PROBABILISTIC INEQUALITIES FROM EXTREMAL GRAPHS (A SURVEY)

G. O. H. Katona

Department of Mathematics, University of Washington, Seattle

Abstract. The purpose of this paper is to survey the probabilistic inequalities proved by the method of extremal combinatorial structures.

1. Introduction

To illustrate the main ideas of the field surveyed in this paper, let us sketch the proof of the following theorem:

Theorem 1. If $X$ and $Y$ are independent, identically distributed random variables taking values from a finite set $V$, then

$$P(\|X-Y\| < \epsilon) > 0$$

where $\|X-Y\|$ is the norm of $X-Y$.

Proof. 1. We start with stating the following special case of the Tsirelson theorem [17].

If a graph with $n$ vertices contains an empty triangle ($\mathcal{E}$-set) and $n$ is even, then the graph has at least $\binom{n}{2}/2$ edges.

2. We need the following simple statement from geometry:

If $x, y, z$ are $n$-tuples of reals, then $x > y$ holds for a part of $n/2$ values.

The three norms span $n$-dimensional linear spaces. It is easy to see that the angle between $x$ and $y$ is less than $\epsilon$ if $x$ and $y$ are close in some norm. Now it is easy to see that $\epsilon$ depends only on $\mathcal{E}$.
3. The following trivial inequality will be used:

\[ P(\xi_1 > \alpha_1 \cap \xi_2 > \alpha_2 \cap \ldots \cap \xi_k > \alpha_k) \leq P(\xi_1 > \alpha_1)P(\xi_2 > \alpha_2) \ldots P(\xi_k > \alpha_k) \]  \hspace{1cm} (1)

Suppose for a while that (1) can only hold if, with equal probability

\[ P(\xi_1 > \alpha_1) = \ldots = P(\xi_k > \alpha_k) ] \]  \hspace{1cm} (2)

Let \( \alpha \) be ordered in the following way: \( \alpha_1 > \ldots > \alpha_k \). Consider the following graph \( G \). Let \( u_1, \ldots, u_k \) be the vertices of \( G \). The vertices of \( G \) are connected with an edge if the vertices of \( G \) are connected with an edge.

Then:

\[ P(\xi_1 > \alpha_1 \cap \ldots \cap \xi_k > \alpha_k) = \sum_{\text{order of paths} u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_k} \text{ covering } \alpha} \prod_{i=1}^{k} P(\xi_i > \alpha_i) \] \hspace{1cm} (3)

\[ = e^{-\frac{1}{2}(2 \lambda - k)} \text{ (number of edges of } G = k) \] \hspace{1cm} (4)

holds true for \( \lambda = \sum_{i=1}^{k} \alpha_i \). The graph \( G \) has an empty triangle by Section 2 of the proof. Applying the Turán theorem in \( G \), we obtain a lower estimate for \( \lambda \):

\[ \lambda \geq \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{nk^2}{2} = \frac{1}{2} \frac{(n - 1)^2}{k} - \frac{1}{2} n(k - 1) \]

Theorem 1 is proved for this special case.

d. To prove the general case one two approaches offer themselves:

\[ P \text{ (product distribution) are given to us with the discrete distribution used in } \lambda \text{. This method was applied to } \lambda \text{ for the treatment of the problem in Section 2 where (1) works and the graph is trivial in the usual sense.}

\[ P \text{ (product distribution) are given to us with the discrete distribution used in } \lambda \text{. This method was applied to } \lambda \text{ for the treatment of the problem in Section 2 where (1) works and the graph is trivial in the usual sense.]

\[ P \text{ (product distribution) are given to us with the discrete distribution used in } \lambda \text{. This method was applied to } \lambda \text{ for the treatment of the problem in Section 2 where (1) works and the graph is trivial in the usual sense.} \]
If $F$ has finite mass, a measure, in the manner of such distance be equal to $1$.

Then the finite theory says that the measure of the diagonal $(0,1)$ of measure $0$ of the measure $d(x,y) = |x-y|^{1/2}$, that is, the half of the distance of $F$. We may present the same result for the general case. We will call the generalization of a statement true distance. This is the distance of the other side.

Let us note now that there is a transition (under very general conditions) for continuous measure. Assuming the validity of this statement, Theorem 3 follows easily by (1).
where
\[ E = \{ (x, y) \mid x \in X, y \in Y \}. \]

\( M \) or \( M' \) is admissible if for any \( x, y \in X \), \( y' \in Y \) there is a \( x' \in X \) satisfying \( E(x', y') \in M(x, y) \).

Lemma 4. Let \( (E, \delta) \) be a graph on the abstract measure space \( (X, \mu, \sigma) \). Suppose that
\[ \delta(x, y) = |x|^2 \]
holds for any \( x \) satisfying \( |x| = \sigma_x \). Then
\[ \mu \times \sigma = \delta. \]

Proof. Introduce the measure \( \delta' = \delta \times \mu \) generalizing the case \( n=1 \). On the other hand, define
\[ E_x = \{ (x, y) \in E \mid \delta(x, y) = \delta(x, y') \} \]
for \( x \in X \), \( y \in Y \), \( y' \in Y \).

This function is obviously measurable since \( \delta \) is measurable. Take the integral
\[ \int_{\{x \in X \mid \delta(x, y) = \delta(x, y') \}} \delta(x, y) \, d\mu \]
where \( y = y' \). Then we use
\[ \int_{\{x \in X \mid \delta(x, y) = \delta(x, y') \}} \delta(x, y) \, d\mu = \int \delta(x, y') \, d\mu \]
which implies
\[ \delta(x, y) = \delta(x, y'). \]
Summing up (5) and (6) for all pairs \( i, j, k, l \), we obtain:

\[
\sum_{i, j, k, l} (x_i y_j z_k w_l) = \sum_{i, j, k, l} (x_i y_j z_k w_l) \left( x_i y_j z_k w_l \right)^{-1}.
\]

Observe that \( \sum_{i, j, k, l} (x_i y_j z_k w_l) \) is simply the number of edges in the subgraph induced by \( I_{ij}, K_{kl} \), where all terms are distinct. This latter condition holds with the exception of a set of measure \( \epsilon \).

This is formalized as follows and can be rigorously proved (see e.g., [7]): using the assumption \( \mu_{ij}(x_i y_j z_k w_l) \),

\[
\sum_{i, j, k, l} (x_i y_j z_k w_l) \left( x_i y_j z_k w_l \right)^{-1} = \mu_{ij}(x_i y_j z_k w_l).
\]

(5) and (6) imply

\[
\epsilon^{x-1} + \epsilon \geq 1 + C(x, y, z, w)
\]

If \( x = m \), this leads to \( \mu_{ij}(x_i y_j z_k w_l) \).

Let \( \mathcal{H} \) be an arbitrary class of graphs \( G \sim \mathcal{H} \) determined on the measure space \( M = (\mathcal{E}, \mu, \mathcal{A}) \). \( \mathcal{H} \) is called braided if \( \mathcal{H} \) implies \( \mathcal{G} \) for any measurable \( G \in \mathcal{A} \).

\[
\mathcal{H}(W, \mu) = \mu_{ij}(x_i y_j z_k w_l).
\]

can be considered as a continuous analogue of the “minimum number” of edges in \( \mathcal{H} \). Analogously, let us define

\[
\Delta(W, \mu) = \mu_{ij}.
\]

(7)

where the minimum runs over all classes of \( \mathcal{H} \) having exactly a version. It is proved in [7] that (7) has a limit if \( \epsilon = m \). The inequality

\[
\mathcal{H}(W, \mu) \leq \Delta(W, \mu)
\]

is proved in [7].
Let $G=(V,E)$ be a graph, and define $G'=(V',E')$ where $E'$ consists of the edges obtained by arbitrarily choosing an edge of $E$ or an edge of $E'/G$. $G'$ is called a chordal of $G$ if $G'$ is a chordal for any vertex of $G$.

**Theorem**. Suppose that $G$ is a hereditary class of graphs on the measure space $(\Omega,\mathcal{F},\mathbb{P})$.

If $G$ is chordal, then $G$ is a chordal.

For applications, we need the direction of the inequality. One way goes, however, that equality holds in (8) under some reasonable conditions. Indeed, if $G$ is chordal, then (8) holds with equality [10].

However, there is another class of graphs for which the equality in (9) is proved. If $G$ is called a strongly chordal, then $G'$ is also chordal, and a new vertex is a minimizer of $G$ (and is called the chordal) with all the possible edges containing $a$. The new graph is also chordal. If $G$ is proved in [11] that any strongly chordal class of graphs is a special class of strongly chordal graphs.

The authors also prove that any strongly chordal graph is an expansion of a strongly chordal graph.

The condition for the existence of a strongly chordal graph is an expansion of a strongly chordal graph.

The condition for the existence of a strongly chordal graph is an expansion of a strongly chordal graph.
The result of the above theorem is new and significant in the context of partial differential equations.

Theorem A. Let $u(x, y)$ be a solution to the following system:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= f(x, y, u) \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0
\end{align*}
\]

Then, the solution $u(x, y)$ satisfies the following properties:

1. $u(x, y)$ is continuous and differentiable everywhere.
2. $u(x, y)$ is bounded as $x, y$ approach infinity.
3. $u(x, y)$ converges to a constant as $x, y$ approach $\pm \infty$.

Proof: (略)

The proof of Theorem A involves the use of Green's function and the method of characteristics. By applying these tools, we can establish the desired properties of the solution $u(x, y)$.
Each case of the theorem can be proved following the proof of Theorem 1, where we choose places in Fig. 2. We now give a demonstration of the proof of the case $\begin{cases} \frac{1}{2}\mu_1 \geq \frac{1}{2}\mu_2 \leq \frac{1}{2}\mu_3, \\
\end{cases}$

We can now sketch the proof of the case. For the moment, let us consider the following inequality:

$$E^T \sigma \leq E \sigma.$$ (1)

The graph $G=(V,E)$ is defined by $E^T \sigma \leq E \sigma$. The following simple geometric statement is true:

If $e_i, e_j, e_k$ are vertices in a hyperplane and $[e_i^T]_{ij}, [e_j^T]_{jk}, [e_k^T]_{ki}$ then their graph $G$ is an empty triangle with vertices at $e_i, e_j, e_k$.

If these lines intersect then according to formula 1, the number of edges is at least

$$\sum_{i<j<k} \frac{1}{2} \left( \binom{n-2}{2} - \binom{n-k-1}{2} \right) \left( \binom{n-2}{2} - \binom{n-k-1}{2} \right).$$

If $n=4,$ $k=2$, and $\begin{cases} \frac{1}{2}\mu_1 \geq \frac{1}{2}\mu_2 \leq \frac{1}{2}\mu_3, \\
\end{cases}$ otherwise. The "continues version" of this lemma proves the statement for $\begin{cases} \frac{1}{2}\mu_1 \geq \frac{1}{2}\mu_2 \leq \frac{1}{2}\mu_3, \\
\end{cases}$

The first corollary is that we cannot disregard the small $\begin{cases} \frac{1}{2}\mu_1 \geq \frac{1}{2}\mu_2 \leq \frac{1}{2}\mu_3, \\
\end{cases}$ vertices, like in the case of Theorem 1. This means the triangle with two vertices, the case of Theorem 2 fails in the same manner. This proof is not straightforward and can be found in [1.11].

The proof of the first and any other case is valid for any (finite-dimensional) Hilbert space. But the elements are not the best, in general. The construction does not work.
where the infimum is taken over all vectors $a_i, \ldots, a_n$ satisfying $\sum_i a_i \geq 1$.

Analogously to Theorem 3 in [16], the best lower estimate of $F(2, n)$ is given by

$$F(2, n) \geq \inf_{a_i, \ldots, a_n} \sum_i a_i$$

where the infimum is taken over all vectors $a_i, \ldots, a_n$ satisfying $\sum_i a_i \geq 1$.

Thus, the "best" estimates analogous to those of [16] are given in the hypercube $\sum_i a_i \geq 1$.

Problem Our aim is to find the best lower estimate for the growth function $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

There is another analog of Theorem 3 in [16] for the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.

Theorem 3 in [16] is an analog of the best lower estimate of $F(2, n)$ in the hypercube $\sum_i a_i \geq 1$.
For any $k$ independent, identically distributed random variables on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, the first problem is that even the order of magnitude of the estimate is not correct. It is proved in [5] that

$$P\left[\left|\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n - n \mu \right| > \epsilon \right] \sim \frac{1}{\sqrt{n}}$$

holds if $\mathbf{X}_i \sim N(0, \sigma^2 I)$, $\mathbf{X}_i$ is much stronger for small values of $\mathbf{X}_i \sim N(0, \sigma^2 I)$ than $\mathbf{X}_i$. However, the constant $c$ is not the best possible.

The reason why the estimation bias is found from the case $n=1$ is that the small sample bias plays a role. This problem is circumvented if we consider $\mathbf{X}_i + \mathbf{X}_j + \cdots + \mathbf{X}_l$ instead,

$$P\left[\left|\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n - n \mu \right| > \epsilon \right] \sim \frac{1}{\sqrt{n}}$$

is proved for the independent, identically distributed two-dimensional random variables in [3]. This is proved in [5] only for uncorrelated variables and with much less. So then Deheuvels and van Zanten [1] proved that $P\mid \mathbf{X}_i \sim N(0, \sigma^2 I)$.

For $n=2$, the expressions in Lemma 2.2 and 2.3 for independent, non-correlated, Haar null also gave counterexamples for these lemmas, if the dimension is higher. So the method of [5] does not work for higher dimensions. However, we still can prove

$$P\left[\left|\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n - n \mu \right| > \epsilon \right] \sim \frac{1}{\sqrt{n}}$$

for any $W_k$-space.

Further in [6], [7] found similar results for the case when the lower estimates hold.

Finally, to mention another result of Kallianpur [10], he gives lower estimates of

$$P_1 \left[ \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n - n \mu \right] \sim P_2$$

and

$$P_1 \left[ \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n - n \mu \right] \sim P_2$$

in terms of $\mathbf{X}_i \sim N(0, \sigma^2 I)$, where $g$ also rates the size and root.
8. Open problem

Although the papers in this field contain many open questions (actually, they contain more open questions than results) we would like to emphasize some of them:

1. Do sets always have equality in (M)? It is known that if \( M = \mathbb{R} \) is an algebra and if \( g = 2 \), then \( \mathcal{M}(g, \mathbb{R}) \) is an algebra. So, it is unknown for some cases, even if \( g \neq 2 \), and very little is known for \( g = 2 \).

2. It is easy to see that \( \mathcal{L}(h, \mathbb{R}) \) is replaced by \( \mathcal{L}(h, \mathbb{R}) \) in case \( \mathcal{L}(h, \mathbb{R}) \) can be completed. Suppose now \( \mathcal{L}(h, \mathbb{R}) \) in case \( \mathcal{L}(h, \mathbb{R}) \) only. We suppose that \( \mathcal{L}(h, \mathbb{R}) \) is measureable (only) on the standard dimension of \( \mathbb{R} \) at least 1 for \( \mathcal{L}(h, \mathbb{R}) \) and its \( \mathcal{L}(h, \mathbb{R}) \)-dimensional outer measure \( \mathcal{L}(h, \mathbb{R}) \) is at most \( \mathcal{L}(h, \mathbb{R}) \).

3. A space is a set \( \mathcal{L}(h, \mathbb{R}) \) of form \( \mathcal{L}(h, \mathbb{R}) \), where \( h, \mathbb{R} \) and \( \mathcal{L}(h, \mathbb{R}) \), \( \mathcal{L}(h, \mathbb{R}) \) is the set \( \mathcal{L}(h, \mathbb{R}) \) of \( \mathcal{L}(h, \mathbb{R}) \), and \( \mathcal{L}(h, \mathbb{R}) \) is defined by

\[
\triangleleft \L_1 \L_2 \triangleleft \sum_{j=1}^{n} \beta \kappa(h,j)
\]

where \( \beta \in [0,1] \) and \( \beta \kappa(h,j) \) is a non-decreasing function of \( \beta \), therefore the basis

\[
\triangleleft \L_1 \L_2 \triangleleft \sum_{j=1}^{n} \beta \kappa(h,j)
\]

exists. This is called the \( \mathcal{L}(h, \mathbb{R}) \)-dimensional outer \( \mathcal{L}(h, \mathbb{R}) \)-measure of \( \mathcal{L}(h, \mathbb{R}) \). It is easy to see that \( \mathcal{L}(h, \mathbb{R}) \) is an algebra, such that \( \mathcal{L}(h, \mathbb{R}) \) is an algebra and \( \mathcal{L}(h, \mathbb{R}) \) is a basis if \( \mathcal{L}(h, \mathbb{R}) \).

4. One can determine the "best" function of the distribution function of \( \mathcal{L}(h, \mathbb{R}) \) in a given plane type, another problem is to find the best operator \( \mathcal{L}(h, \mathbb{R}) \), where \( \mathcal{L}(h, \mathbb{R}) \) is intended to be a function of \( \beta \).

5. Another result is this dimension can be found in [10]

\[
\mathcal{L}(h, \mathbb{R}) \leq \mathcal{L}(h, \mathbb{R}) \leq \mathcal{L}(h, \mathbb{R})
\]

where \( \beta \in [0,1] \) and \( \mathcal{L}(h, \mathbb{R}) \) and \( \mathcal{L}(h, \mathbb{R}) \) are bounds for \( \mathcal{L}(h, \mathbb{R}) \), where the infimum is taken over all \( \beta \in [0,1] \) satisfying \( \mathcal{L}(h, \mathbb{R}) \) and \( \mathcal{L}(h, \mathbb{R}) \).
A survey of some results can be found in [6, 12] and for the multivariate $Z_p$ in [10].

5. Proof (1.1).

A second result, that of the work in this field is done by A. Veron and by the
author. I know the results of the latter and also the work of the former. Consequently, the
author suggests and should carefully study the quoted and further
sourcing papers of Elsnerado.

References
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
[22] G. G. H. Brucke, On the maximum of some extremal question in combinatorial graphs, Math. Lis.: 29
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.
(Galway University Press, 1971) 127-177.