Length of sums in a Minkowski space

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ABSTRACT. Let C be a centrally symmetric compact convex body in $\mathbb{R}^2$ whose center is the origin. It is proved that if none of the elements $a_1, a_2, a_3 \in \mathbb{R}^2$ are inside C then not all the sums $a_i + a_j (i \neq j)$ can be inside C.

1. The main theorem

Let C be a centrally symmetric compact convex body in the two-dimensional plane $\mathbb{R}^2$. For any $x \in \mathbb{R}^2$, let

$$||x||_C = \min_{0 \leq \lambda}\{\lambda : \ x \in \lambda C\}.$$

It is easy to check that

$$0 < ||x||_C \text{ for all } x \in \mathbb{R}^2 \text{ except that } ||0||_C = 0, \qquad (i)$$

$$||\mu x||_C = |\mu| ||x||_C \text{ for all } \mathbb{R}^2 \text{ and } \mu \in \mathbb{R}, \qquad (ii)$$

$$||x + y||_C \leq ||x||_C + ||y||_C \text{ for all } x, y \in \mathbb{R}^2. \qquad (iii)$$

In other words, $|| \cdot ||_C$ is a *norm.* Defining the distance of two points $x, y \in \mathbb{R}^2$ as $||x - y||_C$, we get the so-called *Minkowski metric* on $\mathbb{R}^2$. With respect to this metric, C is the unit ball around the origin, that is, $C = \{x \in \mathbb{R}^2 : \ ||x||_C \leq 1\}$. The space $\mathbb{R}^2$ equipped with the Minkowski metric is called the *Minkowski space* with *gauge body* C (see [8]). We provide two proofs for Theorem 1.1 to illustrate two techniques fro obtaining this basic theorem.

THEOREM 1.1. *Let a_1, a_2, a_3 be elements of a Minkowski space with norm at least 1. Then there is a pair i, j of distinct indices such that $1 \leq ||a_i + a_j||$.*

FIRST PROOF. It can be assumed that the origin is an inner point of C. Otherwise the statement is trivial.

Suppose first that $||a_1|| = ||a_2|| = ||a_3|| = 1$ and denote the set of points satisfying $||u|| = 1$ by U. Its shift $U + a_1 = \{u + a_1 : \ u \in U\}$ is denoted by U'.

2000 *Mathematics Subject Classification.* Primary 46B07; Secondary 60D05.

Gyula O.H. Katona has been supported by the Hungarian National Foundation for Scientific Research grant number T037846, and UVO-ROSTE, Grant 875.630.9.

Wojbor A. Woyczynski has been supported by grants from the U.S. National Science Foundation.

It is easy to see that $U \cap U'$ consist of either two points or two straight line segments. Actually, the former is a special case of the latter, however we show the proof for the first (special case) as a warming up. Let the two points in common be b' and c'. It is obvious that $b = b' - a_1$ and $c = c' - a_1$ are on U. The arc in U' containing the origin and connecting b' and c' is inside U, the other arc connecting them is outside of U. The shift (by a_1) of the arc of U connecting b and c and going through a_1 is an arc of U' outside U.

We show that $c' = -b$. It is obvious that $b'' = b' + a_1$ is in U'. Its mirror image with respect to a_1 (the center of U') is denoted by d. Since $b' - b = a_1 - 0, b' - 0 = b'' - a_1 = a_1 - d$, the triangles $b, b', 0$ and $0, a_1, d$ are congruent. Hence $d = -b$, that is d is the mirror image of b in U. Therefore d is on both U and U', and thus must be either b' or c'. It cannot be b' (otherwise $d = -b' + a_1 = b'$ imples $2b' = a_1$, since both are on U, $|a_1| = 0$ is a contradiction), therefore $d = c' = -b$. (By symmetry we have $c = -b'$.)

Since $b = -c'$ and $c' = c + a_1$ we have $b + c = -a_1$. Hence $a_1, b, -a_1, c$ are in U in this order and the angle between b and c containing $-a_1$ is less than 180^o. We distinguish cases according to the distribution of a_2 and a_3 among the 3 arcs determined by $b', c', -a_1$ in U. Call the arcs between b' and c', between $-a_1$ and b and between c and $-a_1$ by α, β and γ, respectively.

Case 1. At least one of a_2 and a_3 is on α.

By symmetry, we may suppose that a_2 is on α. $a_1 + a_2$ is a shift of a_2 therefore it is on the arc of U' outside in U. Its norm is at least 1.

Case 2. a_2 is on β, a_3 is on γ or a_3 is on β, a_2 is on γ.

By symmetry we can suppose that the first subcase holds. Define w_2 and w_3 by $a_2 = b + w_2$ and $a_3 = c + w_3$, respectively. Hence $a_2 + a_3 = b + c + w_2 + w_3 = -a_1 + w_2 + w_3$. By convexity, a_2 is in the halfplane determined by the points $-a_1$ and b, and not containing 0. Therefore (shift by $-b$) $w_2 = a_2 - b$ is on the same side of the line determined by $-a_1 - b = c$ and 0 as b and a_1. On the other hand, since a_2 is on the same side of b as $-a_1$, this is also true for $w_2 = a_2 - b$. We can conclude that w_2 lies in the angular sector determined by b and c and containing $-a_1$. By symmetry, the same can be said about w_3 (the roles of b and c are interchanged in the verification). Hence $w_2 + w_3$ is in the same angular sector. Shift this angular sector with $-a_1$. It is determined by $b - a_1, 0$ and $c - a_1, 0$ and it contains $-2a_1$. By the convexity, again, this angular sector cannot contain an interior point of C therefore $a_2 + a_3 = w_2 + w_3 - a_1$ cannot be in its interior, proving the statement in this case.

Case 3. Either both a_2 and a_3 are on β or both are on γ.

By symmetry we can suppose that the first subcase holds. Similarly to the previous case, one can see that (by convexity) $w_2 = a_2 - b$, $w_3 = a_3 - (-a_1)$ and $-a$ are on the same side of the line determined by b' and c. The same must hold for the sum $w_2 + w_3 = a_2 + a_3 - b + a_1$. Shift this line by $b - a_1$, that is, by $-2a_1$. The line ℓ thus obtained contains the points $b - a_1$ and $-2a_1$ which are points of $U^{-\prime}$, the shift of U by $-a_1$. Since these two points are on an arc which is outside of U, by convexity, ℓ cannot go through the interior of C. $w_2 + w_3 + (-b + a_1)$ will be on the side of ℓ opposite to C, therefore it cannot be in the interior of U.

If the intersection of U and U' consists of two intervals then let b' and c' denote the middle points of these intervals. The preceding argument can be repeated.

To complete the proof we have to show that the stament is true when the vectors are allowed to be outside U. Define a_i^* as the vector λa_i satisfying $||\lambda a_i|| = 1$.

Let $d_i = a_i - a_i^*$. By symmetry we can suppose that $1 \le ||a_1^* + a_2^*||$. The sum $d_1 + d_2$ is in the (smaller) angular sector σ determined by a_1^* and a_2^*. Consequently, $a_1 + a_2 = d_1 + d_2 + a_1^* + a_2^*$ is in the angular sector σ' obtained by shifting σ by $a_1^* + a_2^*$. It remains to verify that σ has no inner point of C. This is an easy consequence of the convexity of C □

SECOND PROOF. Suppose that $||a_1|| = ||a_2|| = ||a_3|| = 1$. The general case can be reduced to this one as in the previous proof. It is also assumed that C has inner points and U is defined as in the first proof.

Let L be a linear transformation in $\mathbb{R}^2$. It is easy to see that $a \in C$ if and only if $La \in LC$ for any centrally symmetric compact convex body. Hence, if the theorem holds for C then it also holds for LC.

Let $(0, u) \in U, (0 < u)$. There is a line ℓ^1 containing this point and being "above" C. The line containing $(0, -u)$ and parallel to ℓ^1 is denoted by ℓ_1. Similarly, let $(v, 0) \in U(0 < v)$. There is a line ℓ^2 which is "on the right" of C. The line ℓ_2 is parallel to ℓ^2 and contains $(-v, 0)$. It is obvious that there is a linear transformation L which maps the parallelogram defined by $\ell^1, \ell_1, \ell^2, \ell_2$ to the square Q_1 defined by the lines $y = 1, y = -1, x = 1, x = -1$. Therefore we may suppose for the rest of the proof that U is within Q_1 and contains the points $(1, 0), (-1, 0), (0, 1), (0, -1)$. Hence U is between the square Q_1 and the square Q_2 determined by the points $(1, 0), (-1, 0), (0, 1), (0, -1)$.

Two cases will be distinguished.

Case 1. The angle between two of the a vectors, say between a_1 and a_2 is at most 90^o.

Consider the vectors $a_i^* = \mu_i a_i$ lying on Q_2, where $\mu_i \le 1 (i = 1, 2)$. We prove that $a_1^* + a_2^*$ cannot be an inner point of Q_1. By symmetry we can suppose that the coordinates of $a_1^* = (x_1, y_1)$ satisfy $0 \le x_1, 0 \le y_1, y_1 \le x_1$. Of course we know $x_1 + y_1 = 1$. By the condition on the angle between a_1 and a_2 we have the following cases: (i) y_2 positive, x_2 negative, $-x_2 \le y_1, y_2 = x_2 + 1$, (ii) both x_2 and y_2 are non-negative, $x_2 + y_2 = 1$, (iii) y_2 is negative, x_2 is non-negative, $y_1 \le x_2, y_2 = x_2 - 1$.

In case (i) $-x_2 = 1 - y_2$ and $-x_2 \le y_1$ imply $1 \le y_1 + y_2$. In case (ii) $x_1 + y_1 + x_2 + y_2 = 2$ implies that either $x_1 + x_2$ or $y_1 + y_2$ is at least 1. In case (iii) $x_1 + y_1 = 1$ and $y_1 \le x_2$ result in $1 \le x_1 + x_2$. One of the coordinates of the sum $a_1^* + a_2^*$ is at least 1, that is the sum cannot be an inner point of Q_1. If a_i^* is replaced by $a_i (i = 1, 2)$ then the sum which turned out to be at least one was increased (non-decreased), therefore $a_1 + a_2$ cannot be an inner point of Q_1 either, consequently $1 \le ||a_1 + a_2||$ holds.

Case 2. All the angles among a_1, a_2 and a_3 exceed 90^o.

Let a be an arbitrary element of $\mathbb{R}^2$. Let U' the shift of U by a, namely, $U' = U + a = \{u + a : u \in U\}$. Suppose that $U \cap U'$ contains at least two elements. Let the middle points of the two intervals (they can be points) of the intersection $U \cap U'$ be denoted by v and w, respectively. Let α denote the arc between v and w on U which contains the inner points of U'. Similarly, let β denote the arc between v and w on U' which contains the inner points of U. It is easy to see that α and β are mirror images with respect to the point $a/2$, therefore they are congruent. The shift of α by a is denoted by γ. This is an arc in U', disjoint and congruent to α. Hence the angles of β and γ with respect to the center a of U' is at most 180^o. This is trivially true if the intersection $U \cap U'$ does not have two different elements.

We can conclude that the angle vaw containing the arc of U' with inner points in U is at most 180^o.

Define now $a = a_1+a_2+a_3$. Then $a_2+a_3 = a-a_1$, etc., therefore the statement of the theorem can be reformulated: not all three vectors $a-a_1, a-a_2, a-a_3$ can be inner points of U. These vectors all lie in U'. Since their pairwise angles are more than 90^o, they cannot be all within an angle $\leq 180^o$, they cannot all lie on the arc in U' inside U. One of them is either on $U \cap U'$ or outside U. □

The theorem is sharp in the following sense.

PROPOSITION 1.2. *For any given Minkowski space and* a_1 *with* $||a_1|| = 1$ *there exist* a_2, a_3 *satisfying* $||a_2|| = ||a_3|| = ||a_1 + a_2|| = ||a_1 + a_3|| = ||a_2 + a_3|| = 1$.

PROOF. The vectors $a_2 = b, a_3 = c$ from the first proof satisfy the conditions. □

2. Further results, remarks and problems

Theorem 1.1 is rather trivial for the Euclidean space. However this special case was one of the ingredients of a proof of an inequality concerning random vectors (see e.g. [4] and [7]). Our recent generalization for Minkowski spaces (Theorem 1.1) makes it also possible to prove the inequality for Minkowski spaces.

THEOREM 2.1. *Let* ξ *and* η *be independent and identically distributed random elements of a Minkowski space. Then*

$$P(||\xi + \eta|| \geq x) \geq \frac{1}{2}P^2(||\xi|| \geq x). \tag{1}$$

This theorem can be proved by the methods used in the papers [**4**] – [**6**]. Both [**4**] and [**7**] start with the proof of the theorem above for the special case of the Euclidean metric.

The above mentioned papers also prove generalizations and extensions of (1). The basic ingredients of the proofs are (i) a geometric statement analogous to Theorem 1.1 and extremal theorems for graphs. The earlier papers mostly consider Euclidean spaces only. We briefly state what are some open questions for Minkowski spaces.

The statement of Theorem 1.1 is well-known in any dimensional Euclidean space or even in a Hilbert space. Unfortunetely, it is not true for a Minkowski space in $\mathbb{R}^3$ as the following examle shows. Let C be the cube determined by the 8 points having three ± 1s as coordinates. Each of the points $(1, -\frac{1}{3}, -\frac{1}{3}), (-\frac{1}{3}, 1, -\frac{1}{3}), (-\frac{1}{3}, -\frac{1}{3}, 1)$ have norm 1. However the sum of any two has norm $\frac{2}{3}$. It was shown in [**7**] (in a more general context: Lemma 4) that $\frac{2}{3}$ can always be attained.

The references [**3**] – [**6**] contain many analogous results for the Euclidean spaces. One of them is that if $1 \leq |a_1|, |a_2|, \ldots , |a_k|$ holds then at least one of the $k-1$-term subsums has an absolute value of at least 1. Eli Goodman [**2**] asked if this is true for a Minkowski space of dimension at most $k - 1$.

PROBLEM 2.2. [**2**] *Let* M *be a* $k-1$*-dimensional Minkowski space,* $a_1, \ldots , a_k$ *satisfy* $1 \leq ||a_i||(1 \leq i \leq k)$. *Is it true that*

$$1 \leq || \sum_{i=1, \neq j}^{k} a_i ||$$

must hold for some $1 \leq j \leq k$?

For dimension k the generalization of the cube example above serves as a counter-example.

The following definition is needed to formulate the (simplest) problems whose solutions are needed to extend probabilistic results for Minkowski spaces. Let M be a Minkowski space.

$$\delta(k, M) = \min \max_{1 \le i < j \le k} ||a_i + a_j||,$$

where the minimum is taken for all choices of vectors $a_1, \dots, a_k \in M, 1 \le ||a_i||(1 \le k)$.

PROPOSITION 2.3. *If M is a two-dimensional Minkowski space then $\delta(k, M)$ is attained for vectors satisfying* $1 = ||a_1|| = \dots = ||a_k|| = ||a_1 + a_2|| = ||a_2 + a_3|| = \dots = ||a_{k-1} + a_k|| = ||a_k + a_1||$.

PROBLEM 2.4. *Suppose that M is two-dimensional and k is a multiple of 4. Prove that $\delta(k, M)$ is attained for a system of vectors invariant under a rotation of* 90^o.

PROBLEM 2.5. *Determine $\delta(k, M)$ for a two-dimensional Minkowski space with the norm $|| \ ||_p$. (k=4 is easy.)*

PROBLEM 2.6. *Find connections between $\delta(k, M)$ and the modul of convexity defined as*

$$I(\varepsilon, M) = \sup_{\substack{||x|| = ||y|| = 1 \\ ||x - y|| = \varepsilon}} ||x + y||.$$

The interested reader can find many results in the cited literature which are proved for Euclidean space and are waiting for extension to Minkowski spaces.

Finally we mention a related result. We rephrase it in our terminology, which is rather different from the original one.

THEOREM 2.7. [**1**]. *For any n there is an n-dimensional Minkowski space with a strictly convex norm which contains $m = c^n (1 < c)$ vectors $a_1, \dots, a_m$ of norm 1, such that each of the sums $a_i \pm a_j (1 \le i < j \le m)$ have a norm 1.*

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