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ON A PROBLEM OF L. FEJES TÓTH

by

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It is easy to see by induction that n lines in the Euclidean plane determine at most $\binom{n-1}{2}$ bounded domains. L. Fejes Tóth conjectured that an analogous statement holds for n convex sets. We prove in this note a slightly stronger theorem. Instead of convexity we need only that the pairwise intersections of the domain are connected.

Let E denote the Euclidean plane. If $A \subset E$, then $\delta(A)$ denotes the set of boundary points of A. The family of subsets A which are homeomorphic to the closed unit disc is denoted by $\mathcal{D}$. The elements of $\mathcal{D}$ are called *domains*. The family of closed connected sets is denoted by $\mathcal{C}$. A homeomorphic map of an interval is called an *arc*. If an arc is non-empty and is not a single point we call it *non-trivial*. If it is not ambiguous we name the arcs by their endpoints a, b: (a, b) if the arc is open and $[a, b]$ if it is closed.

THEOREM. *If* $A_1, \dots, A_n \in \mathcal{D}$ *and* $A_i \cap A_j \in \mathcal{D}$ $(1 \leq i,\ j \leq n)$ *then*

$$E - \bigcup_{i=1}^{n} A_i$$

has at most $\binom{n-1}{2}$ *bounded connected componets.*

In the proofs we use the following trivial statements without proofs:

(i) Let $A \in \mathcal{D}$ and let $B, C \subset A$, $B \cap C = \varnothing$ be sets in the plane. If $x_1, y_1 \in \delta(B) \cap \delta(A)$. $x_2, y_2 \in \delta(C) \cap \delta(A)$ and x_1, x_2, y_1, y_2 lie on $\delta(A)$ in this order then either B or C is disconnected (Fig. 1).

(ii) Let $A \in \mathcal{D}$. If $B \subset A$ and $T \subset A$ are connected sets satisfying $B \cap T \neq \varnothing$, $B \cap (A-T) \neq \varnothing$ then $B \cap \delta(T) \cap \delta(A-T) \neq \varnothing$ (Fig. 2).

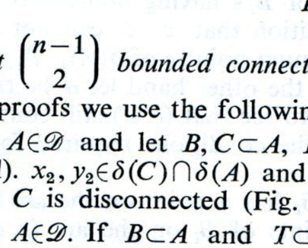

Fig. 1.

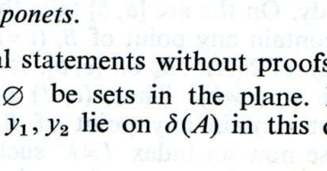

Fig. 2.

Let $A\in\mathscr{D}$, $B_1, \dots, B_m\in\mathscr{C}$ be sets in the plane. The family of those connected components T of

$$A-\bigcup_{i=1}^{n} B_i$$

for which $T\cap\delta(A)$ contains at least two connected components is denoted by $\mathscr{T}_m=\mathscr{T}_m(A; B_1, \dots, B_m)$. The set of the connected components of $T\cap\delta(A)$ $(T\in\mathscr{T}_m)$ is $\mathscr{V}_m=\mathscr{V}_m(A; B_1, \dots, B_m)$.

LEMMA. *Let $A\in\mathscr{D}$, $B_1, \dots, B_m\in\mathscr{C}$ be sets in the plane satisfying $B_i\subset A$ $(1\leqq i\leqq m)$ and $B_i\cap B_j=\emptyset$ $(1\leqq i<j\leqq m)$. Then $|\mathscr{V}_m|$ and $|\mathscr{T}_m|$ are finite and*

$$|\mathscr{V}_m|-|\mathscr{T}_m|\leqq m-1.$$

PROOF. 1. We first prove that there is an index j such that between the extremal points of $\delta(A)\cap B_j$ there is no element of B_i $(i\neq j)$ on $\delta(A)$ (Fig. 3, 4).

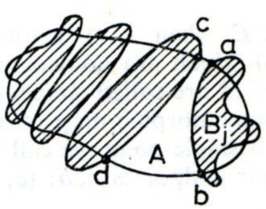

Fig. 3.

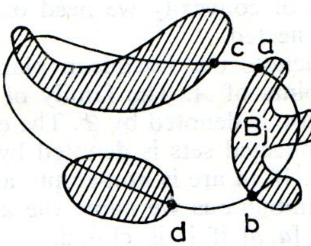

Fig. 4.

Choose k and two points a, b of $\delta(A)\cap B_k$ (and one of the arcs determined by them) in such a way that the number z of B_i's having non-empty intersection with (a, b) is >0 but minimal. If the condition that $z>0$ can not be satisfied, we are ready. On the arc $[a, b]$ take the maximum point c of $\delta(A)\cap B_k$ so that (a, c) does not contain any point of B_i $(i\neq k)$. On the other hand let d be the minimum point $(>c)$ of $\delta(A)\cap B_k$ on $[c, b]$. It is easy to see that (c, d) still contains a point from some B_i $(i\neq k)$. Thus, (c, d) satisfies the conditions required for (a, b), but it does not contain any point of B_k.

Choose now an index $l\neq k$ such that B_l has a point on the arc (c, d). Let e and f be the minimum and maximum points of B_l on the arc (c, d). If (e, f) contains a point of some B_i, $i\neq l$, then (e, f) contains points form at most $z-1$ different B_i, which contradicts the minimality of z. Consequently, (e, f) contains only points of B_l. If there is a point g of B_l outside of $[c, d]$, then we obtain a contradiction by (i), since $c, d\in B_k$, $e, g\in B_l$ and B_k and B_l are disjoint connected sets. It follows that there is no point of B_l on $\delta(A)$ outside of $[e, f]$, and there is no point of B_i $(i\neq l)$ in $[e, f]$. Our statement is proved.

2. Choose j according to section 1 of this proof. Let and b be the minimum and maximum points of $\delta(A)\cap B_j$ (Fig. 3, 4).

We shall use the following notations:

$$\mathscr{T}_{m-1}=\mathscr{T}_{m-1}(A; B_1, \dots, B_{j-1}, B_{j+1}, \dots, B_m),$$
$$\mathscr{V}_{m-1}=\mathscr{V}_{m-1}(A; B_1, \dots, B_{j-1}, B_{j+1}, \dots, B_m).$$

We claim that

(1) $$|\mathscr{V}_m|-|\mathscr{V}_{m-1}|\leqq 2.$$

Denote by c the maximum of the points $\leqq a$ of $\delta(A)\cap(\bigcup_{i\neq j} B_i)$. Similarly, let d denote the minimum of the points $\geqq b$ in $\delta(A)\cap(\bigcup_{i\neq j} B_i)$. Since the B_i's are disjoint, we have $c<a$ and $b<d$. Let us show that there is no element of $\mathscr{V}_m$ in the arc (a, b). In the opposite case there is an arc $\alpha\subset T\cap\delta(A)$ in (a, b) for some $T\in\mathscr{T}_m$. There must be an $\alpha\neq\beta\subset T\cap\delta(A)$ by the definition of τ_m. If β is outside of (a, b), then we obtain the contradiction by (i) becaused $\alpha, \beta\subset T\cap\delta(A)$, $a, b\in\in\delta(A)\cap B_j$ and T and B_j are disjoint and connected. If B is also in (a, b) then there is a point e between α and β satisfying $e\in B_i$ for some i. It follows from the definition of j that $i=j$. We obtain the contradiction by (i) using the points a, e and one point of each of α and β.

It is easy to see that the elements of $\mathscr{V}_m$ and $\mathscr{V}_{m-1}$ lying outside of (c, d) coincide.

Thus $\mathscr{V}_m-\mathscr{V}_{m-1}$ contains at most 2 elements: the arcs (c, a) and (b, d). (1) is proved.

Equality holds in (1) only if both (c, a) and (b, d) belong to some members T_1 and T_2 of $\mathscr{V}_m$, but (c, d) does not belong to any member of $\mathscr{V}_{m-1}$ (Fig. 3).

We prove now that

(2) $$|\mathscr{T}_m|-|\mathscr{T}_{m-1}|\geqq -1.$$

Let T be the connected component of $A-\bigcup_{i\neq j} B_i$ containing (c, d). If there is a point of B_j outside of T, we may apply (ii) with A, B_j and T: $B_j\cap\delta(T)\cap\cap\delta(A-T)\neq\emptyset$. However $\delta(T)\cap\delta(A-T)\subset\bigcup_{i\neq j} B_i$ since the B_i's are closed, thus the fact that $B_j\cap\delta(T)\cap\delta(A-T)\neq\emptyset$ contradicts the disjointness of B_i's. We have shown that $B_j\subset T$. As a consequence we obtain that B_j does not change the elements of $\mathscr{T}_{m-1}-\{T\}$: $\mathscr{T}_{m-1}-\{T\}\subset\mathscr{T}_m$. We have proved (2).

Equality holds in (2) only if $T\in\mathscr{T}_{m-1}$ but no subset of $T\in\mathscr{T}_m$. It means that (c, a), $(b, d)\notin\mathscr{V}_m$, and we have the stronger inequality $|\mathscr{V}_m|-|\mathscr{V}_{m-1}|\leqq 0$. (It can be proved that this situation does not occur.) In this case we have

(3) $$(|\mathscr{V}_m|-|\mathscr{V}_{m-1}|)-(|\mathscr{T}_m|-|\mathscr{T}_{m-1}|)\leqq 1.$$

If both in (1) and (2) there is strict inequality (Fig. 4), then (3) is true, again. We still have to discuss the case when we have equality in (1) (Fig. 3). Then $|\mathscr{T}_m|-|\mathscr{T}_{m-1}|\geqq 1$ can be proved. Indeed, in this case $T\notin\mathscr{T}_{m-1}$, showing that $\mathscr{T}_{m-1}\subset\mathscr{T}_m$. On the other hand $T_1\in\mathscr{T}_m-\mathscr{T}_{m-1}$. (3) is proved.

3. To prove the lemma we use induction over m. For $m=1$ it is easy to see that $|\mathscr{V}_m|=|\mathscr{T}_m|=0$. Suppose that $m>1$ and the lemma is true for $m-1$:

(4) $$|\mathscr{V}_{m-1}|-|\mathscr{T}_{m-1}|\leqq m-2.$$

The lemma follows from

$$|\mathscr{V}_m|-|\mathscr{T}_m|=(|\mathscr{V}_m|-|\mathscr{V}_{m-1}|)-(|\mathscr{T}_m|-|\mathscr{T}_{m-1}|)+(|\mathscr{V}_{m-1}|-|\mathscr{T}_{m-1}|)$$

and from (3) and (4).

PROOF OF THE THEOREM. We use induction over n. For $n=1$ the statement trivially holds. Let us suppose $n+1>1$ and the theorem is true for n, which means that the number of the bounded connected components of $E-\bigcap_{i=1}^{n} A_i$ is at most $\binom{n-2}{2}$. We have to prove that subtracting A_{n+1}, the number of the bounded connected components does not increase by more than $n-1$. Then the statement for $n+1$ follows from the identity $\binom{n-1}{2}+n-1=\binom{n}{2}$.

A_{n+1} does influence only those connected components of $E-\bigcup_{i=1}^{n} A_i$, which are non-disjoint to A_{n+1}. Denote these components by $C_1, \dots, C_u$. The connected components of C_i-A_{n+1} are denoted by $D_{i1}, \dots, D_{iw_i}$. Thus we have to prove that

$$\sum_{i=1}^{n} w_i - u \leqq n-1. \tag{5}$$

Construct a graph G_i whose vertices are on the one hand $D_{i1}, \dots, D_{iw_i}$ and on the other hand the connected components $E_{i1}, \dots, E_{it_i}$ of $C_i \cap A_{n+1}$. Two vertices are connected in G_i iff they have a common non-trivial arc on $\delta(A)$. If they have more such common arcs, then the are connected with more edges. It is easy to see that G_i is a connected graph, since C_i is connected. Denote the number of edges of G_i by v_i. The number of edges of a connected graph is greater than or equal to the number of vertices -1. That is,

$$w_i+t_i-1 \leqq v_i \quad (1 \leqq i \leqq u)$$

whence

$$\sum_{i=1}^{u} w_i - u \leqq \sum_{i=1}^{u} v_i - \sum_{i=1}^{u} t_i. \tag{6}$$

Denote the connected components of $\bigcup_{i=1}^{n} (A_{n+1} \cap A_i)$ by $B_1, \dots, B_m$. Obviously

$$m \leqq n \tag{7}$$

and by definition

$$\sum_{i=1}^{u} t_i = |\mathcal{T}_m|, \quad \sum_{i=1}^{u} v_i = |\mathcal{V}_m|. \tag{8}$$

Now (5) follows from (6), (8), the Lemma and (7). The proof is completed.

Recently M. Geréb, E. Győry and Gy. Szász found some other proofs and generalizations. They will be published in a forthcoming paper.

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