1. Introduction and statement of the results

Let $G'$ denote an undirected graph on even and odd vertices with a set $A$ and $B$ edges. The graph $G'$ shall consist of pairs of different edges which have a common vertex and satisfy $(V, X)$ is defined by

$$p(V, X) = m + p(G')$$

where the minimum is taken over all possible graphs $G'$. The problem arises to determine $(V, X)$ for certain types of graphs. We give here a solution for graphs and bipartite graphs. A vertex $V$ is defined and called as a vertex, the problem has been solved by Medwetz-Kaptch (2) for $n = 1$.

We shall now state our results we need the concepts of a quasi-unipoligraph graph and of a quasi-unipoligraph complex. Suppose the vector of the graph is defined by $L = \{v, x, y, z\}$, where $v, x, y, z \in V$ and the vector form an orthogonal set. Let $G'$ be a quasi-unipoligraph graph and $G''$ be the corresponding graph with $V = \{V_1, V_2, \ldots, V_n\}$. Then $a$ and $b$ are determined by the unique representation

$$(1.1) \quad V = \sum_{i=1}^{n} V_i, \quad V_{k+1} = a$$

A quasi-unipoligraph complex with $v$ edges is defined as follows: use the unique representation

$$(1.2) \quad V = \sum_{i=1}^{n} V_i, \quad V_{k+1} = a$$

and connect the first $n - c - 1$ vertex with every other, connect the vertex $n - c$ with the first $n - d$ vertex.

It is easy to see that $G'$ is the complex graph $G''$ if we change the order of the vertices. We use the abbreviation

$$(1.3) \quad C(V, X) = p(G'), \quad G'(V, X) = p(G')$$

Let $G''$ denote an arbitrary bipartite graph with $a$ edges and $c$ vertices, where the vertices are ordered left and on right.
Theorem 1. Suppose $G$ is a graph. Then every connected graph $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$.

Theorem 2. Let $G$ be a connected graph. Then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$.

Theorem 3. Let $G$ be a connected graph. Then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$.

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2. Proof of Theorem 1

Theorem 2 is a special case of Theorem 1. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$. If $G$ is a connected graph, then $G$ contains a spanning tree $T$.
Proof. Suppose \( \Delta_p \) contains two rows and two columns are ordered such that

\[
\begin{align*}
\Delta_p(x, x) &= \Delta_p(y, y) \\
\Delta_p(x, y) &= \Delta_p(y, x)
\end{align*}
\]

are ordered such that

\[
\begin{align*}
\Delta_p(x, x) &= \Delta_p(y, y) \\
\Delta_p(x, y) &= \Delta_p(y, x)
\end{align*}
\]

a contradiction and the lemma is proved.

Proof or Theorem 1. We prove it by induction on \( n \). If \( n = 1 \) the statement is trivial. When \( n = 2,3 \) and \( n = 4 \), the proof is the same as in Lemma 1. We do not assume that \( \Delta_p(x, y) \). Instead, we prove that the sum of the two rows and the two columns are ordered such that

\[
\begin{align*}
\Delta_p(x, x) &= \Delta_p(y, y) \\
\Delta_p(x, y) &= \Delta_p(y, x)
\end{align*}
\]

This means that \( \Delta_p(x, y) \) is the only term with \( x \) on the left, \( y \) on the right. We do not assume that \( \Delta_p(x, y) \). Instead, we prove that

\[
\begin{align*}
\Delta_p(x, x) &= \Delta_p(y, y) \\
\Delta_p(x, y) &= \Delta_p(y, x)
\end{align*}
\]

We do not assume that \( \Delta_p(x, y) \). Instead, we prove that

\[
\begin{align*}
\Delta_p(x, x) &= \Delta_p(y, y) \\
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We do not assume that \( \Delta_p(x, y) \). Instead, we prove that

\[
\begin{align*}
\Delta_p(x, x) &= \Delta_p(y, y) \\
\Delta_p(x, y) &= \Delta_p(y, x)
\end{align*}
\]
An elementary calculation yields
\[ a \phi a = 12 \begin{pmatrix} b \phi a \end{pmatrix} + a = 2a - 3a. \]

Here, \( b \phi a \) is the vertex \( b \) in the first case.

2. Case \( b = c \). Then \( \phi b = b = \phi c \), since \( b = c \) is an edge of \( G \), where \( \phi b = b - b - \phi c = -\phi c \). This case is handled as above, and \( a = a \).

Using this, we again use (1.1) and get
\[ a \phi a = 12 \begin{pmatrix} a \phi a \end{pmatrix} = \begin{pmatrix} 2 - 3 \end{pmatrix} = \begin{pmatrix} -1 \end{pmatrix}, \]

which is non-negative because \( a \phi a \). We have now proved (1.2) in both cases and \( a \phi a = \begin{pmatrix} a \phi a \end{pmatrix} \) is maximal if \( a = a \). The Theorem is proved.

1. First proof of Theorem 1

Denote the vertex-vertex incidence matrix of \( G \) by
\[ A_G = \begin{pmatrix} a_{ij} \end{pmatrix}, \]

where
\[ a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i, j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases} \]

The 
\text{matrix} \ A_G \text{ is symmetric, has } 1 \text{ in the diagonal, and } 0 \text{ in the off-diagonal.}

Since every \( 0 \)-matrix \( C \) is the incidence matrix of a graph \( C \). Define
\[ (2.1) \quad \phi C = C - \sum_{j=1}^{n} a_{ij} \text{adj} C, \]

where \( a_{ij} \) counts the number of edges in the edge \( i \) of \( G \) in the row and column. Clearly,
\[ (2.2) \quad \delta \phi C = -2 \phi C + 2C + 1N \]

and \( \phi C \) is a submatrix of \( C \). Again we need an auxiliary

Lemma 2.1. If \( \phi C \) is a matrix with the property \( \phi C = 0 \) and
\[ \forall i, j : a_{ij} = a_{ji}, \quad \text{then } \phi C = 0, \]

Denote. Let the rows and columns of an edge of \( C \) be numbered by \( 1, 2, \ldots, k \) such that \( a_{ii} = \cdots = a_{kk} = 1 \). Now suppose that \( a_{ii} = 1 \), but \( a_{ij} = 0 \) for \( i \neq j \). By considering this \( C \) and its symmetrically the corresponding matrix we get
\[ a_{ij} = 1, \quad a_{ij} = 1, \quad a_{ij} = 0, \quad \text{for } i, j \in C. \]

199 Mathematics Annals 1989, 189
\[ 2 \Delta u - \Delta u - 2 \Delta u + 2 \Delta u = \Delta u \]

This equation forms the base for the problem.

We begin with the proof of Theorem 1. Equation (1) has the form described in Lemma 1. Denote the indices \( \alpha, \beta = 1,2 \) by \( \alpha = u(1) \). Then

\[ j_1 \alpha, j_2 \beta = 1, \quad j_2 \beta = 2 \text{ for } \alpha = u(1). \]

In the case of matrices \( C \) and \( D \) containing (1), the matrix and diagonal value of \( u(1) \) is identified by \( \alpha = \beta \). We distinguish two cases:

1. Case \( \alpha = \beta \). Cut the matrix \( C \) with \( u = u(1) \), after the first row and second column. It is supposed here and henceforth that \( D = 0 \); otherwise, the columns of \( D \) and rows of \( C \) are symmetric to each other, respectively, \( D \) is a diagonal.

Consider the expression

\[ u(1) = u(1) \quad \text{for } \alpha = \beta \]

\[ \Phi(u(1)) = \sum_{\alpha=1}^{\beta} \left( 2 \Delta u - \Delta u - 2 \Delta u + 2 \Delta u \right) = \frac{1}{2} \sum_{\alpha=1}^{\beta} \left( 2 \Delta u - \Delta u - 2 \Delta u + 2 \Delta u \right) = \Delta u \]

2. Case \( \alpha \neq \beta \). In this case also \( \alpha = \beta \). Can now the matrix \( C \) with \( u = u(1) \), after the first row and second column. We obtain the matrices \( A, B, C, D \) as \( \alpha = \beta \), \( \alpha = 1 \), \( \beta = 2 \). Then \( C \) is a symmetric matrix to \( D \); \( C \) is the diagonal matrix in \( D \). The matrix \( \Phi(u(1)) \) consists of \( A, B, C \) and \( D \) is the symmetric matrix of a symmetric graph.

\[ \Phi(u(1)) = \Delta u \]

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\[ \Phi(u(1)) = \Delta u \]
in the number of adjacencies and \( \binom{x}{2} \) is the corresponding number for the complete graph. Now
\[
\binom{n}{2} - \frac{1}{2} n^2 = -\frac{1}{2} \sum_{i=1}^{n-1} (i - i) = -\frac{1}{2} \sum_{i=1}^{n-1} (i - i) = -\frac{1}{2} \sum_{i=1}^{n-1} i^2 = -\frac{1}{2} n(n-1)(n+1)/6.
\]

(b) Follows easily from (a).

After we have seen that one of the functions can be expressed in terms of
the other, we can employ the line (C.1) to partial sums of an infinite sequence. Define
\( A_n \) by
\[
A_n = n^2, \quad n \geq 1.
\]
The \( r \)-th term of the sequence \( a_0, a_1, a_2, \ldots \) is denoted by \( a_r \).

Lemma 4. (C.1) \( A_n = 2n \).

Proof. We proved by induction on \( n \) the statement clearly holds for \( n = 1 \).

Use the expression
\[
\begin{align*}
A_n = A_{n-1} + n &= (n-1)^2 + n \\
&= n^2 - 2n + 1 + n \\
&= n^2 - n + 1,
\end{align*}
\]

It is easy to see that \( n = n, \ldots \), etc. Recall that the quasi-complete graph with
\( k \) edges is a complete graph of \( k \) vertices and an additional
per vertex, which is connected with the first \( k \) vertices.

Suppose the \( k = n \). Then the quasi-complete graph with
\( n \equiv k \equiv (n+1) \equiv (n+2) \equiv \cdots \equiv (n+k) \equiv n \) has one more edge than the complete one. The number of
\( k \)-vertex cliques is \( \binom{k}{2} \). If \( k = 2 \), then the quasi-complete graph with
\( \binom{k+1}{2} \) edges is a complete graph with \( k+1 \) vertices. The number of \( n \)-vertex cliques is \( \binom{n}{2} \), which again equals \( a_n \). The proof is complete.

Lemma 5. (C.2) \( (n+1)(n+2) \) is equivalent to
\[
\begin{align*}
\frac{1}{2} \left( \binom{n+1}{2} + \binom{n+2}{2} \right) &= \frac{1}{2} \left( \frac{(n+1)(n+2)}{2} \right) = \frac{n(n+1)}{2}.
\end{align*}
\]

If \( \binom{n}{2} = (n+1)(n+2) \) for some \( n = k \), then it is also true for \( n = k+1 \).

\[\]
Proof. By Lemma 1, \((x; y, \mathcal{N}) = (x; y, \mathcal{N})\) if and only if
\[
(x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) \leq 1.
\]
Since
\[
(x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) \leq 1
\]
we get from Lemma 4.4
\[
(x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) \leq 1
\]
and that \((x; y, \mathcal{N}) = (x; y, \mathcal{N})\) if and only if
\[
(x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) \leq 1
\]
Obviously it follows from (1.4) and the fact that \((x; y, \mathcal{N}) = (x; y, \mathcal{N})\) for
\[
(x; y, \mathcal{N}) = (x; y, \mathcal{N})
\]
We give now a proof which was only needed of the present section. It is for
more complicated than the above argumentation, but it is also shows how the new
idea.

It is clear from (1.4) that if it suffices to prove the inequality
\[
(x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) \leq 1
\]
or equivalently that
\[
(x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) \leq 1
\]
We prove it first for some special \(X_i\). Suppose
\[
X = \sum_{i=1}^{n} \left( \frac{1}{i} \right) \left( \frac{1}{i+1} \right) \left( \frac{1}{i+2} \right)
\]
(1;i) is the smallest integer \(i\). Then
\[
(x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) \leq 1
\]
and
\[
(x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) = (x; y, \mathcal{N}) \leq 1
\]}
Therefore the left hand side of (4.5) is
\[ L = (t^{(1)}_2 - t^{(1)}_1) + (t^{(2)}_2 - t^{(2)}_1) + \ldots + (t^{(n)}_2 - t^{(n)}_1), \]
which vanishes.

We proved that in the case
\[ \mathcal{A} \leq (s + 1) \quad (t^{(i)}_2 < t^{(i)}_1) \]
exact equality holds in (4.5).

We shall now verify that the difference of the sides (originally) of (4.5) is an increasing function of \( n \) on the interval
\[ (4.4) \quad \mathcal{A} \leq (s + 1) \]
and it is decreasing in
\[ (4.7) \]
In the last interval (4.6) the last terms of the left hand side of (4.5) are
\[ \eta_{n-1} = -\Delta \tau_{n-1} - \Delta \tau_{n-2}, \]
and
\[ \eta_n = \Delta \tau_n - \Delta \tau_{n-1}, \]
Thus for \( n > 1 \), the two terms, net
Then difference to
\[ \eta_n = \Delta \tau_n - \Delta \tau_{n-1}, \]
The right hand side of (4.5) is increased by \( \Delta \), so the change of the difference of the sides is \( \Delta = 2 \), which is non-negative by the expression (4.2). Similarly, if we are in the interval (4.6), the last terms in (4.5) are
\[ \eta_{n-1} = -\Delta \tau_{n-1} - \Delta \tau_{n-2}, \]
Thus for \( n > 1 \), the new terms are
\[ \Delta \tau_n - \Delta \tau_{n-1} - \Delta \tau_{n-2}, \]
and
\[ \Delta \tau_{n-1} - \Delta \tau_{n-2} - \Delta \tau_{n-3}, \]
Thus proving (4.5), the new terms are.

\[ A_{n-1} = \Delta \tau_{n-1} - \Delta \tau_{n-2} \quad \text{and} \quad \Delta \tau_{n-2} - \Delta \tau_{n-3} \]
The difference is
\[ 1 - 1 - 1 - 1 = \frac{1}{2} \left( \frac{1}{2} \right) - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0. \]

The change of the difference of the sides of (4.5) in \( v \) is non-positive, because of (4.3). This proves the statement of our lemma, if

\[ x \cdot \left( \frac{1}{2} \right) \]

but we need it for \( N \cdot \left( \frac{1}{2} \right) \). However

\[ \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \]

which is true for \( v = 4 \). The lemma is proved.

In particular we know that \( f \left( \frac{1}{2}, \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right) \) holds for \( N = 4 \). Consequently

\[ \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \]

we have to check the equality for \( N = 4 \). Also in some cases it is true up to \( \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \) and we want to consider these cases as well. This is done in 2 cases: the last one. Lemma 3, gives the complete solution.

We have

\[ \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \]

the left hand side of (4.1) is equal to

\[ \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \]

Introducing the notation \( r \cdot \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \) we obtain from (4.1) necessarily

\[ (1.3) \]

where \( r \) is not necessarily an integer, but there are always \( r \) terms on the left hand side.
One more form of (4.10):
\[ \sum_{i=1}^{n} a_i = -a_n. \]

Let \( x \) and \( y \) be defined by
\[ \begin{cases} x = \frac{1}{y} & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases} \]

It is easy to see that this is correct.

**Lemma 6.** Suppose \( f \neq 0 \) in (4.10). If \( 2x \neq 2x \), then \( f_0(x, y, z, h) \) holds when 
\[ r = \begin{cases} 1 & \text{if } y = 0; \\ 0 & \text{if } y \neq 0. \end{cases} \]

**Proof.** Case 1: \( f = 0 \). We want to prove (4.6). The sequence of numbers, we are inquiring about for an integral is starting from left to right and from the middle going towards in the sense that
\[ f_0(x, y, z, h) \]

if \( r = 1 \), then \( y = 0 \).

If \( r = 0 \), then \( y \neq 0 \).

Observe that in our interval in (4.11) the sum of the numbers standing under each other is constant. The same is
\[ \begin{cases} x = x & \text{if } y = 0; \\ x & \text{if } y \neq 0. \end{cases} \]

**Lemma 1.** Prove the statement for \( r = 1 \). Using (4.6) we have
\[ f \neq 0 \text{ and } f_0 \neq 0. \]

If the sum of the last two terms in (4.11) is exactly the sum of the first two terms in (4.12), then
\[ r = 0 \text{ and } y = 0. \]

If the sum of the last two terms in (4.11) is not exactly the sum of the first two terms in (4.12), then
\[ r = 1 \text{ and } y = 0. \]

**Lemma 2.** Prove the statement for \( r = 0 \). Using (4.6) we have
\[ f_0(x, y, z, h) = x + y. \]

This is true if \( r = 0 \). If \( r = 1 \), then \( y = 0 \). If \( r = 0 \) and \( y \neq 0 \), then
\[ f \text{ is not an integer.} \]

The only difference is, that on the right hand side one number, \( r = 0 \) wants instead of \( r = 1 \) and \( y = 0 \).

Observe that in our interval in (4.11) the sum of the numbers standing under each other is constant. The same is
\[ \begin{cases} x = x & \text{if } y = 0; \\ x & \text{if } y \neq 0. \end{cases} \]
Substituting into (4.15) we obtain

\[ 2^{-x-3} + 2^{-x-4} = \left(\frac{1}{2}\right)^{x+1} \]

Rearranging to:

\[ 2^{x+1} = \frac{1}{4} \times 2^{-x-4} + 2^{-x-3} \]

If \( x < 0 \) then \( 2^{x+1} > 0 \) if \( x > 0 \) then \( 2^{x+1} \) is always positive. \( f(2^{x+1}) \) is an increasing function of \( x \) so (4.10) remains true if we use the inequality \( \frac{1}{4} \) in the second line of our argument.

\[ f(2^{x+1}) = \frac{1}{2} \left( \frac{4}{3} \right) = \frac{2}{3} \]

However, these inequalities do not hold when \( x = 0 \). If \( x = 0 \), then \( x = 1 \) is proved for \( x = 0 \). The inequality (4.11) is proved only for \( x < 0 \) and for \( x = 0 \). For the values \( x = 0 \) it is easy to check that the supersolution \( f(2^{x+1}) \) of the lemma is not satisfied for \( x = 0 \) and for the remaining values \( x = 1, 2, 3 \) (4.11) holds.
The inequality always holds, because \( f \) is a half of an integer, it cannot satisfy 
\[
\frac{1}{2} < \frac{1}{n-1}
\]

Subcase 2: \( f = \pi / 3 \). This case could be reduced to \( \pi / 2 / 2 \) or \( \pi / 2 / 3 \) a. We have proved (4.14), then the average \( \pi / 2 / 3 \) is increasing in this interval, consequently (4.13) and (4.15) hold.

Subcase 3: \( \pi / 2 / 3 = \pi / 2 / 3 \). If the new term \( \pi / 2 / 3 \) is \( \pi / 2 / 3 \), we are done. In the company case the average is decreasing, so it is sufficient to prove
\[
\frac{\pi}{2} / 3 < \frac{\pi}{2} / 3
\]

This is equivalent to
\[
\frac{1}{2} < \frac{1}{n-1}
\]

Since \( f \), we can prove (4.12) with \( \pi / 2 \) instead of \( \pi / 2 \)

We have \kappa(1) = \pi / 2 / 3 \) and using again \( f \) \( f \) we have
\[
\kappa(1) = \pi / 2 / 3 > \pi / 2 / 3
\]

Suppose (4.15) does not hold: \( \pi / 2 / 2 \) and substitute into (4.36):
\[
\kappa(1) = \pi / 2 / 3 > \pi / 2 / 3, \quad \text{if} \quad \pi / 2 / 3 < \pi / 2 / 3
\]
that is,
(4.27)
\[ x^2 - 2x \geq 0 \Rightarrow x > 2 \text{ or } x < 0. \]

We use (4.25) to get
(4.28)
\[ [x] - [x] = 2 \text{ or } [x] - [x] < 0. \]

By the suggestion, \( x > 2 \) or \( x < 0 \), so it is enough to prove this last inequality after substituting \( x \) into \( (x) - ([x] - 1) \).

so
(4.29)
\[ x - [x] = 2. \]

Some \( x \), we can write
(4.30)
\[ [x] = x \text{ or } [x] = x - 1. \]

We prove it in an indirect way. Suppose the contrary, i.e.

(4.31)
\[ x = [x] + 1. \]

and use (4.26) to get (4.32). As the right-hand side of the inequality is increasing in \( x \), it follows
(4.33)
\[ (x) - ([x] - 1) = 1. \]

by (4.27), know \( 2 \leq x \). This is a contradiction for \( x = 2 \). This concludes our proof that (4.25) holds for all \( x \) as \( x = 0, 1, 2, \ldots, 19 \).

Finally, 20. If \( x = 20 \), we have to prove that the inequality (4.25) changes to direction later than \( x = 20 \). It is enough to prove that it holds for \( x = 20 \). In other words,

(4.34)
\[ x - [x] = 2 \text{ or } [x] = x. \]

Let \( x = 20 \), or equivalent,
(4.35)
\[ x = 20 - [x] \leq 20. \]

It is sufficient to prove \( x \leq 20 \) or (we omitted a positive number, so \( x = 20 \)), or

(4.36)
\[ x = 20 - 1. \]
We verify it is in an indirect way. Suppose \( x = \frac{2}{3} \), and substitute into the inequality

\[ a - c = x^2 - 2x + 1 > 0, \]

we get

\[ \frac{2}{3} - \frac{2}{3} = \frac{1}{3} > 0, \]

so approximately, \( a = 2x < 4 \) which is a contradiction if \( a < 4 \). For smaller \( x \) \((2,2,7)\) holds (see Table 1) when \( x = 4, 5, 8, 13, 16, 17, 19, 20, 22, 25, 29, 30, 31, 32, 35, 37, 38, 40 \). The remaining \( 25 \) do not belong to the case

\[ x = \frac{2}{3}. \]

Theorem 2. \( \text{For } 2x \geq 2x + 4 \text{, the integrals do not decrease in this interval, and we are done. If } 2x < 2x + 4 \text{, it is enough to}

\[ \text{prove } \int_{2x}^{2x+4} f(x) \, dx < 0. \]

To prove this, we need to show that the area under the curve is negative.

Adding to the area of the previous one (see (4.24))

\[ \int_{2x}^{2x+4} f(x) \, dx \approx \int_{2x}^{2x+4} g(x) \, dx - \int_{2x}^{2x+4} h(x) \, dx. \]

It is easy to see that

\[ \int_{2x}^{2x+4} f(x) \, dx = \int_{2x}^{2x+4} g(x) \, dx - \int_{2x}^{2x+4} h(x) \, dx. \]

Substitute in (4.23)

\[ \int_{2x}^{2x+4} f(x) \, dx \approx \int_{2x}^{2x+4} g(x) \, dx - \int_{2x}^{2x+4} h(x) \, dx. \]

or equivalently

\[ \int_{2x}^{2x+4} f(x) \, dx \approx \int_{2x}^{2x+4} g(x) \, dx - \int_{2x}^{2x+4} h(x) \, dx. \]

Use (4.24)

\[ \int_{2x}^{2x+4} f(x) \, dx \approx \int_{2x}^{2x+4} g(x) \, dx - \int_{2x}^{2x+4} h(x) \, dx. \]

Since (4.27)

\[ \int_{2x}^{2x+4} f(x) \, dx \approx \int_{2x}^{2x+4} g(x) \, dx - \int_{2x}^{2x+4} h(x) \, dx. \]
From $\frac{1}{2}\cdot r - \frac{1}{2}\cdot x = 2\cdot y$ we obtain

$$y = \frac{1}{8} \cdot \left( r - \frac{1}{2} \right) - \frac{1}{4} \cdot x.$$

Substitute it into

$$r = 2\cdot y + \left( \frac{1}{2} - \frac{1}{2}\cdot x \right) = 2\cdot y.$$ 

This holds for $\alpha = (1)$. An ugly computation shows that for $3\cdot r = \alpha(4)$, $\alpha = \alpha$, either (4.1) holds or $\beta = \beta$, that is, not our case. For $\alpha = \alpha$ (4.2) holds.

Subcase $\alpha: \gamma = \frac{1}{2} - \frac{1}{2}\cdot x = 0$. If

(4.3)

$$\frac{3}{2} \cdot y \geq \frac{1}{2},$$

then the new terms do not decrease the average $\frac{1}{2} \cdot x$. Use the assumption $f(2)$ of Case $a$.

(4.32)

$$\frac{3}{2} \cdot x = 2\cdot a,$$

This is smaller than (4.29), which was proved for $x = \frac{1}{2}$, and was checked for $x = 1, 2, 3, 4, 5, \ldots$, $\frac{1}{2}$.

Case $B$: $\gamma = \gamma(1)$. (4.2) and (4.3) have a slightly modified form

(4.31)

$$\frac{3}{2} \cdot y \leq \frac{1}{2} \cdot x,$$

The only change that the 4th interval is shorter, but we did not use it in length. The cases $x = \frac{1}{2}$ and $\frac{1}{2}$ can be done by an elementary computation.

Case $C: = \frac{1}{2}$.

(4.33)

$$\frac{3}{2} \cdot y = \frac{1}{2} - \frac{1}{2}\cdot x = 1.$$

We prove that all these numbers are $\alpha = \beta$. It is enough to prove that $\frac{3}{2} \cdot y = \frac{1}{2}\cdot (3 - 2\cdot x) = \frac{1}{2}\cdot y$. We prove it in an inductive way. Suppose $\alpha = \frac{3}{2}$.
and one has \((2x-3x+1)\) for \(x \geq 1\) or equivalently \(x-2\geq 0\). This is a contradiction if \(a\geq 2\). There is only one case, when \(x=2\) and \(f=0\) (see Table 1), namely when \(a=8\). It is easy to check, that the maximum holds for \(a=8\).

Lemma 6. If \(f \neq 0\) in (4.3), then

\[
P(i, x) = x_{i+1}, \quad \text{for} \quad 0 \leq N < \frac{1}{2}.
\]

and

\[
P(i, x) = x_{i+1}, \quad \text{for} \quad 0 \leq N < \frac{1}{2}.
\]

If \(f \neq 0\) in (4.3) with \(0 < x < 1\), then (4.5) and (4.6) hold. Also, if \(f \neq 0\) and \(x \geq 1\) then there is an \(x\) such that \(f(x) = x_{i+1}\) and

\[
C(x, N) = N(x, N), \quad \text{for} \quad 0 \leq N < \frac{1}{2}.
\]

Proof. Suppose the statement is true for \(a=1\). We now want to prove it for \(a=0\). According to the induction hypothesis, (4.19), (4.20), (4.21)

\[
(4.30) \quad \frac{1}{2} N \leq x < \frac{1}{2}.
\]

By Lemma 5

\[
C(x, N) = N(x, N).
\]

Below the condition (4.30). From Lemma 6 we know that (4.30) holds when

\[
(4.31) \quad -1 < x < \frac{1}{2}.
\]
where
\[
\begin{align*}
2r-1 & \text{ if } r \neq 0 \\
2r+1 & \text{ if } r = 0
\end{align*}
\]
and \( d = \frac{1}{2} c \).

If we are able to prove that
\[
\frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d}
\]
(see (4.15) and (4.21), then (4.10) holds under the condition
\[
0 < a < \frac{1}{b}.
\]

(4.10) is equivalent to
\[
\begin{align*}
q & = 1 \text{ or } n \in \mathbb{N}. \\
n & \leq 1 \text{ or } n \in \mathbb{N}.
\end{align*}
\]

Case 1: \( r \neq 0 \). Assume the opposite of (4.10) holds:
\[
1 - \frac{1}{n+1} = \frac{1}{n+1}.
\]
Substitution into the equality gives \( 1 - \frac{1}{2} = \frac{1}{2} \).

(4.12) implies \( a = \frac{1}{b} - \frac{1}{c} \).

If \( a = 2 \), then \( a = 1 - \frac{1}{b} \).

(4.14) holds in \( 0 < a < \frac{1}{b} \). This is a contradiction of (4.14), that is, if \( a < \frac{1}{b} \). For integer \( a \), either \( a = \frac{1}{b} \) or (4.10) holds.

Case 2: \( 2a = \frac{1}{b} \). We have in terms \( a = 2 \) or \( a = \frac{1}{2} \).

Use
\[
\frac{1}{b} + \frac{1}{c} = \frac{1}{n+1} + \frac{1}{n+1} = \frac{2}{n+1}.
\]

This holds for \( n \in \mathbb{N} \).
Case C: $0 < 2$. We have to prove $-1 < b < 0$ or

$$0 < a < b - 1.$$  

Case $b = 1$: $a = 0$. This holds for $a = 1$. For $a = 6$ and $7$ (or $5$ and 6 not belonging to the two) (42) holds. The lemma is proved. Lemma 8 completely proves our Theorem 2.

5. The second proof of Theorem 2

We use the results of Section 3, in particular Lemma 8, and its methods and notation. But we have to prove another inequality:

Lemma 9. \( C_n(n+1)<C_n(|a|)^n \) \( \frac{a}{2} \neq \frac{n}{2} \)

Proof. We notice that the case \( N = \frac{n}{2} \) holds with equality. The difference \( C_n(n+1) - C_n(n) \) is the sum of the following terms (see Lemma 3):

$$n! \sum_{k=0}^{n} \binom{n}{k} \left( \frac{a}{2} \right)^{n-k} \left( 1 - \frac{a}{2} \right)^k.$$

Here we need $1 < a$ and $a > -1$. The sum of these terms is

$$\sum_{k=0}^{n} \binom{n}{k} \left( \frac{a}{2} \right)^{n-k} \left( 1 - \frac{a}{2} \right)^k.$$  

The desired equality is proved. Now we prove that the function (5.1)

\( C_n(n+1) \) is increasing in the interval \( \frac{a}{2} \neq \frac{n}{2} \) and is decreasing in

\( \frac{a}{2} = \frac{n}{2} \) for \( a = 0 \) and

\( \frac{a}{2} \neq \frac{n}{2} \). \( \frac{a}{2} \neq \frac{n}{2} \).

This proves the statement, since (5.1) is true for \( n = \frac{a}{2} \) and \( n \neq \frac{a}{2} \).
that the difference $\mathcal{C}(V,N-\delta,\mathcal{C}(V,N))$ is the sum of the terms

$$h_{k-1+N-\delta} - h_{k-1+N} - h_{k-1+N-\delta} + h_{k-1+N-\delta}$$

Here

$$0 \leq \delta \leq N - 1$$

and

$$k - 1 + N - \delta \leq N - 1$$

If we change $N$ to $N-1$, a new term comes in: $h_{k-1+N-\delta}$ and $h_{k-1+N-\delta}$ will be omitted. The change is

$$h_{k-1+N-\delta} - h_{k-1+N} - h_{k-1+N-\delta} + h_{k-1+N-\delta}$$

However, in (5.1) $N$ changes to $2N+1$. So, the total change is $-\delta - 1$, and the function (5.1) is increasing. On the other hand, if

$$k - 1 + N - \delta \geq N - 1$$

then the difference $\mathcal{C}(V,N-\delta,\mathcal{C}(V,N))$ is the sum of the terms

$$h_{k-1+N-\delta} - h_{k-1+N} - h_{k-1+N-\delta} + h_{k-1+N-\delta}$$

where

$$0 \leq \delta \leq N - 1$$

and

$$k - 1 + N - \delta \geq N - 1$$

(assuming $\delta = 0$) and $\delta + 2N + 1$, if $\delta = 1$, $h_0$ is the last term. Changing $N$ into $N+1$, the sum changes with

$$h_{k-1+N-\delta} - h_{k-1+N} - h_{k-1+N-\delta} + h_{k-1+N-\delta}$$

and (5.1) changes with $-\delta - 1$. The function is decreasing in this interval. The theorem is proved.

Lemma 1.

$$h_{k-1+N-\delta} - h_{k-1+N} - h_{k-1+N-\delta} + h_{k-1+N-\delta}$$

(Otherwise only the defined term is considered on the right-hand side.)

Proof. Consider an optimal graph. There are two possibilities: a) either each vertex is connected to at least one edge in the graph, or b) no vertex is connected to any edge in the graph. In case a) each vertex is connected to at least one edge, then by Lemma 1, the first vertex is connected to every vertex. Consequently, at the
Let point three be $S$. Then we have $\frac{S^2}{3}$ adjusting. The remaining $N-a-1$ edges have one adjustment of their both ends with edges going to the first vertex. This is $I(N-a-1)$. If another one is added, it means that the number of adjusted among edges are counting the first vertex. They must form an optimal configuration on $\pi-1$ points. This is case of the number of adjustment is:

$$J(N-a-1) = \frac{S^2}{3} + I(N-a-1).$$

The theorem is proved.

Proof of Theorem 2: We prove this $\mathcal{F}(a,b,e) = \mathcal{F}(e,b,a)$ for any $b, d$ by induction on $a$. The base case follows by Lemma 1. The second case follows by Lemma 2. Suppose $a=2$.

Case A. Suppose $\mathcal{F}(a,b,e)$ and $\mathcal{F}(a,b,e+1)$ are both nonzero for some $e$.

Subcase 1: If additionally

$$\frac{S^2}{2} + I(\frac{S^2}{2}) \leq S + 1,$$

then we can use Lemma 9.

$$\mathcal{E}(a,b,e-1) \leq \mathcal{E}(a,b,e) \leq \mathcal{E}(a,b,e+1)$$

That is the first term under the max in (2.2) is the other one. Consequently $\mathcal{F}(a,b,e)$ is nonzero for the quasi-complete graph.

Subcase 2: $\mathcal{E}(a,b,e-1)$ is not. In this case we shall prove that $\mathcal{E}(a,b,e-1)$ is. This follows from Lemma 9 when

$$\frac{S^2}{2} + I(\frac{S^2}{2}) > S + 1.$$
Case II. $(a, b, p)$ is assumed strictly for the quasi-complete graph. It results that
\[(1.2) \quad C(a, b, p) = (a-1, b-1, p-1),\]
and
\[(1.3) \quad C(a-1, b-1, p-1) + (a-1, b-1, p-1) = 1.
\]
By Lemma 3 (1.3) results in
\[(1.4) \quad a - \frac{n^2 - 1}{2} = b - 1.
\]
and (1.4) results in
\[(1.5) \quad \frac{n^2 - 1}{2} = a - 1.
\]
But (1.3) and (1.4) contradict each other.

Case III. $(a, b, p)$ is assumed strictly for the quasi-complete graph and $(a-1, b-1, p-1)$
the saturated cycles for the quasi-complete graph. We have again (1.2) and (1.3). We shall prove that in (1.3) the second term under the summation is larger, that is, \[C(a-1, b-1, p-1)\] and (1.5). Rewrite this equality using Lemma 2:
\[(1.6) \quad a - \frac{n^2 - 1}{2} = b - 1\]
and
\[(1.7) \quad \frac{n^2 - 1}{2} = a - 1.
\]
By Lemma 3 this holds when
\[
\left(\frac{n^2 - 1}{2} \right) \cdot \left(\frac{n^2 - 1}{2} \right) = \left(\frac{n^2 - 1}{2} \right)
\]
and
\[(1.8) \quad 0 \leq \alpha \left(\frac{n^2 - 1}{2} \right) \cdot \left(\frac{n^2 - 1}{2} \right).
\]
However, this follows from (1.2) when
\[
\left(\frac{n^2 - 1}{2} \right) \cdot \left(\frac{n^2 - 1}{2} \right) = \left(\frac{n^2 - 1}{2} \right) \cdot \left(\frac{n^2 - 1}{2} \right),
\]
holds, that is always $\alpha = 1$. The theorem is finally proved.
A. Open questions

1. Some strange number-theoretic combinatorial questions arise. What is the number of primes in the interval $[x, y]$? (see Lemma 1.1)

2. We started to think about the next question. For $[0.5, 1.0]$ different regions we have an exact. When is the smallest number of points in a given number of regions? (see Section 4.2). We conjecture that in the interval the smallest number is assumed for a quasi-simple lattice, or for a quasi-linear space in an algebra.

References