# Bounds on maximal families of sets not containing three sets with $A \cap B \subset C, A \not\subset B$


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## 1 Introduction

Let $[n] = \{1, 2, \dots, n\}$ be a finite set and $\mathcal{F} \subset 2^{[n]}$ a family of its subsets. In the present paper, $\max |\mathcal{F}|$ will be investigated under certain conditions on the family $\mathcal{F}$. The well-known Sperner's Theorem ([**?**]) was the first such discovery.

**Theorem 1.1** *If $\mathcal{F}$ is a family of subsets of $[n]$ without inclusion ($F, G \in \mathcal{F}$ implies $F \not\subset G$) then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

*holds, and this estimate is sharp as the family of all $\lfloor \frac{n}{2} \rfloor$-element subsets achieves this size.*

There are a very large number of generalizations and analogues of this theorem. Here we will mention only some results where the conditions on $\mathcal{F}$ exclude certain configurations which can be expressed by inclusion only

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(i.e. the conditions can be stated without using intersections, unions, etc.) The first such generalization was obtained by Erdős [**?**]. The family of $k$ distinct sets with mutual inclusions, $F_1 \subset F_2 \subset \dots F_k$ is called a *chain of length* $k$, which we denote simply by $P_k$. For any "small" family of sets $\mathcal{P}$, with specified inclusions between pairs of sets, let $\mathrm{La}(n, \mathcal{P})$ denote the size of the largest family $\mathcal{F}$ of subsets of $[n]$ which contains no $\mathcal{P}$ as a subfamily. In the rest of the paper, the specified $\mathcal{P}$s will be denoted by normal upper case letters. Erdős extended Sperner's Theorem as follows:

**Theorem 1.2** [**?**] $\mathrm{La}(n, P_{k+1})$ *is equal to the sum of the* $k$ *largest binomial coefficients of order* $n$*. This bound is tight as the middle* $k$ *layers of the Boolean lattice form a family of this size which contains no* $P_{k+1}$*.*

Now consider families other than chains. Let $V_r$ denote the $r$*-fork*, which is a family of $r+1$ distinct sets: $F \subset G_1, F \subset G_2, \dots, F \subset G_r$. The quantity $\mathrm{La}(n, V_r)$ was first (asymptotically) determined for $r = 2$.

**Theorem 1.3** [**?**]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq \mathrm{La}(n, V_2) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n}\right).$$

This was recently generalized:

**Theorem 1.4** [**?**], cf. also [**?**]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{r}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq \mathrm{La}(n, V_{r+1}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

The family of four distinct subsets satisfying $A \subset C, A \subset D, B \subset C$ is called and denoted by $N$. Another recent result is the following one:

**Theorem 1.5** [**?**]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq La(n, N) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right)$$

*holds.*

The goal of the present paper is to investigate what happens if $V_2$ is excluded in an "induced way" that is only when the two "upper" sets are not related by inclusion. In other words, $V_2$ is excluded unless it is a $P_3$. There is a standard notation for such a poset: $1 \oplus 2$ denotes the poset on 3 elements, $a, b, c$ where $a < b, a < c$ and no other comparabilities occur. Let $\mathrm{La}^\sharp(n, V_2) = \mathrm{La}^\sharp(n, 1 \oplus 2)$ denote the size of the largest family $\mathcal{F}$ of subsets of $[n]$ containing no three distinct members $F, G_1, G_2 \in \mathcal{F}$ such that $F \subset G_1, F \subset G_2, G_1 \not\subset G_2$, in other words, the family contains no $1 \oplus 2$ as a subposet. We prove the following sharpening of Theorem 1.3.

**Theorem 1.6**

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{1}{n}+\Omega\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}^\sharp(n, V_2) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right).$$

Since $\mathrm{La}(n, V_2) \le \mathrm{La}^\sharp(n, V_2)$, the lower estimate is a consequence of the lower estimate in Theorem 1.3, we have to prove only the upper estimate. Although it seems to be a small modification, the proof (at least the one we found) is much more difficult than the proof of the upper estimate in Theorem 1.3. We will point out later what the differences are, why this case is more difficult. The method of the proof is a further extension of the proof used in [**?**].

The reader might be puzzled by the origin of the factors 1 and 2 in the respective second terms from the lower and upper estimates in Theorems 1.3 and 1.6. The construction proving the lower estimates (see [**?**] and [**?**]) is based on choosing the largest possible family $A_1, \dots, A_m$ of $\lceil \frac{n+1}{2} \rceil$-element sets such that $|A_i \cap A_j| < \lceil \frac{n-1}{2} \rceil$ $(i \ne j)$. The best known construction (see [**?**]) gives only about the half of the trivial upper estimate. This is a well known open problem of coding theory. The problem will be reduced to middle sized sets in Section 2. Section 3 will give a sketch of the main idea of the proof, with details given in Section 4.

Note that because of the symmetry of the Boolean lattice and reflection around the middle layer, this paper also proves the analogous result for the size of Boolean families which contain no three distinct sets $A \cup B \supset C, A \not\subset B$.

## 2 Reduction to middle sized sets

Observe that the main part of a large family must be near the middle since the total number of sets far from the middle is small. More precisely, let $0 < \alpha < \frac{1}{2}$ be a fixed real number. The total number of sets $F$ (for a given $n$) of size satisfying

$$|F| \notin \left[ n\left(\frac{1}{2} - \alpha\right), n\left(\frac{1}{2} + \alpha\right) \right] \tag{2.1}$$

is very small. It is well-known (see e.g. [**?**], page 232) that for a fixed constant $0 < \beta < \frac{1}{2}$

$$\sum_{i=0}^{\beta n} \binom{n}{i} = 2^{n(h(\beta)+o(1))}$$

holds where $h(x)$ is the binary entropy function: $h(x) = -x \log_2 x - (1 - x)\log_2(1-x)$. Therefore using symmetry and the above fact, the total number of sets satisfying (2.1) is at most

$$2 \sum_{i=0}^{\left\lfloor n\left(\frac{1}{2}-\alpha\right)\right\rfloor} \binom{n}{i} \leq 2^{n\left(h\left(\frac{1}{2}-\alpha\right)+o(1)\right)} = \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right) \tag{2.2}$$

where $0 < h\left(\frac{1}{2} - \alpha\right) < 1$ is a constant. We will prove the following theorem in Section 4.

**Theorem 2.1** *If $\mathcal{F}$ satisfies the conditions of Theorem 1.6 and all members $F \in \mathcal{F}$ satisfy*

$$n\left(\frac{1}{2} - \alpha\right) \leq |F| \leq n\left(\frac{1}{2} + \alpha\right) \tag{2.3}$$

*then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n(1-2\alpha)} + O\left(\frac{1}{n^2}\right)\right). \tag{2.4}$$

Let us show that Theorem 2.1 implies Theorem 1.6.

If $\mathcal{F}$ is a family of distinct subsets of $[n]$, $0 < \alpha < \frac{1}{2}$ then let $\mathcal{F}_\alpha$ denote the subfamily consisting of sets satisfying (2.3). On the other hand, let $\mathcal{F}_{\overline{\alpha}}$ denote

$\mathcal{F}-\mathcal{F}_\alpha$. If $\mathcal{F}$ satisfies the conditions of Theorem 1.6, then, by Theorem 2.1, (2.4) gives an upper estimate on $|\mathcal{F}_\alpha|$. On the other hand

$$|\mathcal{F}_{\overline{\alpha}}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right) \tag{2.5}$$

holds by (2.2). Since $|\mathcal{F}| = |\mathcal{F}_\alpha| + |\mathcal{F}_{\overline{\alpha}}|$, (2.4) and (2.5) imply that (2.4) is true not only for $\mathcal{F}_\alpha$, but also for $\mathcal{F}$, itself. Since it holds for every positive $\alpha$, it must hold for $\alpha = 0$, too, proving Theorem 1.6.

# 3 Plan of the proof

A family $\mathcal{G}$ is *connected* if for any pair $(G_0, G_k)$ of its members there is a sequence $G_1, \ldots, G_{k-1}$ ($G_i \in \mathcal{G}$) such that either $G_i \subset G_{i+1}$ or $G_i \supset G_{i+1}$ holds for $0 \leq i < k$. If the family is not connected, the maximal connected subfamilies are called its *connected components*. A *full chain* in $2^{[n]}$ is a family of sets $A_0 \subset A_1 \subset \ldots \subset A_n$ where $|A_i| = i$. Let us mention that the number of full chains in $2^{[n]}$ is $n!$. We say that a (full) chain $\mathcal{C}$ *goes through* a family $\mathcal{F}$ if they intersect, that is, if $\mathcal{F} \cap \mathcal{C} \neq \emptyset$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_K$ be the components of $\mathcal{F}$, satisfying the conditions of Theorem 2.1. The number of full chains going through $\mathcal{F}_i$ is denoted by $c(\mathcal{F}_i)$. Observe that a full chain cannot go through two distinct components since this would force them to be connected. Therefore, the following inequality holds:

$$\sum_{i=1}^{K} c(\mathcal{F}_i) \leq n!. \tag{3.1}$$

What can these components be? It is obvious that they have a tree-like form: each $\mathcal{F}_i$ has a maximal member which can contain several members, each of which can contain other members, and so on. For comparison, let us mention that in the case of the Sperner theorem, each component consists of exactly one set. In the case of Theorem 1.3, a component has a maximal member which contains an unlimited number of other members but they in turn cannot contain additional members; the longest chain that such a structure can contain has length two. In the present case, not only is the number of members unlimited so is the height of the longest included chain; this makes the proof more difficult.

We will give a good lower estimate

$$f(n,\alpha) \le \frac{c(\mathcal{F}_i)}{|\mathcal{F}_i|} \tag{3.2}$$

which will hold for all components $\mathcal{F}_i$. (3.1) and (3.2) imply

$$f(n,\alpha)\sum_{i=1}^{K}|\mathcal{F}_i| \le \sum_{i=1}^{K}\frac{c(\mathcal{F}_i)}{|\mathcal{F}_i|}|\mathcal{F}_i| = \sum_{i=1}^{K} c(\mathcal{F}_i) \le n!.$$

Hence the final result will be

$$|\mathcal{F}| = \sum_{i=1}^{K}|\mathcal{F}_i| \le \frac{n!}{f(n,\alpha)} \tag{3.3}$$

what will prove Theorem 2.1 using the appropriate $f(n,\alpha)$.

The proof of the lower estimate (3.2) is based on the sieve, more precisely on a very primitive version. The number of chains going through $\mathcal{F}_i$ can be lower bounded by the sum of the number of chains going through the members $F \in \mathcal{F}_i$ minus the sum of the chains going through two members $F, G \in \mathcal{F}_i$ where $F$ and $G$ are comparable. This sum can be partitioned into $|\mathcal{F}_i|$ sums, where one sum consists of the number of chains going through a fixed $F \in \mathcal{F}_i$ minus the sum (over $G$) of the number of chains going through $F$ and another member $G$ such that $F \subset G$. If this sum is lower bounded by $f(n,\alpha)$, it implies (3.2) The proof of this latter estimate uses two facts: (i) For a given $F$, there is at most one set $G$ with $F \subset G$ on each level. (ii) The number of these $G$s cannot exceed $2n\alpha$. (i) is obtained from the condition of Theorem 1.6, while (ii) is a consequence of the condition (2.3).

Let us note that the restriction (2.3) was introduced because the estimate obtained from the first two terms of the sieve is too weak when either small or large sets are present in the family.

# 4 Details of the proof

An order opposite to the previous section will be used.

Let $c(F)$ denote the number of full chains going through the set $F$ and $c(F,G)$ the number of full chains going through both $F$ and $G$. (This is obviously 0 if the two sets are incomparable.) If $\mathcal{G}$ is a family of subsets with $F \in \mathcal{G}$, define $d(\mathcal{G}, F)$ by

$$d(\mathcal{G}, F) = c(F) - \sum_{G \in \mathcal{G}:\ F \subset G} c(F, G).$$

**Lemma 4.1** *Suppose $0 < \alpha \le \frac{1}{8}$ and $n \ge 16$. Let $\mathcal{F}_i$ be a connected component of a family satisfying the conditions of Theorem 2.1. Then*

$$\frac{n!}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \left( 1 - \frac{2}{n(1-2\alpha)} - \frac{128}{n^2} \right) \le d(\mathcal{F}_i, F)$$

*holds for every component $F_i$.*

**Proof.**

$$d(\mathcal{F}_i, F) = c(F) - \sum_{G \in \mathcal{F}_i:\ F \subset G} c(F, G)$$

$$= |F|!(n - |F|)! - \sum_{G \in \mathcal{F}_i:\ F \subset G} |F|!(|G| - |F|)!(n - |G|)!$$

$$= \frac{n!}{\binom{n}{|F|}} \left( 1 - \sum_{G \in \mathcal{F}_i:\ F \subset G} \frac{1}{\binom{n-|F|}{n-|G|}} \right). \tag{4.1}$$

It is easy to give a good lower estimate for the first factor. Let us investigate the second one:

$$1 - \sum_{G \in \mathcal{F}_i:\ F \subset G} \frac{1}{\binom{n-|F|}{n-|G|}}. \tag{4.2}$$

There is at most one $G \in \mathcal{F}_i$ of a given size by the condition of Theorem 1.6. Moreover $|F| < |G|$ holds. Therefore

$$1 - \frac{1}{\binom{n-|F|}{1}} - \frac{1}{\binom{n-|F|}{2}} - \frac{1}{\binom{n-|F|}{3}} - \dots \tag{4.3}$$

is a lower estimate on (4.2) On the other hand, since the number of possible sizes (different from the size of $F$) is at most $2\alpha n$, the number of negative terms in (4.2) is at most $2\alpha n$. Using this observation, a further lower estimate is obtained from (4.3):

$$1 - \frac{1}{\binom{n-|F|}{1}} - \frac{1}{\binom{n-|F|}{2}} - \frac{2\alpha n}{\binom{n-|F|}{3}}. \tag{4.4}$$

Using the inequality $n - |F| \geq n\left(\frac{1}{2} - \alpha\right)$ in (4.3) gives the next estimate:

$$1 - \frac{2}{n\,(1-2\alpha)} - \frac{2}{\left(n\left(\frac{1}{2}-\alpha\right)-1\right)^2} - \frac{12\alpha n}{\left(n\left(\frac{1}{2}-\alpha\right)-2\right)^3}. \tag{4.5}$$

Here $n\left(\frac{1}{2} - \alpha\right) - 2 \geq \frac{n}{4}$ holds by the conditions $0 < \alpha \leq \frac{1}{8}, n \geq 16$. Substitute this into (4.5) and the final lower estimate yields that:

$$\begin{aligned} d(\mathcal{F}_i, F) &\geq 1 - \frac{2}{n\,(1-2\alpha)} - \frac{2}{\left(\frac{n}{4}\right)^2} - \frac{12\alpha n}{\left(\frac{n}{4}\right)^3} \\ &= 1 - \frac{2}{n\,(1-2\alpha)} - \frac{32}{n^2} - \frac{12 \cdot 64 \cdot \alpha}{n^2} \\ &\geq 1 - \frac{2}{n\,(1-2\alpha)} - \frac{128}{n^2} \end{aligned}$$

is obtained where $\alpha \leq \frac{1}{8}$ was used. □

**Lemma 4.2** *Under the assumptions of Lemma 4.1*

$$\frac{n!}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \left(1 - \frac{2}{n(1-2\alpha)} - \frac{128}{n^2}\right) \leq \frac{c(\mathcal{F}_i)}{|\mathcal{F}_i|}$$

*holds for every $\mathcal{F}_i \in F$.*

**Proof.** We start with the inequality

$$\sum_{F \in \mathcal{F}_i} c(F) - \sum_{F,G \in \mathcal{F}_i, F \subset G} c(F,G) \leq c(\mathcal{F}_i). \tag{4.6}$$

This is an easy version of the sieve. Let us show it for the sake of completeness. If a chain $\mathcal{C}$ counted in $c(\mathcal{F}_i)$ satisfies $|\mathcal{C} \cap \mathcal{F}_i| = r > 0$ then it is counted $r$ times in the first term of the left hand side and $\binom{r}{2}$ times in the second term. $r - \binom{r}{2} \leq 1 (0 < r)$ shows that the left hand side of (4.6) counts every chain fewer times than the right hand side does.

Group together the terms on the left hand side of (4.6) containing $F$ as the "smaller set".

$$\sum_{F \in \mathcal{F}_i} c(F) - \sum_{F,G \in \mathcal{F}_i, F \subset G} c(F,G) = \sum_{F \in \mathcal{F}_i} \left( c(F) - \sum_{G \in \mathcal{F}_i:\ F \subset G} c(F,G) \right)$$

$$= \sum_{F \in \mathcal{F}_i} d(\mathcal{F}_i, F) \tag{4.7}$$

(4.6) and (4.7) imply

$$\sum_{F \in \mathcal{F}_i} d(\mathcal{F}_i, F) \le c(\mathcal{F}_i).$$

Using Lemma 4.1 for each $F \in \mathcal{F}_i$

$$|\mathcal{F}_i| \frac{n!}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \left(1 - \frac{2}{n(1-2\alpha)} - \frac{128}{n^2}\right) \le c(\mathcal{F}_i)$$

is obtained which is equivalent to the statement of the lemma. □

Lemma 4.2 shows that the function in (3.2) can be chosen to be

$$f(n, \alpha) = \frac{n!}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \left(1 - \frac{2}{n(1-2\alpha)} - \frac{128}{n^2}\right).$$

Based on this, (3.3) gives

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{1}{1 - \frac{2}{n(1-2\alpha)} - \frac{128}{n^2}}. \tag{3.4}$$

Applying the inequality

$$\frac{1}{1-x} \le 1 + x + 2x^2 \quad \left(x \le \frac{1}{2}\right)$$

in (3.4) with

$$x = \frac{2}{n(1-2\alpha)} + \frac{128}{n^2}$$

the upper estimate in Theorem 2.1 is obtained, finishing the proofs.

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