A Simple Proof of a Theorem of Miller

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Commissioned by the Managing Editors

Received August 18, 1995

Dedicated to the memory of Ed C. Milnor

A new direct proof is given for the following theorem of Miller. An interesting
intersection family of an arbitrary set has at least \(\binom{2m+1}{m}\) members.

Let \(F\) be a finite set of \(m\) elements and \(\mathcal{F}\) a family of subsets of \(F\) such that any subset of \(F\) of size \(\ell\) is contained in \(\ell - 1\) members of \(\mathcal{F}\), where \(\ell < m\). On the other

hand, the empty set is contained in \(m\) members of \(\mathcal{F}\). Miller [1] proved the following theorem.

Theorem 1. An interesting intersection family of an arbitrary set has at least

\[
\binom{m+1}{\frac{m+1}{2}}
\]

members.

Miller's proof was not very complicated, but it used a non-trivial theorem of the present author [2]. Here we show that this was not necessary, as our elementary proof uses the cyclic method [3].

[1] The work was supported by the Hungarian National Foundation for Scientific Research (grant numbers 7-299 and 7-210). The European Community (Commission of Science, Research and Development) under Contract (C1Ê1003) 72788, and D254728.
First we give a lemma which is a weighted version of the problem on the "cycle". Let $f$ be a cyclic permutation of $X$.

Lemma 1. Let $H_1, \ldots, H_n$ be an intersecting Sperner family of intervals in $X$. Then

\[
\sum_{\lambda} \left( \begin{array}{c} n \\ \lambda \end{array} \right) s^n \left( \begin{array}{c} n+1 \\ \lambda + 1 \end{array} \right) \quad (1)
\]

Proof. Consider the intervals starting from a given element to the right along $X$. Obviously at most one of them can be a $H_r$, since the family is Sperner. Therefore $r$ is odd. This enables us to scale $x = 1$, so we may suppose that $r = 1$. Two cases will be distinguished.

1. $r = 1$. Suppose that $H_r$ ends at the right element along $X$. The Sperner property implies $|H_r| = \frac{1}{n} \left( \begin{array}{c} n \\ \lambda \end{array} \right)$, consequently we have $|H_r| = \frac{1}{n} \left( \begin{array}{c} n \\ \lambda \end{array} \right) s^n \left( \begin{array}{c} n+1 \\ \lambda + 1 \end{array} \right)$, and the right-hand side of (1) is the same as the left-hand side of (1), so the left-hand side of (1) is at most $\left( \frac{1}{n} \left( \begin{array}{c} n \\ \lambda \end{array} \right) s^n \left( \begin{array}{c} n+1 \\ \lambda + 1 \end{array} \right) \right)^{-1}$, proving the lemma for this case.

2. $r = 1$. At most one of the complementing all different intervals can occur among the $H_r$, therefore at most $n \ell$ of the intervals on the left side of (1) can be $\left( \begin{array}{c} n+1 \\ \lambda + 1 \end{array} \right)$, so the left-hand side of (1) is at most

\[
\frac{1}{n} \left( \begin{array}{c} n \\ \lambda \end{array} \right) s^n \left( \begin{array}{c} n+1 \\ \lambda + 1 \end{array} \right)
\]

It is easy to see that the latter expression is equal to the right-hand side of (1).

Proof of the Theorem. Let $A_1, \ldots, A_n$ be an intersecting Sperner family on $X$. Consider the

\[
\sum_{\lambda} \left( \begin{array}{c} n \\ \lambda \end{array} \right) s^n \left( \begin{array}{c} n+1 \\ \lambda + 1 \end{array} \right) \quad (2)
\]

For those pairs for which $A_1$ is an interval along $X$. First fix $A_i$. It is easy to see that $A_i$ is an interval in exactly $s^n \left( \begin{array}{c} n+1 \\ \lambda + 1 \end{array} \right)$ cyclic permutations of $A_i$. Hence (1) is equal to

\[
\sum_{\lambda} \left( \begin{array}{c} n \\ \lambda \end{array} \right) s^n \left( \begin{array}{c} n+1 \\ \lambda + 1 \end{array} \right) = \sum_{\lambda} \left( \begin{array}{c} n+1 \\ \lambda + 1 \end{array} \right) \quad (3)
\]
Theorem 1. The lemma given an upper bound on the sum with a given $w$. That is, (2) is upper bounded by
\[
(\sigma - \varphi n) \begin{pmatrix} \frac{e^x}{y} \\ \frac{e^y}{z} \end{pmatrix}
\]
Comparing this with (3), the following inequality is obtained:
\[
\log (w + 1) = \left( \frac{e^x}{y} \right) \left( \frac{e^y}{z} \right)
\]
completing the proof.

REFERENCES