldsymbol{p}')$ can be converted to exactly the same form as above.

(c) Finally, suppose that $|U^*| = d - 1$. In this case the same argument can be adapted, by observing that in the transformed matrix $C(G', \boldsymbol{p}')$ the top-left $d \times d$-matrix consists of the row of the loop together with the rows of the $d-1$ edges v_1u, for $u \in U^*$. □

5 Local completability of planar graphs in $\mathbb{R}^2$

In this section we consider the local completability problem of semi-simple planar graphs in $\mathbb{R}^2$, in both the Gram matrix and rectangular matrix models.

Lemma 1 implies that if a graph $G = (V, E)$ is c-independent in $\mathbb{R}^2$, then $i(X) \leq 2|X| - 1$ for all non-empty $X \subseteq V$ and $|E(H)| \leq 2|V(H)| - 4$ for all bipartite subgraphs H with at least three vertices. Note that in a planar graph the latter sparsity condition is guarenteed by Euler's formula and hence every bipartite planar graph will satisfy both necessary conditions.

We shall prove that all planar bipartite graphs are c-independent.

5.1 Planar bipartite graphs

Consider a planar bipartite graph $G = (V, E)$. We may suppose that G is a maximal planar bipartite graph (a planar quadrangulation). The c-independence of G follows by induction on the number of vertices by using the fact that every planar quadrangulation can be obtained from a pair of incident edges by repeated applications of the vertex splitting operation (with $|U^*| = 2 = d$) introduced in Section 4, see [1]. Hence Lemma 17 implies the following.

Theorem 18. *All planar bipartite graphs are c-independent in $\mathbb{R}^2$.*

This implies the following characterization of locally completable planar bipartite graphs in the rectangular matrix model.

Theorem 19. *A planar bipartite graph is locally completable in the rectangular matrix model in $\mathbb{R}^2$ if and only if it is a quadrangulation.*

Figure 6: (a) Looped wheel W_5'. (b) A 3-connected $(2,1)$-sparse simple planar graph which is not c-independent in $\mathbb{R}^2$. This graph G can be obtained from two copies of W_5' by identifying four edges (including the loop) and then removing the loop. By the circuit elimination axiom, the edge set of G contains a circuit of the 2-dimensional completability matroid.

5.2 Planar graphs

Let $G = (V, E)$ be a semi-simple graph. We say that G is $(2,1)$-*sparse* if $i(X) \leq 2|X| - 1$ holds for all non-empty subsets $X \subseteq V$. Lemma 1 says that $(2,1)$-sparsity is a necessary condition for c-independence. Since planarity forces c-independence for bipartite graphs in $\mathbb{R}^2$, it is natural to ask whether $(2,1)$-sparsity characterizes the c-independence of general planar graphs. We give a negative answer to this question by constructing $(2,1)$-sparse planar graphs which are not c-independent in $\mathbb{R}^2$.

The *looped wheel* W_n' with n vertices is the wheel W_n on n vertices with one loop attached at the center vertex (see Figure 6(a)). Theorem 16 and Lemma 10 imply that W_n' is c-independent in $\mathbb{R}^2$ if and only if n is even. Hence W_n' is a $(2,1)$-sparse planar graph which is not c-independent for all odd n. We can construct simple examples by replacing the loop in W_{2m+1}' by any simple planar minimally locally completable graph (so that the central vertex of W_{2m+1}' becomes a cut-vertex). Simple 3-connected examples can also be constructed, see Figure 6(b).

6 Sufficient conditions for global completability

In this section we consider the global completability of graphs. Recall that a graph G is globally completable in $\mathbb{R}^d$ if $(G, \boldsymbol{p})$ is globally completable for all d-dimensional generic configurations $\boldsymbol{p}$.

6.1 Completability stress and global completability

Let $G = (V, E)$ be a semi-simple graph. We define the *completability function* $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ by

$$f_G(p) = \big(\ldots, \langle p_u, p_v \rangle, \ldots\big) \qquad (p \in \mathbb{R}^{dV}).$$

Then f_G is smooth. Notice also that the completion matrix $C(G, p)$ is the Jacobian of f_G at p.

Following [20] we say that $\boldsymbol{\omega} : e \in E \mapsto \omega_e \in \mathbb{R}$ is a *completability stress* of (G, p) if

$C(G,p)^\top \boldsymbol{\omega} = 0$, that is, for each $u \in V$

$$\sum_{v \in N_G(u)} \omega_{uv} p_v = 0. \tag{7}$$

The *stress matrix* associated to $\boldsymbol{\omega}$ is the $|V| \times |V|$-matrix Ω, where each column and each row are associated with a vertex in V and each entry is given by

$$\Omega[u,v] = \omega_{uv}. \tag{8}$$

The following was observed in [20].

Proposition 20. *Let $G = (V,E)$ be a graph and $(G,\boldsymbol{p})$ be a d-dimensional framework such that $\boldsymbol{p}(V)$ linearly spans $\mathbb{R}^d$. Then for any completability stress $\boldsymbol{\omega} : E \to \mathbb{R}$ of $(G,\boldsymbol{p})$,*

$$P(\boldsymbol{p})\Omega = 0. \tag{9}$$

In particular, the rank of Ω is at most $n-d$.

Singer and Cucuringu [20] conjectured that having a maximum rank stress matrix implies global completability. Our next result verifies their conjecture.

Theorem 21. *Let G be a finite graph and let $\boldsymbol{p} : V \to \mathbb{R}^d$ be generic. Then G is globally completable if there is a completability stress $\boldsymbol{\omega}$ of $(G,\boldsymbol{p})$ with* $\operatorname{rank} \Omega = n-d$.

Our proof of Theorem 21 is an adaptation of the proof by Connelly [2] for the sufficiency of Theorem 3.

We say that $(G,\boldsymbol{q})$ (or $\boldsymbol{q}$) is a *linear image* of $(G,\boldsymbol{p})$ (or $\boldsymbol{p}$) if there is a $d\times d$-matrix A such that $q_v = Ap_v$ for all $v \in V$.

Proposition 22. *Let $G = (V,E)$ be a graph, $(G,\boldsymbol{p})$ be a d-dimensional framework such that $\boldsymbol{p}(V)$ linearly spans $\mathbb{R}^d$, and $\boldsymbol{\omega}$ be a completability stress of $(G,\boldsymbol{p})$. If* $\operatorname{rank} \Omega = n-d$, *then any other configuration $\boldsymbol{q}$ for which ω is a completability stress of $(G,\boldsymbol{q})$ is a linear image of $\boldsymbol{p}$.*

Proof. By Proposition 20, $P(\boldsymbol{p})\Omega = 0$ and $P(\boldsymbol{q})\Omega = 0$. Since $\boldsymbol{p}(V)$ linearly spans $\mathbb{R}^d$, rank $P(\boldsymbol{p}) = d$. Since the kernel of Ω has dimension d, the row vectors of $P(\boldsymbol{q})$ are spanned by those of $P(\boldsymbol{p})$, which implies that there is a $d \times d$-matrix A such that $P(\boldsymbol{q}) = AP(\boldsymbol{p})$. In other words, $q_v = Ap_v$ holds for all $v \in V$. $\square$

We say that $\boldsymbol{p} : V \to \mathbb{R}^d$ *lies on a conic at infinity* if there is a non-zero symmetric $d \times d$-matrix B satisfying

$$p_u^\top B p_v = 0 \qquad \text{for } uv \in E. \tag{10}$$

Proposition 23. *Let $(G,\boldsymbol{p})$ be a d-dimensional framework, and suppose that $\boldsymbol{p}$ does not lie on a conic at infinity. Then, if $(G,\boldsymbol{q})$ is a linear image of $(G,\boldsymbol{p})$ and is equivalent to $(G,\boldsymbol{p})$, then $(G,\boldsymbol{q})$ is congruent to $(G,\boldsymbol{p})$.*

Proof. By assumption there exists a $d \times d$ matrix A such that $q_v = Ap_v$ for $v \in V$. Since the inner product is preserved for each $uv \in E$, we have

$$0 = \langle q_u, q_v \rangle - \langle p_u, p_v \rangle = p_u^\top (A^\top A - I_d) p_v.$$

However, since $\boldsymbol{p}$ does not lie on a conic at infinity, $A^\top A - I_d = 0$ holds. In other words, A is an orthogonal matrix and $(G, \boldsymbol{q})$ is congruent to $(G, \boldsymbol{p})$. □

Proposition 24. *Let $(G, \boldsymbol{p})$ be a d-dimensional framework such that each vertex is incident with at least d edges. If $\boldsymbol{p}$ is generic, then $\boldsymbol{p}$ does not lie on a conic at infinity.*

Proof. We proceed by induction on d. The claim is clear if $d = 1$ so may assume that $d \geq 2$.

Take any symmetric matrix $B = (b_{ij})$ of size $d \times d$. Note that condition (10) is a system of $|E|$ linear equations for b_{ij}, where each coefficient is a polynomial of coordinates of $\boldsymbol{p}$. Hence, if there exists a map $\boldsymbol{q} : V \to \mathbb{R}^d$ that does not lie on a conic at infinity, then any generic $\boldsymbol{p}$ will not lie on a conic at infinity.

Let v be a vertex, and take $\boldsymbol{q}$ such that $\boldsymbol{q}(v) = (0, \ldots, 0, 1)^\top$ and $\boldsymbol{q}$ restricted to $V \setminus \{v\}$ is generic. Let $\boldsymbol{b}_d$ be the d-th row vector of B. For any $u \in N_G(v)$, condition (10) implies that $0 = q_v^\top B q_u = \langle \boldsymbol{b}_d, q_u \rangle$. Since $\{q_u \mid u \in N_G(v)\}$ linearly spans $\mathbb{R}^d$, we get $\boldsymbol{b}_d = 0$. (We take $v \in N_G(v)$ if there is a loop at v.)

Define $\hat{\boldsymbol{q}} : V \setminus \{v\} \to \mathbb{R}^{d-1}$ such that $\hat{\boldsymbol{q}}(u)$ is the $(d-1)$-dimensional vector obtained from $\boldsymbol{q}(u)$ by removing the last coordinate, and let B' be a $(d-1) \times (d-1)$ symmetric matrix obtained from B by removing its last row and column. Since $\boldsymbol{b}_d = 0$, we have $0 = q_u^\top B q_w = \hat{q}_u^\top B' \hat{q}_w$ for any edge uw in $G - v$. Hence by induction we get $B' = 0$, which in turn implies $B = 0$. □

We also need one more claim from [2].

Proposition 25 (Connelly [2])**.** *Let $f_i : \mathbb{R}^a \to \mathbb{R}$ be a polynomial with integer coefficients for $1 \leq i \leq b$, $f : \mathbb{R}^a \to \mathbb{R}^b$ be $f = (f_1, \ldots, f_b)$, $\boldsymbol{p} \in \mathbb{R}^a$ be generic over $\mathbb{Q}$, and $\boldsymbol{q} \in f^{-1}(f(\boldsymbol{p}))$. Then there are open neighborhoods $N_{\boldsymbol{p}}$ of $\boldsymbol{p}$ and $N_{\boldsymbol{q}}$ of $\boldsymbol{q}$ in $\mathbb{R}^a$ and a diffeomorphism $g : N_{\boldsymbol{q}} \to N_{\boldsymbol{p}}$ with $g(\boldsymbol{q}) = \boldsymbol{p}$ such that for all $x \in N_{\boldsymbol{q}}$, $f(g(x)) = f(x)$.*

Proof of Theorem 21. Since $(G, \boldsymbol{p})$ has a non-zero completability stress, the minimum degree of G is at least d. Let $(G, \boldsymbol{q})$ be a framework equivalent to $(G, \boldsymbol{p})$. Then $\boldsymbol{q} \in f_G^{-1}(f_G(\boldsymbol{p}))$, and Proposition 25 implies that there are open neighborhoods $N_{\boldsymbol{p}}$ of $\boldsymbol{p}$ and $N_{\boldsymbol{q}}$ of $\boldsymbol{q}$ in $\mathbb{R}^{dV}$ and a diffeomorphism $g : N_{\boldsymbol{q}} \to N_{\boldsymbol{p}}$ with $g(\boldsymbol{q}) = \boldsymbol{p}$ and $f_G(g(x)) = f_G(x)$ for $x \in N_{\boldsymbol{q}}$. By taking differentials, we get $C(G, \boldsymbol{p})A = C(G, \boldsymbol{q})$, where A is the Jacobian of g at $\boldsymbol{q}$. Therefore $\boldsymbol{\omega}^\top C(G, \boldsymbol{q}) = \boldsymbol{\omega}^\top C(G, \boldsymbol{p})A = 0$. In other words $\boldsymbol{\omega}$ is a completion stress of $(G, \boldsymbol{q})$. Since rank $\Omega = n - d$, Proposition 22 implies that $(G, \boldsymbol{q})$ is a linear image of $(G, \boldsymbol{p})$. Since $\boldsymbol{p}$ does not lie on a conic at infinity by Proposition 24, $(G, \boldsymbol{q})$ is congruent to $(G, \boldsymbol{p})$ by Proposition 23. □

6.2 Operations Preserving Global Completability

Theorem 3 tells us that having a maximum rank (rigidity) stress matrix is both necessary and sufficient for global rigidity of graphs. The analogous result does not hold for global completability since, as pointed out by Singer and Cucuringu [20], the converse of Theorem 21 does not hold in general. The following result can be used to construct an infinite family of counterexamples for any d.

Theorem 26. *Let G be a generically globally completable graph in $\mathbb{R}^d$, and let G' be a graph obtained from G by a simple 0-extension. Then G' is generically globally completable in $\mathbb{R}^d$.*

Proof. Take a generic $\boldsymbol{p} : V(G) \to \mathbb{R}^d$ such that $(G, \boldsymbol{p})$ is globally completable in $\mathbb{R}^d$, and let $\boldsymbol{p}' : V(G') \to \mathbb{R}^d$ be a generic configuration obtained by extending $\boldsymbol{p}$. We show that $(G', \boldsymbol{p}')$ is globally completable.

To see this take any $\boldsymbol{q}' : V(G') \to \mathbb{R}^d$ such that $(G', \boldsymbol{q}')$ is equivalent to $(G', \boldsymbol{p}')$. Since $(G, \boldsymbol{p})$ is globally completable, we may assume $q'_u = p_u = p'_u$ for any $u \in V(G)$.

Let v be the new vertex. Then for $u \in N_{G'}(v)$ we have $\langle p'_u, p'_v \rangle = \langle q'_u, q'_v \rangle$, which means $\langle p_u, p'_v - q'_v \rangle = 0$ for all $u \in N_{G'}(v)$. Since $\{p_u \mid u \in N_{G'}(v)\}$ linearly spans $\mathbb{R}^d$, we get $p'_v - q'_v = 0$, which in turn implies $\boldsymbol{p}' = \boldsymbol{q}'$. □

We next investigate the double 1-extension operation. Lemma 15 tells us that the double 1-extension operation preserves local completability. In the proof we gave a special configuration $\boldsymbol{p}$ that achieves the maximum rank of the completability matrix, which in turn implies the generic local completability of the underlying graph G.

On the other hand, in the case of global completability, even if the rank of a stress matrix of $(G, \boldsymbol{p})$ is maximum for a non-generic configuration $\boldsymbol{p}$, G may not be globally completable. Theorem 27 below allows us to avoid this problem by showing that, if we add the hypothesis that $(G, \boldsymbol{p})$ is locally completable, then there does exist a generic $\boldsymbol{q}$ so that $(G, \boldsymbol{q})$ has a maximum rank stress matrix. We can then use Theorem 21 to deduce that G is globally completable.

Theorem 27 can be proved by using a very similar technique to that used by Connelly and Whiteley [4] to obtain an analogous result for global rigidity.

Theorem 27. *Let $(G, \boldsymbol{p})$ be a d-dimensional framework with a completability stress $\boldsymbol{\omega}$, and suppose that $(G, \boldsymbol{p})$ is locally completable and Ω has rank $|V(G)| - d$. Then there is a generic $\boldsymbol{q}$ and a completability stress $\boldsymbol{\omega}'$ of $(G, \boldsymbol{q})$ such that Ω' has rank $|V(G)| - d$.*

A stress $\boldsymbol{\omega}$ is called *nowhere zero* if $\boldsymbol{\omega}(e) \neq 0$ for all $e \in E(G)$. We also need the following.

Lemma 28. *Let $(G, \boldsymbol{p})$ be a generic d-dimensional framework with a completability stress $\boldsymbol{\omega}$. Suppose that the rank of Ω is $|V| - d$. Then there is a nowhere zero completability stress $\boldsymbol{\omega}'$ of $(G, \boldsymbol{p})$ such that the rank of Ω' is $|V| - d$.*

Proof. Let $e \in E(G)$ such that $\omega(e) = 0$. Then ω (restricted to $E(G) - e$) is a completability stress of $(G - e, \boldsymbol{p})$ such that the rank of Ω is equal to $|V(G)| - d$. By Theorem 21 $(G - e, \boldsymbol{p})$ is globally completable, and hence is locally completable. This in particular implies that the completability matrix of $(G - e, \boldsymbol{p})$ is full rank, and hence there is a completability stress $\boldsymbol{\omega}'$ of $(G, \boldsymbol{p})$ for which $\boldsymbol{\omega}'(e) \neq 0$. Since the rank of Ω is $|V(G)|-d$, the rank of the stress matrix of $\boldsymbol{\omega}+\epsilon\boldsymbol{\omega}'$ is $|V(G)|-d$ for sufficiently small ϵ. Applying the same argument for all $e \in E(G)$, the desired completability stress can be obtained. □

Theorem 29. *Let G' be a graph obtained from a graph G by a double 1-extension. Suppose that there is a generic $\boldsymbol{p} : V(G) \to \mathbb{R}^d$ and a completability stress $\boldsymbol{\omega} : E(G) \to \mathbb{R}$ of $(G, \boldsymbol{p})$ such that the rank of Ω is $|V(G)|-d$. Then there is a generic $\boldsymbol{p}^* : V(G') \to \mathbb{R}^d$ and a completability stress $\boldsymbol{\omega}^* : E(G') \to \mathbb{R}$ of $(G', \boldsymbol{p}^*)$ such that the rank of Ω^* is $|V(G')| - d$. In particular, G' is globally completable.*

Proof. Suppose that the double 1-extension is performed along edge ab by adding new vertices v_1 and v_2 and new edges $v_1u_1^1, \ldots, v_1u_1^{d-1}$ and $v_2u_2^1, \ldots, v_2u_2^{d-1}$. We define a configuration $\boldsymbol{p}' : |V(G')| \to \mathbb{R}^d$ as given in the proof of Lemma 15 i.e., $\boldsymbol{p}'(v_1) = \boldsymbol{p}(b)$, $\boldsymbol{p}'(v_2) = \boldsymbol{p}(a)$, and $\boldsymbol{p}'(u) = \boldsymbol{p}(u)$ for $u \in V(G)$.

By Lemma 28 we may assume that $\omega_{ab} \neq 0$. Based on $\boldsymbol{\omega}$, we define a completability stress $\boldsymbol{\omega}' : E(G) \to \mathbb{R}$ by

$$\boldsymbol{\omega}'_e = \begin{cases} \omega_{ab} & \text{if } e = av_1 \text{ or } bv_2 \\ -\omega_{ab} & \text{if } e = v_1v_2 \\ \omega_e & \text{if } e \in E(G) \setminus \{ab\} \\ 0 & \text{otherwise.} \end{cases}$$

Due to the special configuration $\boldsymbol{p}'$, one can easily check that $\boldsymbol{\omega}'$ is a completability stress of $(G', \boldsymbol{p}')$. Now the stress matrix Ω' is

$$\begin{array}{c|c|c|c|c|c|}
 & V(G)\setminus\{a,b\} & a & b & v_1 & v_2 \\ \hline
V(G)\setminus\{a,b\} & \multicolumn{3}{c|}{} & 0 & 0 \\ \cline{5-6}
a & \multicolumn{3}{c|}{*} & \omega_{ab} & 0 \\ \cline{5-6}
b & \multicolumn{3}{c|}{} & 0 & \omega_{ab} \\ \hline
v_1 & 0 & \omega_{ab} & 0 & 0 & -\omega_{ab} \\ \hline
v_2 & 0 & 0 & \omega_{ab} & -\omega_{ab} & 0 \\ \hline
\end{array}$$

If we add the column of v_1 to that of b and add the column of v_2 to that of a, the matrix is changed to

$$\begin{array}{c|c|c|c|c|c|}
 & V(G)\setminus\{a,b\} & a & b & v_1 & v_2 \\ \hline
V(G)\setminus\{a,b\} & \multicolumn{3}{c|}{} & 0 & 0 \\ \cline{5-6}
a & \multicolumn{3}{c|}{\Omega} & \omega_{ab} & 0 \\ \cline{5-6}
b & \multicolumn{3}{c|}{} & 0 & \omega_{ab} \\ \hline
v_1 & 0 & 0 & 0 & 0 & -\omega_{ab} \\ \hline
v_2 & 0 & 0 & 0 & -\omega_{ab} & 0 \\ \hline
\end{array}$$

where the top left block turns out to be Ω. Since $\omega_{ab} \neq 0$, rank $\Omega' = |V(G)| - d + 2 = |V(G')| - d$.

In the proof of Lemma 15, we showed that $(G', \boldsymbol{p}')$ is also locally completable. Hence, by Theorem 27, there is a generic $\boldsymbol{p}^*$ and a completability stress $\boldsymbol{\omega}^*$ of $(G', \boldsymbol{p}^*)$ such that the rank of Ω^* is $|V(G')|-d$. By Theorem 21, G' is globally completable. □

Singer and Cucuingu [20] obtained the following combinatorial characterization of global completability of graphs in $\mathbb{R}^1$. (They proved the equivalence between (i) and (ii). The equivalence between (ii) and (iii) is straightforward.)

Theorem 30. *The following statements are equivalent for a graph G:*

(i) *G is globally 1-completable;*

(ii) *G is connected and contains an odd cycle;*

(iii) *G can be constructed from the graph consisting of one vertex with one loop by a sequence of (1-dimensional) simple 0-extensions, double-1-extensions, and edge additions.*

Note that Theorem 30 does not follow from Theorems 26 and 29, as G may not have a nowhere zero completability stress. On the other hand, Theorem 30 implies that the double-1-extension operation preserves global rigidity in $\mathbb{R}^1$. We have examples showing that this is not the case in $\mathbb{R}^2$.

7 More links between completability and rigidity

We first show that we may focus on frameworks whose vertices lie on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid \langle x, x\rangle = 1\}$ when considering infinitesimal completability in $\mathbb{R}^d$. For a configuration $\boldsymbol{p} : V \to \mathbb{R}^d \setminus \{0\}$, the *central projection* of $\boldsymbol{p}$ onto $\mathbb{S}^{d-1}$ is a configuration $\boldsymbol{q} : V \to \mathbb{S}^{d-1}$ with $\boldsymbol{q}_i = p_i/|p_i|$ for $i \in V$. We say that $(G, \boldsymbol{q})$ is the *central projection* of $(G, \boldsymbol{p})$ onto $\mathbb{S}^{d-1}$.

Lemma 31. *Let $(G, \boldsymbol{p})$ be a framework in $\mathbb{R}^d$ with $\boldsymbol{p} : V \to \mathbb{R}^d \setminus \{0\}$. Then $(G, \boldsymbol{p})$ is infinitesimally completable in $\mathbb{R}^d$ if and only if the central projection $(G, \boldsymbol{q})$ of $(G, \boldsymbol{p})$ onto $\mathbb{S}^{d-1}$ is infinitesimally completable in $\mathbb{R}^d$.*

Proof. Let $\mathbf{s} : (\mathbb{R}^d)^{|V|} \to (\mathbb{R}^d)^{|V|}$ be a linear map defined by, for $\dot{\boldsymbol{p}} : V \to \mathbb{R}^d$ and $i \in V$, $s \circ \dot{p}_i = \dot{p}_i/|p_i|$. Then, for any $\dot{\boldsymbol{p}} : V \to \mathbb{R}^d$ and $i, j \in V \times V$,

$$\langle p_i, \dot{p}_j\rangle + \langle p_j, \dot{p}_i\rangle = |p_i||p_j|(\langle q_i, s \circ \dot{p}_j\rangle + \langle q_j, s \circ \dot{p}_i\rangle).$$

So $\langle p_i, \dot{p}_j\rangle + \langle p_j, \dot{p}_i\rangle = 0$ if and only if $\langle q_i, s \circ \dot{p}_j\rangle + \langle q_j, s \circ \dot{p}_i\rangle = 0$ and s induces a linear bijection between the spaces of infinitesimal c-motions of $(G, \boldsymbol{p})$ and $(G, \boldsymbol{q})$. □

7.1 The Pogorelov map

In this subsection we give a brief description of a method used by Pogorelov [17] to transform a framework on the upper hemisphere $\mathbb{S}^{d-1}_+ = \{x \in \mathbb{R}^d \mid \langle x, x\rangle = 1, \langle \boldsymbol{e}, x\rangle > 0\}$ to a framework on its tangent space $\mathbb{E}^{d-1} = \{x \in \mathbb{R}^d \mid \langle \boldsymbol{e}, x\rangle = 1\}$ at the point $\boldsymbol{e} = (0, 0, \ldots, 0, 1) \in \mathbb{R}^d$, in such a way that the dimension of the space of infinitesimal motions is preserved. The reader is referred to Saliola and Whiteley [18] and Ismestiev [6] for more details.

Given a point $x \in \mathbb{S}_+^{d-1}$, let $T_x = \{y \in \mathbb{R}^d \mid \langle y, x \rangle = 0\}$ be the tangent space of $\mathbb{S}_+^{d-1}$ at x. Following Pogorelov [17], see also [18], we define an *infinitesimal motion* of a framework $(G, \boldsymbol{p})$ on $\mathbb{S}_+^{d-1}$ to be a map $\boldsymbol{t} : i \in V \mapsto t_i \in T_{p_i}$ satisfying

$$\langle p_i, t_j \rangle + \langle p_j, t_i \rangle = 0 \qquad (ij \in E). \tag{11}$$

Thus an *infinitesimal motion* of $(G, \boldsymbol{p})$ on $\mathbb{S}_+^{d-1}$ is an infinitesimal c-motion in $\mathbb{R}^d$ which satisfies the additional constraint that the infinitesimal velocity of each point p_i lies in the tangent hyperplane at p_i.

On the other hand, an *infinitesimal motion* of a framework $(G, \boldsymbol{q})$ on $\mathbb{E}^{d-1}$ is a map $\boldsymbol{m} : V \to M = \{y \in \mathbb{R}^d \mid \langle \boldsymbol{e}, y \rangle = 0\}$ satisfying

$$\langle p_i - p_j, m_i - m_j \rangle = 0 \qquad (ij \in E). \tag{12}$$

Thus an *infinitesimal motion* of $(G, \boldsymbol{q})$ on $\mathbb{E}^{d-1}$ is an infinitesimal motion in $\mathbb{R}^d$ which preserves the edge lengths and the fact that the points lie on $\mathbb{E}^{d-1}$.

We can transform a framework $(G, \boldsymbol{p})$ on $\mathbb{S}_+^{d-1}$ to a framework on $\mathbb{E}^{d-1}$ using the map $\phi : \mathbb{S}_+^{d-1} \to \mathbb{E}^{d-1}$ defined by

$$\phi : x \mapsto \frac{x}{\langle \boldsymbol{e}, x \rangle}.$$

Note that ϕ is bijective and its inverse is given by $\phi^{-1}(x) = x/|x|$.

We next define a map between the spaces of infinitesimal motions of $(G, \boldsymbol{p})$ and $(G, \phi \circ \boldsymbol{p})$. For $x \in \mathbb{S}_+^{d-1}$, define $\psi_x : T_x \to M$ by

$$\psi_x : t \mapsto \frac{t - \langle t, \boldsymbol{e} \rangle \boldsymbol{e}}{\langle \boldsymbol{e}, x \rangle}. \tag{13}$$

It is easy to check that ψ_x is a linear isomorphism whose inverse is given by

$$\psi_x^{-1} : m \mapsto \langle \boldsymbol{e}, x \rangle m - \langle m, x \rangle \boldsymbol{e}. \tag{14}$$

Given an infinitesimal motion $\boldsymbol{t}$ of $(G, \boldsymbol{p})$ we can now define an infinitesimal motion $\psi(\boldsymbol{t})$ of $(G, \phi \circ \boldsymbol{p})$ by putting $\psi(t_i) = \psi_{p_i}(t_i)$ for all $i \in V$. Then ψ is a linear bijection between the spaces of infinitesimal motions of $(G, \boldsymbol{p})$ in $\mathbb{S}_+^{d-1}$ and $(G, \phi \circ \boldsymbol{p})$ in $\mathbb{E}^{d-1}$.

7.2 An application to completability

We will use the Pogorelov map to obtain another relationship between completability in $\mathbb{R}^d$ and rigidity in $\mathbb{R}^{d-1}$. As an application we obtain a sufficient condition for completability in $\mathbb{R}^2$ which extends Corollary 7.

Let $(G, \boldsymbol{p})$ be a framework in $\mathbb{R}^d$ whose vertices lie on $\mathbb{S}_+^{d-1}$. Since $\mathbb{R}^d = T_{p_i}^{\perp} \oplus T_{p_i}$ for each $i \in V$, a map $\dot{\boldsymbol{p}} : V \mapsto \mathbb{R}^d$ can be decomposed into the direct sum of two maps $\boldsymbol{s} : i \in V \mapsto s_i \in \mathbb{R}$ and $\boldsymbol{t} : i \in V \mapsto t_i \in T_{p_i}$, such that

$$\dot{p}_i = s_i p_i + t_i \qquad (i \in V). \tag{15}$$

Let $L \subseteq E$ be the set of loops of G. We may substitute equation (15) into (2) to obtain

$$\begin{aligned} 0 = \langle p_i, \dot{p}_j \rangle + \langle p_j, \dot{p}_i \rangle &= (s_i + s_j)\langle p_i, p_j \rangle + \langle p_i, t_j \rangle + \langle p_j, t_i \rangle \qquad (ij \in E \setminus L) \\ 0 = \langle p_i, \dot{p}_i \rangle &= s_i |p_i|^2 = s_i \qquad (ii \in L) \end{aligned} \tag{16}$$

On the other hand, a calculation (from [18]) shows that

$$\begin{aligned} \langle \phi(p_i) - \phi(p_j), \psi_{p_i}(t_i) - \psi_{p_j}(t_j) \rangle &= \frac{\langle p_i, t_i \rangle}{\langle \boldsymbol{e}, p_i \rangle^2} - \frac{\langle p_i, t_j \rangle + \langle p_j, t_i \rangle}{\langle \boldsymbol{e}, p_i \rangle \langle \boldsymbol{e}, p_j \rangle} + \frac{\langle p_j, t_j \rangle}{\langle \boldsymbol{e}, p_j \rangle^2} \\ &= -\frac{\langle p_i, t_j \rangle + \langle p_j, t_i \rangle}{\langle \boldsymbol{e}, p_i \rangle \langle \boldsymbol{e}, p_j \rangle} \end{aligned}$$

where the last equation follows from the facts that $t_i \in T_{p_i}$ and $t_j \in T_{p_j}$. Putting $\bar{q}_i = \phi(p_i) \in \mathbb{E}^{d-1}$ and $\bar{u}_i = \psi_{p_i}(t_i) \in M$, we obtain

$$\begin{aligned} \langle p_i, \dot{p}_j \rangle + \langle p_j, \dot{p}_i \rangle &= (s_i + s_j)\langle p_i, p_j \rangle - \langle \boldsymbol{e}, p_i \rangle \langle \boldsymbol{e}, p_j \rangle \langle \phi(p_i) - \phi(p_j), \psi(t_i) - \psi(t_j) \rangle \\ &= \frac{1}{|\bar{q}_i||\bar{q}_j|}[(s_i + s_j)\langle \bar{q}_i, \bar{q}_j \rangle - \langle \boldsymbol{e}, \bar{q}_i \rangle \langle \boldsymbol{e}, \bar{q}_j \rangle \langle \bar{q}_i - \bar{q}_j, \bar{u}_i - \bar{u}_j \rangle]. \end{aligned} \tag{17}$$

Since ψ_x is a linear bijection between T_x and M, it induces a linear bijection ψ between $\{\dot{\boldsymbol{p}} \in (\mathbb{R}^d)^{|V|} \mid \dot{p} : i \in V \mapsto \dot{p}_i \in \mathbb{R} \oplus T_{p_i}\}$ and $\{(\boldsymbol{s}, \bar{\boldsymbol{u}}) \mid \boldsymbol{s} : V \to \mathbb{R}, \bar{\boldsymbol{u}} : V \to M\}$. It now follows from (2), (16), and (17) that the space of infinitesimal c-motions $\dot{\boldsymbol{p}}$ of $(G, \boldsymbol{p})$ is linearly isomorphic to the space of solutions $(\boldsymbol{s}, \bar{\boldsymbol{u}})$ to the system of equations

$$\begin{aligned} \langle \bar{q}_i - \bar{q}_j, \bar{u}_i - \bar{u}_j \rangle - \frac{\langle \bar{q}_i, \bar{q}_j \rangle}{\langle \boldsymbol{e}, \bar{q}_i \rangle \langle \boldsymbol{e}, \bar{q}_j \rangle}(s_i + s_j) &= 0 \qquad (ij \in E \setminus L) \\ s_i &= 0 \qquad (ii \in L) \end{aligned} \tag{18}$$

associated to the framework $(G, \bar{\boldsymbol{q}})$ on $\mathbb{E}^{d-1}$, where $\bar{\boldsymbol{q}} = \phi \circ \boldsymbol{p}$.

By taking the natural projection of $\mathbb{E}^{d-1}$ onto $\mathbb{R}^{d-1}$, we can transform (18) into a system of equations associated to a framework $(G, \boldsymbol{q})$ in $\mathbb{R}^{d-1}$. Let $\boldsymbol{q}, \boldsymbol{u} : V \to \mathbb{R}^{d-1}$ be such that $\begin{pmatrix} q_i \\ 1 \end{pmatrix} = \bar{q}_i$ and $\begin{pmatrix} u_i \\ 0 \end{pmatrix} = \bar{u}_i$ for all $i \in V$. Then (18) becomes

$$\begin{aligned} \langle q_i - q_j, u_i - u_j \rangle - (\langle q_i, q_j \rangle + 1)(s_i + s_j) &= 0 \qquad (ij \in E \setminus L) \\ s_i &= 0 \qquad (ii \in L) \end{aligned} \tag{19}$$

Let $R'(G, q)$ be the $|E| \times d|V|$-matrix representing this linear system of equations. Then

$$R'(G, \boldsymbol{q}) = [R(G, \boldsymbol{q}) \quad -S(G, \boldsymbol{q})] \tag{20}$$

where $R(G, \boldsymbol{q})$ is the rigidity matrix of $(G, \boldsymbol{q})$ and $S(G, \boldsymbol{q})$ is the $|E| \times |V|$-matrix in which the row associated with each $ij \in E$ is

$$0 \dots 0 \quad \overbrace{\langle q_i, q_j \rangle + 1}^{i} \quad 0 \dots 0 \quad \overbrace{\langle q_i, q_j \rangle + 1}^{j} \quad 0 \dots 0.$$

The equivalence of (2) and (20) gives us the following result.

Proposition 32. *Let $(G, \boldsymbol{p})$ be the framework on $\mathbb{S}^d_+$ and let $(G, \boldsymbol{q})$ be the framework in $\mathbb{R}^{d-1}$ with $\phi(p_i) = \begin{pmatrix} q_i \\ 1 \end{pmatrix}$ for $i \in V$. Then* $\operatorname{rank} C(G, \boldsymbol{p}) = \operatorname{rank} R'(G, \boldsymbol{q})$.

Note that it is easy to deduce Corollary 7 from Lemma 31 and Proposition 32.

Lemma 31 and Proposition 32 tell us that the infinitesimal completability of a framework $(G, \boldsymbol{p})$ in the half-plane $\{x \in \mathbb{R}^d \mid \langle x, \boldsymbol{e} \rangle > 0\}$ is determined by the rank of an $|E| \times d|V|$ matrix whose first $(d-1)|V|$ columns are given by the rigidity matrix of an associated framework $(G, \boldsymbol{q})$ in $\mathbb{R}^{d-1}$. We can use this to obtain the following sufficient condition for generic local completability in $\mathbb{R}^2$.

Lemma 33. *Let $G = (V, E)$ be a graph. If there exists a partition $\{V_1, V_2, \ldots, V_t\}$ of V such that $G[V_i]$ is locally completable in $\mathbb{R}^1$ for all $1 \leq i \leq t$, and $G - \bigcup_{i=1}^t E(G[V_i])$ is connected, then G is locally completable in $\mathbb{R}^2$.*

Proof. Let $G_1 = (V, E - \bigcup_{i=1}^t E(G[V_i]))$ and $G_2 = (V, \bigcup_{i=1}^t E(G[V_i]))$. By Proposition 32 it suffices to show that there is $\boldsymbol{q} : V \to \mathbb{R}$ such that the rank of $R'(G, \boldsymbol{q})$ is $2n - 1$. Take any t distinct numbers $a_1, \ldots, a_t$. By using the partition $\{V_1, \ldots, V_t\}$, we shall define $\boldsymbol{q}$ by $q_i = a_j$ for $i \in V_j$. Then by definition $R'(G, \boldsymbol{q})$ can be written as follows.

$$\begin{array}{r|c|c|} \cline{2-3} E(G_1) & I(\overrightarrow{G}_1) & * \\ \cline{2-3} E(G_2) & 0 & I(G_2) \\ \cline{2-3} \end{array}$$

where $I(\overrightarrow{G}_1)$ is the incidence matrix of an arbitrarily oriented G_1 and $I(G_2)$ is the edge-vertex incidence matrix of G_2. Since G_1 is connected, the rank of $I(\overrightarrow{G}_1)$ is equal to $n - 1$. On the other hand, the rank of $I(G_2)$ is equal to n if and only if each connected component of G_2 contains an odd cycle, equivalently G_2 is locally completeble in $\mathbb{R}^1$ by Theorem 16. ☐

The special case of Corollary 7 when $d = 2$ follows from Lemma 33 by considering the partition of the vertex set into single vertices. This suggests that Lemma 33 may extend to all dimensions if we replace the connectivity condition of $G - \bigcup_{i=1}^t E(G[V_i])$ by the condition that $G - \bigcup_{i=1}^t E(G[V_i])$ is rigid in $\mathbb{R}^{d-1}$.

8 Concluding Remarks

A complete characterization of locally completable semi-simple graphs in $\mathbb{R}^2$ remains open. One possible approach is to strengthen the necessary condition for c-independence of a graph $G = (V, E)$ given in Lemma 1 by considering families of subgraphs of G which cover E. Such covers have played an important role in various rigidity problems, see for example the formula for the rank function of the 2-dimensional rigidity matroid given by Lovász and Yemini in [15], or the necessary conditions for independence in the 3-dimensional rigidity matroid given in [8, 9].

Let $H = (V, E)$ be a *non-trivial* bipartite graph i.e. H has bipartition (A, B) with $|A|, |B| \geq 2$. A *cover* of $\mathcal{F} = \{X_1, X_2, \ldots, X_s\}$ is a family of subsets of V with $|X_i \cap A|, |X_i \cap B| \geq 2$ for all $1 \leq i \leq s$ and each edge of H induced by at least one set in $\mathcal{F}$. We say that $\mathcal{F}$ is 2-*thin* if $|X_i \cap X_j \cap A|, |X_i \cap X_j \cap B| \leq 1$ for all $1 \leq i < j \leq s$. A *hinge* of $\mathcal{F}$ is a pair of vertices $\{x, y\}$ with $X_i \cap X_j = \{x, y\}$ for some $i \neq j$. Let $H(\mathcal{F})$ be the set of all hinges of $\mathcal{F}$. The *hinge graph* of $\mathcal{F}$ is the bipartite graph with bipartition $(\mathcal{F}, H(\mathcal{F}))$ in which X_i and h are incident if h is contained in X_i. The *degree* $\deg(h)$ of a hinge h is given by its degree in the hinge graph of $\mathcal{F}$. The *value* of $\mathcal{F}$ is defined as

$$\mathrm{val}(\mathcal{F}) = \sum_{i=1}^{s}(2|X_i| - 4) - \sum_{h \in H(\mathcal{F})}(\deg(h) - 1). \tag{21}$$

We say that $\mathcal{F}$ is k-*degenerate* if for all $\mathcal{F}' \subseteq \mathcal{F}$, the hinge graph of $\mathcal{F}'$ contains a vertex $X \in \mathcal{F}'$ of degree at most k.

Lemma 34. *Suppose $G = (V, E)$ is a non-trivial bipartite graph and $\mathcal{F}$ is an 8-degenerate 2-thin cover of G. Then the rank of $\mathcal{C}_2(G)$ is at most* $\mathrm{val}(\mathcal{F})$.

Proof. We use induction on $|\mathcal{F}|$. If $|\mathcal{F}| = 1$ then $H(\mathcal{F}) = \emptyset$ and the lemma follows from Lemma 1. Hence suppose that $|\mathcal{F}| \geq 2$. We may assume that $uv \in E$ for all hinges $\{u, v\}$ of $\mathcal{F}$ since adding such an edge uv to G will not change $\mathrm{val}(\mathcal{F})$ or the fact that $\mathcal{F}$ an 8-degenerate 2-thin cover, and can only increase the rank of $\mathcal{C}_2(G)$. Since $\mathcal{F}$ is 8-degenerate, we can choose an $X \in \mathcal{F}$ which contains at most 8 hinges of $\mathcal{F}$. Let E^* be set of edges of G whose end-vertices are the hinges of $\mathcal{F}$ which are contained in X, and E' be the set of edges of G which are induced by X and do not belong to E^*. Let $G' = G - E'$ and let B' be a base of $\mathcal{C}_2(G')$ which contains E^*. (Lemma 4 implies that any set of at most eight non-loop edges is independent in $\mathcal{C}_2(G')$.) Since $\mathcal{F}' = \mathcal{F} \setminus \{X\}$ is an 8-degenerate 2-thin cover of G' we may use induction, to deduce that $|B'| \leq \mathrm{val}(\mathcal{F}')$. Let B be a basis of $\mathcal{C}_2(G)$ which contains B'. Then Lemma 1 implies that $|B \setminus B'| \leq 2|X| - 4 - |E^*|$ and hence $|B| \leq \mathrm{val}(\mathcal{F})$. ☐

Lemma 34 immediately gives the following necessary condition for c-independence.

Corollary 35. *Suppose G is a bipartite graph. If G is c-independent in $\mathbb{R}^2$ then $|E(H)| \leq$ $\mathrm{val}(\mathcal{F})$ for all non-trivial subgraphs H of G and all 8-degenerate, 2-thin covers $\mathcal{F}$ of H.*

For all examples of c-dependent bipartite graphs we know, their c-dependence can be demonstrated by a 2-degenerate, 2-thin cover.

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