TWO APPLICATIONS (FOR SEARCH THEORY AND TRUTH FUNCTIONS) OF SPERNER TYPE THEOREMS

by

G. O. H. KATONA (Budapest)

To the memory of A. Rényi

"I am constantly wondering what kind of knowledge I should try to acquire. Recently, Rényi told me that certainly, valdes only in mathematics and suggested that I should always make a convene of what to me seems the leading expert on numbers and probability in Athens."

From Rényi's "Dialogues on Mathematics"

§ 1.

Assume a finite set \( X = \{x_1, \ldots, x_n\} \) is given and we are looking for an unknown \( x \in X \). We have informations of type

\[ \mathcal{C}, \mathcal{B}, \mathcal{C}, \mathcal{B}^c \]

where \( \mathcal{A} \)'s are subsets of \( X \). If one of the sets

\[ (1) \]

is empty, then after knowing \( x \in B \) or \( x \notin B \) it may occur that \( x \in C \) or \( x \notin C \) does not contain any new information. For example, if \( BC = \emptyset \), then \( x \notin B \) contains the information \( x \notin C \). In the contrary case, if none of the sets (1) is \( \emptyset \), then we need the information "\( x \in C \) or \( x \notin C \)", independently of the answer of the question "\( x \in B \) or \( x \notin B \)". We say, following M. A. Maszewski [1] that \( B \) and \( C \) are qualitatively independent, if none of the sets (1) is \( \emptyset \). Rényi [2] asked what is the maximal number of pairwise qualitatively independent subsets \( B_1, \ldots, B_n \) of an \( n \)-element set \( X \). He solved in [2] the question for even \( n \) in the following way: The statement "none of \( B, B^c, \mathcal{C}, \mathcal{B} \) is empty" is equivalent to the statement "none of \( B, B^c, C, \mathcal{B} \) is contained in another one". That means, if \( B_1, \ldots, B_n \) are pairwise qualitatively independent, then none of \( B_1, \ldots, B_n \) is contained in another one. The well-known theorem of Sperner [3] says that the maximal number of such subsets is

\[ \left( \begin{array}{c} n \\ \frac{n}{2} \end{array} \right) . \]

It follows \( 2m \leq \left( \begin{array}{c} n \\ \frac{n}{2} \end{array} \right) \) and

\[ m \leq \left( \begin{array}{c} n \\ \frac{n}{2} \end{array} \right) . \]
If $n$ is even, this is the best possible upper estimation, since we can choose
\( \left( \begin{array}{c} n \\ 2 \end{array} \right) \) 2 qualitatively independent sets, taking arbitrary one of each comple-
mentary pair of \( \frac{n}{2} \)-tuples.

In this paper we solve the case of odd $n$.

**Theorem 1.** If $B_1, \ldots, B_n$ are pairwise qualitatively independent subsets
of a set of $n$ elements, then
\[
m \leq \left\lfloor \frac{n}{2} \right\rfloor - 1
\]
and this is the best possible estimation.

**Proof.** If $B$ and $C$ are qualitatively independent, then $B$ and $\overline{C}$ are
qualitatively independent, too. If $|B_i| > \frac{n}{2}$ we may change $B_i$ for $B_i$;
$B_1, \ldots, B_n, \ldots, B_n$ are qualitatively independent. Thus we may assume
$B_1, \ldots, B_n$ are chosen in such a way that
\[|B_i| \leq \left\lfloor \frac{n}{2} \right\rfloor \quad (1 \leq i \leq n).
\]

2. Define $k = \min_{1 \leq i \leq n} |B_i|$. Assume $B_i$'s are indexed in such a way
that for some $p$
\[k = |B_1| = \ldots = |B_p| < |B_j| \quad (p < i \leq n).
\]
Denote by $C(B_1, \ldots, B_p)$ the family \( \{C_1, \ldots, C_p\} \) of sets $C$
satisfying \( |C| = k + 1 \) and $C \supset B_i$ for some $1 \leq i \leq p$. If $n$, $k$, and $p$ are given the minimum
of $r$ is determined in [4] and [5]. However we do not need this exact minimum
here, we need only a simple estimation for $r$, which is determined by Strum–
ker [3]:
\[p \frac{n - k}{k + 1} \leq r.
\]
The number of pairs $(B, C)$, where $1 \leq i \leq p$, $B \subset C$, \( |C| = k + 1 \) is
$p(n - k)$. On the other hand, a fixed $C$ can contain $k + 1$ $B_i$:
\[p(n - k) \leq r(k + 1)
\]
which is equivalent to (3).
3. \( C_1, \ldots, C_n, B_{p+1}, \ldots, B_n \) are pairwise qualitatively independent, if
\[ k < \frac{n}{2} \]

It is trivial for two of them.

\( C_i \cap C_j \) is not empty since \( C_i \supseteq B_i, C_j \supseteq B_j \) for some \( u, v (1 \leq u, v \leq p) \) and \( C_i \cap C_j \supseteq B_i \cap B_j \) is not empty. \( C_i \cap C_j \) can not be empty because \( C_i \) has \( k + 1 \) elements, \( C_j \) has \( n - k - 1 \) elements and they can be complementer sets only if \( C_i = C_j \), that is if \( j = i \). The total number of elements in \( C_i \) and \( C_j \) is \( 2n - 2k - 2 \). They can be disjoint only if
\[ 2n - 2k - 2 \leq n - 1 \]
as there is an element of \( C_i \cap C_j = C_i \cap C_j \). From (4) it follows \( \frac{n - 1}{2} \leq k \)
which contradicts our supposition. \( C_i \cap C_j \) can not be empty. \( C_i \cap B_j \)
\( (1 \leq i \leq p, p < j \leq m) \) is not empty since \( C_i \supseteq B_i \) for some \( u (1 \leq u \leq p) \)
and \( C_i \cap B_j \supseteq B_i \cap B_j \) is not empty. \( C_i \) has \( k + 1 \), \( B_j \) has \( n - k \) elements.
Thus they can not be complementer sets as \( k + 1 + n - k > n, C_i \cap B_j \neq 0 \).
We have similarly \( C_i \cap B_j = 0 \). Finally let us verify that \( C_i \) and \( C_j \) have also a common element. The total number of their elements is \( 2n - 2k - 1 \).
\( C_i \cap B_j = C_i \cap B_j \) has at least one element. Thus, if \( C_i \) and \( B_j \) are disjoint, we have
\[ 2n - 2k - 1 \leq n - 1 \]
This inequality contradicts our supposition \( k \leq \frac{n}{2} \).

4. Now we prove if \( B_1, \ldots, B_n \) are pairwise independent and \( n \) is maximal, then
\( |B_1| = \ldots = |B_n| = \frac{n}{2} \).

Suppose the contrary, \( k = \min \{ |B_1|, \ldots, |B_n| \} < \frac{n}{2} \). We may apply the result of Section 3: \( C_{p+1}, \ldots, C_n, B_{p+1}, \ldots, B_n \) are pairwise independent. However,
\[ \frac{n}{k + 1} \geq 1 \]
\[ \frac{n}{k + 1} \geq 1, C_1, \ldots, C_n, B_{p+1}, \ldots, B_n \] have more members than \( B_1, \ldots, B_n \) in contradiction with the maximality of \( B_1, \ldots, B_n \).
Thus, \( k \geq \frac{n}{2} \)
and (2) ensure the validity of the statement.

5. \( B_1, \ldots, B_n \) have the same number of elements \( \frac{n}{2} \) and \( B_i \cap B_j = 0 \)
\( (1 \leq i, j \leq m) \). We may apply the next theorem of Károlyi-Chao Ko–Rado [6].
If \(|B_1| = \ldots = |B_n| = l\), where \(B_1, \ldots, B_n\) are pairwise non-disjoint subsets of a set of \(n\) elements, then
\[
m \leq \binom{n-1}{l-1}.
\]

In our case
\[
m \leq \binom{n-1}{\left\lfloor \frac{n-1}{2}\right\rfloor}.
\]

The proof is completed.

**Open Problems.** 1. Determine the maximal \(m\) for which there exists a family \(B_1, B_2, \ldots, B_n\) satisfying
\[
|B_i \cap B_j| \geq r, \quad |B_i \cap B_j| \geq r, \quad |B_i \cap B_j| \geq r, \quad (1 \leq i, j \leq m).
\]

where \(r \geq 1\) is a fixed integer.

2. Determine the maximal \(m\) for which there exists a family \(B_1, B_2, \ldots, B_n\) satisfying
\[
H(B_i, B_j) \geq r, \quad (1 \leq i, j \leq m),
\]

where
\[
H(B_i, B_j) = -|B_i \cap B_j| \log |B_i \cap B_j| - |B_i \cap B_j| \log |B_i \cap B_j| - \cdots - |B_i \cap B_j| \log |B_i \cap B_j|.
\]

and \(r\) is a positive real number.

The first problem is solved for \(r = 1\) in Theorem 1. The second problem is solved by Theorem 1 for \(r = -3 \frac{1}{n} \log \frac{1}{n} - \frac{3}{n} \log \frac{3}{n} - 3\).

§ 2.

A logical or truth function is an \(n\)-dimensional function defined on the \(n\)-dimensional 0, 1 vectors and taking on the values 0, 1. A truth function \(f\) is said to be monotonically increasing if \(f(x_1, \ldots, x_n) = 1\) and \(x_1 \leq x'_1, \ldots, x_n \leq x'_n\) imply \(f(x_1, \ldots, x_n) = 1\).

(4) \((x_1 \land x_2 \land \ldots \land x_n) \lor \ldots \lor (x_{n-1} \land x_n \land \ldots \land x_n)\)

is called a disjunctive-normal form, where \(x_{n-1} = x_n\) or \(1 - x_n\) and \(0 \land 0 = 0, 0 \lor 0 = 0, 0 \lor 1 = 1, 1 \lor 0 = 1, 1 \land 1 = 1 (\land = "\&"), 0 \lor 0 = 0, 0 \lor 1 = 1, 1 \lor 0 = 1, 1 \lor 1 = 1 (\lor = "\lor")\). Every truth function has a disjunctive-normal form which is equivalent to it. We may produce such a form in the following way. Fix a 0, 1 vector \(a = (a_1, \ldots, a_n)\) satisfying \(f(a) = 1\). We cor.
respond an expression $z_i \land z_{i+1} \land \ldots \land z_n$, where $z_i = x_i$ if $a_i = 1$ and $z_i = 1 - x_i$ if $a_i = 0$. It is easy to see that $z_1 \land \ldots \land z_n = 1$ if and only if $x_i = a_i \forall i \leq n$. These expressions $z_1 \land \ldots \land z_n$ stand in the place of the bracket-expressions in (5) for all $\varepsilon$ satisfying $f(\varepsilon) = 1$. It is easy to see that this function is identical to $f$.

A disjunctive-normal form is minimal if it has a minimal number of variables (with multiplicity). Assume $f$ is a monotonically increasing function. It is easy to see that we can omit the terms of the form $z = 1 - x$ from its disjunctive-normal form. Thus, a minimal disjunctive-normal form of a monotonically increasing function has the form

$$\bigvee_{(\varepsilon_1, \ldots, \varepsilon_n)} \bigvee_{(x_1, \ldots, x_n)} \bigwedge_{1 \leq i \leq n} (x_i \land \varepsilon_i) \land \bigwedge_{1 \leq i \leq n} \bigwedge_{1 \leq j \leq i} (x_j \land \varepsilon_i) \land x_i.$$ 

(6)

On the other hand, if the index set of one bracket has a proper subset, which is the index set of an other bracket, it can be omitted.

Summarizing what has been said, the minimal disjunctive-normal form of a monotonically increasing function may be determined by a family of subsets of the $n$ indices not containing each other. (For the interested reader see [7]).

By this manner the question what is the maximum of the number of variables (with multiplicity) in the minimal disjunctive-normal form of a truth function of $n$ variables is reduced to the problem what is the maximum of the sum of the number of elements in a family consisting of subsets of an $n$ element set not containing each other. By formula: $\max \sum_{i} |A_i|$, where $A_i \subseteq A_j \forall i = j$.

We solve the problem in a more general form.

**Theorem 2.** Let $g(k)$ be a real function defined on natural numbers. If $A_1, \ldots, A_n$ are subsets of a set $\{1, \ldots, n\}$ with the property $A_i \subseteq A_j \forall i = j$ then

$$\max_{\sum_{i} |A_i|} g(k)$$

attains its maximum for the family of all subsets of $\max_{\sum_{i} |A_i|} k \subseteq \{1, \ldots, n\}$ elements.

**Proof.** First let us prove the Lebedev–Mesalkin inequality ([8],[9]).

A family $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n$ of subsets with $|B_i| = i \forall 0 \leq i \leq n$ is called a complete chain. The total number of complete chains is $\alpha!$. The number of complete chains containing $A_1 \subseteq B_2 \subseteq \ldots \subseteq B_n$ is $|A_n|! (\forall \subseteq |A_n|)!$. It is
easy to see that the complete chains containing $A_i$ are different from the complete chains containing $A_j$ ($i \neq j$) (using $A_i \not\subset A_j$). Thus, we obtain
\[ \sum_{i=1}^{n} |A_i| (n - |A_i|) \leq n^2. \]

It follows the desired inequality
\[ \sum_{i=1}^{n} \frac{1}{|A_i|} \leq 1. \]  

2. We have to maximize $\sum |g(A_i)|$ under the condition (7). This
is trivial:
\[ \sum |g(A_i)| = \sum \frac{|g(A_i)|}{\binom{n}{|A_i|}} \leq \sum \frac{\binom{n}{z}}{\binom{n}{z}} \leq \binom{n}{z/n}. \]
where $z$ is defined by $g(z) = \max g(k) \binom{n}{k}$

3. The estimation is the best possible as $\sum |g(A_i)| = g(z) \binom{n}{z}$ for the
family of all the sets of $z$ elements. The proof is completed.

Examples. 1. If $g(k) = 1 (0 \leq k \leq n)$, then Theorem 2 gives the original
Sperner theorem.

2. If $g(k) = k (0 \leq k \leq n)$ we obtain the inequality
\[ \sum |A_i| \leq \binom{n}{\frac{n}{2}} \]  

since
\[ \max_{1 \leq k \leq n} \binom{n}{k} = \max_{1 \leq k \leq \binom{n}{\frac{n-1}{2}}} \binom{n}{k-1} = \frac{n-1}{2} \binom{n}{\frac{n-1}{2}} = \frac{n}{2} \binom{n}{\frac{n}{2}}. \]
where $(x)$ denotes the least integer $\geq x$. (8) gives the solution of the problem
induced by the minimal disjunctive-normal form of a truth function. Let us
notice that there exists a function which has not a "shorter" disjunctive-
normal form: the function which has value 1 iff the number of one is $\geq \frac{n}{2}$
in the vector.

3. This example is worthy of formulation as a new theorem.
Theorem 3. (Iterated Sperner theorem.) Let \( A_1, \ldots, A_n \) be subsets of a set of \( n \) elements satisfying \( A_j \cap A_k \neq \emptyset \) (1 \( \leq j, k \leq m, j \neq k \)). Let further \( R_1, \ldots, R_m \) be subsets of \( A_1 \) (1 \( \leq i \leq m \)) satisfying \( R_j \cap R_k \neq \emptyset \) (1 \( \leq j, k \leq m, j \neq k \)). Then the number of subsets

\[ |m| \leq \left[ \binom{n}{\frac{2n}{3}} \right] \left[ \binom{3}{3} \right] \]

and the estimation is the best possible.

Proof. By the Sperner theorem we have

\[ m \leq \left[ \frac{|A_j|}{2} \right] \]

Choose the function \( g(k) = \left( \left\lfloor \frac{k}{2} \right\rfloor \right) \). Then, by Theorem 2

\[ |m| \leq \left[ \frac{z}{2} \right] \left[ \frac{3}{2} \right] \]

where \( z \) is defined by

\[ \left( \left\lfloor \frac{z}{2} \right\rfloor \right) = \max \left[ \left\lfloor \frac{k}{2} \right\rfloor \right] \]

Here we have

\[ \left[ \frac{n}{k} \right] \left( \left\lfloor \frac{k}{2} \right\rfloor \right) = \frac{n(n-1) \cdots (n-k+1)}{k!} \left( \left\lfloor \frac{k}{2} \right\rfloor \right) \]

and

\[ \left[ \frac{n}{k+1} \right] \left( \left\lfloor \frac{k+1}{2} \right\rfloor \right) = \frac{n(n-1) \cdots (n-k-1)}{k!} \left( \left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \]

The coefficient satisfies the inequality

\[ \frac{n-k}{2} \leq 1 \text{ if } k < \frac{2n - 2}{3} \text{ and } k \text{ is even or } k < \frac{2n - 1}{3} \text{ and } k \text{ is odd} \]

\[ \frac{n-k}{2} + 1 \leq 1 \text{ if } k \geq \frac{2n - 2}{3} \text{ and } k \text{ is even or } k \geq \frac{2n - 1}{3} \text{ and } k \text{ is odd} \]
The maximal $k$ having a coefficient $>1$ is $\left\lfloor \frac{2n}{3} \right\rfloor + 1$.

Hence we obtain the optimal $Z$:

$$Z = 2n + 1 \quad \left( \begin{array}{c} 2n \\ 3 \end{array} \right)$$

The theorem follows from (10), (11) and (12) using

$$\left( \begin{array}{c} 2n \\ 3 \end{array} \right) - \left( \begin{array}{c} n \\ 3 \end{array} \right)$$

It is easy to generalize the theorem to obtain the $r$ times iterated Sperner theorem.

REFERENCES


(Received May 9, 1971)