Extensions of the Erdös-Ko-Rado Theorem

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One of the most useful extremal properties on collections of subsets of a set is the following result of Erdös, Ko and Rado [1].

THEOREM 1. Given a collection $\mathcal{F} = \{A_1, \ldots, A_N\}$ of subsets of an n-element set S satisfying

$$A_i \cap A_j \neq \emptyset, \quad A_i \not\supset A_j \quad (i \neq j), \quad |A_i| \leq k \leq \frac{n}{2},$$

then

$$N \leq \binom{n-1}{k-1}.$$

A short proof of this theorem was obtained by one of the authors [2]. In this paper we extend the ideas of [2] to obtain a proof of a stronger theorem.

THEOREM 2. Under the hypotheses

$$A_i \cap A_j \neq \emptyset, \quad A_i \not\supset A_j \quad (i \neq j), \quad |A_i| \leq \frac{n}{2}$$

of Theorem 1

$$\sum \frac{1}{\binom{n-1}{|A_i|-1}} \leq 1 . \tag{1}$$

We also describe several ways in which the hypothesis of the theorem can be weakened without disturbing the conclusion, along with a second extremely short proof of Theorem 2 from the "LYM theorem" based on a theorem of Kruskal, which also proves a slightly stronger statement.

FIRST PROOF of Theorem 2. Let Δ be the set of all possible arrangements of the n elements around a circle. For each arrangement α and each member A_i of $\mathcal{F}$ let $f(\alpha, A_i) = \frac{1}{|A_i|}$ if A_i contains consecutive elements for that arrangement and let $f(\alpha, A_i)$ be zero otherwise. Then by summing $f(\alpha, A_i)$ over both arguments in the two possible orders, we obtain

$$\sum_{\alpha}\Big(\sum_{\substack{A_i \text{ consecu-}\\ \text{tive in } \alpha\\ A_i\in\mathcal{F}}} \frac{1}{|A_i|}\Big) = \sum_{A_i\in\mathcal{F}}\Big(\sum_{\substack{\alpha\\ A_i \text{ consecu-}\\ \text{tive in } \alpha}} \frac{1}{|A_i|}\Big) = \sum_{A_i\in\mathcal{F}} \frac{1}{|A_i|}\Big(\sum_{\alpha} 1\Big) =$$

$$= \sum_{A_i\in\mathcal{F}} \frac{1}{|A_i|}\,|\Delta|\,\frac{n}{\binom{n}{|A_i|}} = |\Delta| \sum_{i=1}^{N} \frac{1}{\binom{n-1}{|A_i|-1}}, \tag{2}$$

since A_i will by symmetry be consecutive in a proportion $\frac{n}{\binom{n}{|A_i|}}$ of the orderings $\alpha\in\Delta$. The result will therefore follow if in each α the sum $\sum \frac{1}{|A_i|}$ on the left hand side of (2) is ≤ 1 . This in turn follows from the following lemma:

<u>LEMMA.</u> If A_i is consecutive in an arrangement α then the total number of A_j's consecutive in α cannot exceed $|A_i|$.

<u>PROOF of lemma</u>. Every consecutive A_j must intersect A_i. The intersection cannot be A_i or A_j. Thus the intersection $A_i \cap A_j$ is a set of the first some elements or the last some elements of A_i. For $j \neq j'$, $A_i \cap A_j \neq A_i \cap A_{j'}$ holds. We have $2(|A_i| - 1)$ possibilities. However, $A_i \cap A_j$ and $A_i \cap A_{j'}$ cannot give a partition of A_i (using $|A_j|, |A_{j'}| \leq \frac{n}{2}$). Thus, only $|A_i| - 1$ of the $2(|A_i| - 1)$ possibilities can be simultaneously realized, which proves the lemma.

For each α the number of terms in the sum $\sum \frac{1}{|A_i|}$ on the left hand side of (2) is therefore no greater than the smallest $|A_i|$. Each sum is therefore ≤ 1 and the entire sum is $\leq |\Delta|$, which proves the theorem.

SECOND PROOF of Theorem 2. Let $\mathcal{F}$ be fixed and let p_i denote the number of cyclic arrangements α in which an $A \in \mathcal{F}$ of size i $(1 \leq i \leq \frac{n}{2})$ has consecutive elements and there is no smaller $A \in \mathcal{F}$ with this property. Denote by u_i the number of A's with $A \in \mathcal{F}$, $|A| = i$. We prove now the inequality

$$\sum_{i=1}^{w} u_i i!\,(n-i)! \leq \sum_{i=1}^{w} p_i i \qquad \left(1 \leq w \leq \left[\frac{n}{2}\right]\right). \tag{3}$$

Here, on the left hand side we count the number of pairs (A,α), where $A \in \mathcal{F}$, $1 \leq |A| \leq w$, α is a cyclic arrangement and the elements of A are consecutive in α . It is easy to see that the number of cyclic arrangements permuting a given A into consecutive members is $i!(n-i)!$, where $|A| = i$. Let us count the same thing in a different way. Fixing an arrangement α which has an i-element $A \in \mathcal{F}$ of consecutive members but not smaller, then there are at most i $A \in \mathcal{F}$ with consecutive members by the lemma. This gives an upper bound on the number of above pairs, it is the right hand side of (3).

Let
$$a_i = \frac{\left[\frac{n}{2}\right] - i}{i} - \frac{\left[\frac{n}{2}\right] - i - 1}{i+1} = \frac{\left[\frac{n}{2}\right]}{i(i+1)}$$
if $1 \le i < \left[\frac{n}{2}\right]$ and $a_{\left[\frac{n}{2}\right]} = 1$.

Take the linear combinations of the inequalities (3) by the coefficients a_w $\left(1 \le w \le \left[\frac{n}{2}\right]\right)$. The resulting inequality is

$$\sum_{i=1}^{\left[\frac{n}{2}\right]} u_i (i-1)!(n-i)! \le \sum_{i=1}^{\left[\frac{n}{2}\right]} p_i, \tag{4}$$

where $\sum p_i = (n-1)!$ and (4) is equivalent to the statement of the theorem. The proof is completed.

The preceding theorem shows that the E.K.R. bound will not be exact if A_i's are not all of size k. If we are only interested in attaining the E.K.R. bound we can relax some of the constraints in the theorem. In particular, it is not necessary to require that the A_i's be an antichain (that is, unordered by inclusion). We give several examples of weakenings of the theorem, among a wide spectrum of possibilities. Further weakening possibilities are described in reference [5].

<u>THEOREM 3.</u> <u>Let the elements of S be colored in two colors and suppose that the A_i's satisfy</u> $|A_i| \le k \le \frac{n}{2}$, $A_i \cap A_j \ne \emptyset$ <u>and either</u> $A_i \not\subset A_j$ <u>or</u> $A_j - A_i$ <u>is multicolored</u> $(i \ne j)$. <u>Then</u>

$$N \le \binom{n-1}{k-1}.$$

<u>PROOF</u>. If one partitions the subsets of S into symmetric rectangles, one direction representing the elements of each color, the number of A's in a rectangle is bounded by the number of rank k subsets B in it, that contain them; since k is a bound on their rank and no two can be in a row or column. The B's are pairwise non-disjoint, thus having size k, by Theorem 1 they satisfy the E.K.R. bound. The number of A's is $\le$ the number of B's, from which the theorem follows.

We call a 2-coloring of a set balanced, if the excess of elements of either class over the other is ≤ 1.

THEOREM 4. Suppose the elements of S are 2-colored twice; and that S_1 and S_2 are the sets colored in the two colors in the first coloring, while the second coloring is balanced in S_1 and S_2. Then, if the A_i's satisfy $|A_i| \leq k \leq \frac{n}{2}$, $A_i \cap A_j \neq \emptyset$ and either $A_i \not\subset A_j$ or $(A_j - A_i)$ is multicolored in the first color or $(A_j - A_i)$ is unbalanced in the second color for each (i,j) $(i \neq j)$, then

$$N \leq \binom{n-1}{k-1}.$$

PROOF. The result follows directly from the form of an explicit partition of 2^S into symmetrical rectangles. For details and further similar results see reference [5].

The value of the extension of E.K.R. theorem to Theorem 2 can be seen from the following consequence.

THEOREM 5. Let $f(k)$ be an arbitrary real function. Then under the hypotheses of Theorem 2

$$\sum_{i=1}^{N} f(|A_i|) \leq \operatorname*{Max}_{0 \leq j \leq \frac{n}{2}} f(j) \binom{n-1}{j-1}. \tag{5}$$

PROOF. Let g be a value for which $f(g)\binom{n-1}{g-1}$ is maximal. Then obviously

$$\sum_{i=1}^{N} \frac{1}{\binom{n-1}{|A_i|-1}} = \sum_{i=1}^{N} \frac{f(|A_i|)}{f(|A_i|)\binom{n-1}{|A_i|-1}} \geq \frac{\sum_{i=1}^{N} f(|A_i|)}{f(g)\binom{n-1}{g-1}}$$

holds. Hence (5) follows by (1). The proof is completed.

We conclude with a third independent proof of Theorem 1. This method in fact proves a stronger statement, since one can use it to completely characterize the numbers of sets of a given size which can occur in an E.K.R. family. The technique is based on a theorem (see [3] and [4]) which solves a similar problem for simplicial complexes.

Assume that the elements of our set S are the integers 1, 2, ..., n. This induces a natural ordering on subsets of S, obtained by associating sets with sequences of zeros and ones and ordering these sequences lexicographically. For any antichain $\mathcal{F} \subseteq 2^S$ we define the compression of $\mathcal{F}$ to be the antichain $C(\mathcal{F})$ obtained in the following way: suppose that $\mathcal{F} = \mathcal{F}_k \cup \mathcal{F}_{k+1} \cup \ldots \cup \mathcal{F}_j$ where $\mathcal{F}_i$ denotes the sets in $\mathcal{F}$ of size i $(k \leq j)$. Begin by taking the last $|\mathcal{F}_k|$ k-subsets of S in the lexicographic ordering just obtained. Then take the last $|\mathcal{F}_{k+1}|$ (k+1)-sets which do not contain sets already chosen. Continue in this way until the procedure has been applied to all levels. The resulting family of sets (obviously an antichain) is defined to be $C(\mathcal{F})$. Clearly $\mathcal{F}$ and $C(\mathcal{F})$ have the same number of sets at each level. The above mentioned theorem states that "compressed" antichains have the smallest "shadows" at higher levels. For any antichain $\mathcal{F}$ in 2^S let $N_p(\mathcal{F})$ denote the number of p-sets which contain at least one member of $\mathcal{F}$.

THEOREM 6. ([3] and [4].) *If $\mathcal{F}$ is any antichain in 2^S then $N_p(\mathcal{F}) \geq N_p(C(\mathcal{F}))$.*

The original version of Theorem 6 was proved for sets of uniform size, but the extension to arbitrary antichain is immediate. For arbitrary antichain, it is non-trivial to show that the compression $C(\mathcal{F})$ actually exists - i.e. that there are always enough sets to choose at each level. However this is guaranteed by Theorem 8. We refer the reader to [6] for further discussion of these ideas.

The third proof of Theorem 1 consists of verifying the following statement:

THEOREM 7. *If an antichain $\mathcal{F} \subseteq 2^S$ has the Erdös-Ko-Rado property (i.e. $|A| \leq \frac{n}{2}$ for all $A \in \mathcal{F}$ and no two members of $\mathcal{F}$ are disjoint), then so does its compression $C(\mathcal{F})$. In fact, every member of $C(\mathcal{F})$ contains the element 1.*

From this result it follows that the numbers $|\mathcal{F}_i|$ have exactly the same properties as those corresponding to arbitrary Sperner families on a set of $n - 1$ elements. Theorem 1 follows immediately.

To prove Theorem 7, first consider the case where all of the members of $\mathcal{F}$ have the same size. Then proving that every member of $C(\mathcal{F})$ contains 1 is equivalent to showing $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Hence the theorem is true in this case, by the original E.K.R. theorem * (see footnote on the next page).

If $\mathcal{F}$ has sets of more than one size, let k denote the minimum of those sizes. By Theorem 6, the operation of compressing $\mathcal{F}$ can only decrease the number of k-sets which contain members of $\mathcal{F}$ as subsets. Prior to compression, each of these subsets must contain the element 1, by the remark in the preceding paragraph. Here the same is true after compression, and the result follows.

Using Theorem 7, it is possible to completely characterize the numbers $|\mathcal{F}_i|$. To do so, we use the following notation. If m is a positive integer, then for each k we can write

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_i}{i}$$

with

$$a_k > a_{k-1} > \dots > a_i \geq i > 0,$$

and this expression is unique. Define

$$\partial_k(m) = \binom{a_k}{k-1} + \dots + \binom{a_i}{i-1},$$

THEOREM 8. Let $a_i, a_{i+1}, \dots, a_k$ be a sequence of non-negative integers, with $k \leq \frac{n}{2}$. Then there exists an antichain $\mathcal{F} \subseteq 2^S$ with the E.K.R. property if and only if

$$a_i + \partial_i(a_{i+1} + \partial_{i+1}(a_{i+2} + \dots + \partial_{k-2}(a_{k-1} + \partial_{k-1}(a_k))\dots)) \leq \binom{n-1}{i-1}.$$

This is essentially the statement that an E.K.R. family exists with parameters $a_i, a_{i+1}, \dots, a_k$ if and only if a Sperner family with the same parameters exists on a set of $n-1$ elements. A characterization of these numbers using Theorem 6 was obtained by Daykin [8] (see also [9] and [6]). This characterization yields the above result.

Of course Theorem 8 gives a better characterization of the numbers $|\mathcal{F}_i|$ than Theorem 2 does. In fact, it is not hard to deduce Theorem 2 from Theorem 8.

Our theorems do not give information about the situation if the

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* We could prove the E.K.R. theorem by these methods too: just consider the family $\overline{\mathcal{F}}$ of complements of members of $\mathcal{F}$. Then $\mathcal{F} \cup \overline{\mathcal{F}}$ is an antichain, and the inequality $|\mathcal{F}| \leq \binom{n-1}{k-1}$ follows immediately from Theorem 6 (see Daykin [7]).

subsets of size $> \frac{n}{2}$ are also allowed. The next theorem gives an inequality for this case.

THEOREM 9. Given a collection $\mathcal{F} = \{A_1, \ldots, A_N\}$ of subsets of an n element set S satisfying $A_i \cap A_j \neq \emptyset$, $A_i \not\supset A_j$ $(i \neq j)$, then

$$\sum_{\substack{A \in \mathcal{F} \\ |A| \leq \frac{n}{2}}} \frac{1}{\binom{n}{|A|-1}} + \sum_{\substack{A \in \mathcal{F} \\ |A| > \frac{n}{2}}} \frac{1}{\binom{n}{|A|}} \leq 1 . \tag{6}$$

PROOF. The proof follows the first proof of Theorem 2. The weight-function is

$$f(\alpha, A_i) = \begin{cases} \dfrac{n - |A_i| + 1}{|A_i|} & \text{if } A_i \text{ contains consecutive elements in } \alpha \text{, and } |A_i| \leq \frac{n}{2}; \\ 1 & \text{if } A_i \text{ contains consecutive elements in } \alpha \text{, and } |A_i| > \frac{n}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

In this case the equations (2) have the following form:

$$\sum_{\alpha} \Big(\sum_{\substack{A_i \text{ consecu-} \\ \text{tive in } \alpha \\ A_i \in \mathcal{F}}} f(\alpha, A_i) \Big) = \sum_{A_i \in \mathcal{F}} \Big(\sum_{\substack{\alpha \\ A_i \text{ consecu-} \\ \text{tive in } \alpha}} f(\alpha, A_i) \Big) =$$

$$= \sum_{\substack{A_i \in \mathcal{F} \\ |A_i| \leq \frac{n}{2}}} \frac{n - |A_i| + 1}{|A_i|} \Big(\sum_{\substack{\alpha \\ A_i \text{ consecu-} \\ \text{tive in } \alpha}} 1 \Big) + \sum_{\substack{A_i \in \mathcal{F} \\ |A_i| > \frac{n}{2}}} \sum_{\substack{\alpha \\ A_i \text{ consecu-} \\ \text{tive in } \alpha}} 1 =$$

$$= \sum_{\substack{A_i \in \mathcal{F} \\ |A_i| \leq \frac{n}{2}}} \frac{n - |A_i| + 1}{|A_i|} |\Delta| \frac{n}{\binom{n}{|A_i|}} + \sum_{\substack{A_i \in \mathcal{F} \\ |A_i| > \frac{n}{2}}} |\Delta| \frac{n}{\binom{n}{|A_i|}} =$$

$$= n |\Delta| \sum_{\substack{A_i \in \mathcal{F} \\ |A_i| \leq \frac{n}{2}}} \frac{1}{\binom{n}{|A_i| - 1}} + n |\Delta| \sum_{\substack{A_i \in \mathcal{F} \\ |A_i| > \frac{n}{2}}} \frac{1}{\binom{n}{|A_i|}} . \tag{7}$$

Hence (6) follows, if we prove the inequality

$$\sum_{\substack{A_i \text{ consecu-} \\ \text{tive in } \alpha}} f(\alpha, A_i) = \sum_{\substack{A_i \text{ consecu-} \\ \text{tive in } \alpha \\ |A_i| \leq \frac{n}{2}}} \frac{n - |A_i| + 1}{|A_i|} + \sum_{\substack{A_i \text{ consecu-} \\ \text{tive in } \alpha \\ |A_i| > \frac{n}{2}}} 1 \leq n, \tag{8}$$

and apply it for (7).

To prove (8) we use the ideas of the proof of the lemma.

1. Assume first, that there is no A_i with consecutive members in α with $|A_i| \le \frac{n}{2}$. In this case we have to prove, as the terms are 1, that the number of consecutive subsets is $\le$ n. Consider the consecutive subsets beginning with a fixed element; at most one A_i can occur among them, since $A_i \not\subset A_j$. Thus the number of consecutive A_i's is $\le$ the number of elements $(= n)$.

2. Assume, there is an $|A_i| = r \le \frac{n}{2}$ with consecutive elements and there is no smaller one. We can suppose without loss of generality that $A_i = \{s_1, \ldots, s_r\}$. Consider the sets A_j consisting of consecutive elements and beginning with s_u or ending with s_{u-1} $(2 \le u \le r)$. Every A_j belongs at least to one of these classes because of $A_i \not\supset A_j$, $A_i \not\subset A_j$ and $A_i \cap A_j \ne \emptyset$. By $A_j \not\supset A_k$ there is at most one subset beginning with s_u and there is at most one ending with s_{u-1}. On the other hand, one of the two sets beginning with s_u or ending with s_{u-1} must be of size $> \frac{n}{2}$ by $A_j \cap A_k \ne \emptyset$. Thus, the sum of their weights is at most $\frac{n-r+1}{r} + 1$, and the entire sum of all weights is maximally $\frac{n-r+1}{r} + (r-1)\frac{n-r+1}{r} + r - 1$. This proves (8) and the theorem.

It is easy to see that this method gives a proof of the Lubell--Meshalkin-Yamamoto inequality ([10],[11],[12]). In this case $f(\alpha, A_i)$ is simply 1 or 0, and the proof of (8) consists of section 1.

<u>THEOREM 10.</u> <u>Under the conditions of Theorem 9</u>

$$N \le \binom{n}{[\frac{n}{2}]+1} .$$

This theorem is a trivial consequence of Theorem 9. It is a special case of a theorem of Milner [13], and it was independently proved in [14], too.

D.E.Daykin kindly called our attention to the fact that Theorem 2 is published earlier by Béla Bollobás [15]. His proof is similar to our first proof, however, our lemma is somewhat stronger. This difference enabled us to prove Theorem 9.

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