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INTERSECTION THEOREMS FOR SYSTEMS OF FINITE SETS
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Let \( k \leq m \), and \( M \) be a finite set of cardinal number \( m \). Determine the largest number \( n \) such that there exists a system of \( n \) sets \( a_v \) satisfying the conditions

\[
a_v \subseteq M, \quad a_{\mu} \neq a_v, \quad |a_\mu a_v| \leq k \quad (\mu < v < n),
\]

where \( |a| \) is the cardinal number of \( a \).

If \( m + k \) is even, then the system consisting of the sets \( a \) such that

\[
a \subseteq M \quad \text{and} \quad |a| = \frac{1}{2}(m + k)
\]

has the required properties. P. ERDÔS, CHAO KO and R. RADO have guessed, that this system contains the maximum possible number of sets [1].

In this note I prove this conjecture, and determine the extremal system also in the case when \( m + k \) is odd. For the proof I use a theorem (Theorem 2) which is also interesting in itself.

Notations:

The letters \( a, b, c, d, e \) denote finite sets of non-negative integers, all other lower-case letters denote non-negative integers. If \( k \leq l \), then \([k, l)\) denotes the set

\[
\{k, k+1, \ldots, l-1\} = \{t : k \leq t < l\}.
\]

The obliteration operator \( \sim \) serves to remove from any system of elements the element above which it is placed. Thus \([k, l) = \{k, k+1, \ldots, l\} \). The cardinal number of the set \( a \) is denoted by \( |a| \); inclusion, union, difference and intersection of sets are denoted by \( a \subseteq b \), \( a+b \), \( a-b \), \( a \cdot b \).

If \( k \leq l \leq m \), \( S(k, l, m) \) denotes the set of all systems \( \{a_0, a_1, \ldots, a_n\} \) such that

\[
a_v \subseteq [0, m), \quad |a_v| = l \quad (v < n),
\]

\[
a_\mu \neq a_v, \quad |a_\mu a_v| \leq k \quad (\mu < v < n).
\]

Put \( A = \{a_0, \ldots, a_n\} \), where \( |a_v| = l \quad (v < n) \). \( A^g \) or \( \{a_0, \ldots, a_n\}^g \) denotes the system of sets \( b_v \) such that \( |b_v| = g \), \( b_\mu \neq b_v \) \( (\mu < v < |A^g|) \), and for some \( \mu \) \( b_v \subseteq a_\mu \).

Let us consider \( A = \{a_0, \ldots, a_n\} \), where \( a_v \) are arbitrary sets. Denote by \( A_1 \) the subsystem of sets \( a_v \) satisfying the conditions \( a_v \subseteq A \) and \( |a_v| = l \).

**Theorem 1.** If \( 1 \leq g \leq l, \ 1 \leq k \leq l \) and \( g+k < l \), further \( \epsilon > 0 \), then there exists a system \( A = \{a_0, \ldots, a_n\} \in S(k, l, m) \) for which

\[
\frac{|A^g|}{n} < \epsilon.
\]
PROOF. Let \( m \equiv k \) be a non-negative integer. If \( a_0, a_1, \ldots, a_n \) are distinct sets such that
\[
[0, k) \subset a \subset [0, m) \quad \text{and} \quad |a| = l,
\]
then \( A = \{a_0, \ldots, \hat{a}_n\} \in S(k, l, m) \) and \( n = \binom{m-k}{l-k} \). Clearly \( |A^e| = \binom{m}{g} \) (in fact it is easy to see that \( |A^e| = \binom{m}{g} \)), and
\[
\frac{m}{g} \cdot \frac{m-k}{l-k}
\]
can be arbitrarily small, if \( m \) is sufficiently large, because \( g < l - k \).

THEOREM 2. If \( 1 \equiv g \equiv l, 1 \equiv k \equiv l \) and \( g + k \equiv l \), further \( A = \{a_0, \ldots, \hat{a}_n\} \in S(k, l, m) \) then
\[
(1) \quad n \cdot \frac{\binom{2l-k}{g}}{\binom{2l-k}{l}} \equiv |A^e|.
\]

REMARK. From \( g \equiv l - k \) and \( l \equiv g \) it easily follows that
\[
(2) \quad \frac{\binom{2l-k}{g}}{\binom{2l-k}{l}} \equiv 1
\]
and equality holds if and only if \( g + k = l \) or \( g = l \).

PROOF of Theorem 2. If \( g = l \), the theorem is trivial (moreover always equality holds). In what follows we consider the case \( g < l \).

We distinguish three cases.

Case 1: \( 2l - k \equiv m \).

By counting in two different ways the number of pairs \((a_v, c)\) where \( c \in A^e \) and \( c \subset a_v \), we obtain
\[
(3) \quad n \binom{l}{g} \equiv |A^e| \binom{m-g}{l-g}.
\]

We have to prove that
\[
\frac{\binom{l}{g}}{\binom{m-g}{l-g}} \equiv \frac{\binom{2l-k}{g}}{\binom{2l-k}{l}}.
\]
This is trivial, if $2l-k \equiv m$, moreover equality holds only in case $2l-k = m$. Hence we obtain that equality holds in (3) only if $m = 2l-k$ and every $c$ is included in $\binom{m-g}{l-g}$ distinct sets $a_v$, that is $A$ is the system of all subsets of $[0, m)$. Thus equality in Case 1 can hold only in this way.

Case 2: $g = 1$.

Since $g+k \equiv l$, we have $k \equiv l-1$. There are two cases: $k=l$ and $k=l-1$. If $k=l$, then $n=1$ and we can choose $m=k$. Here $2l-k = k = m$, therefore we have Case 1. Assume next $k = l-1$. If we have a system $A = \{a_0, \ldots, a_n\} \in S(l-1, l, m)$ such that every set of $l-1$ is included in at most in $a_v$, then consider the set $a_0a_1$. Clearly $|(a_0a_1)a_v| \equiv l-1$, on the other hand $|(a_0a_1)a_v| < l-2$ is impossible, because in this case we should have

$$|a_0a_v| \equiv (a_0a_1)a_v + 1 < (l-2) + 1 = l-1.$$ 

Thus $|a_0a_1a_v| = l-2$.

We have $a_0 - a_1 \subset a_v$ for every $v$, because of $|a_0a_1| = l-1$ and $|(a_0a_1)a_v| = l-2$. Similarly $a_1 - a_0 \subset a_v$. Here $a_0a_1, a_0 - a_1, a_1 - a_0$ are disjoint sets, therefore

$$a_v = (a_0 - a_1) + (a_1 - a_0) + a_0a_1 - \lambda_v,$$

where $\lambda_v$ is an element of $a_0a_1$. From this results $n \equiv l+1$, and every element is contained at most in $l$ sets $a_v$. Thus $n \frac{l}{l} \equiv |A^1|$, since every $a_v$ has exactly $l$ elements.

If the system $A$ is such that there is a set $c$ satisfying $|c| = l-1$, which is included at least in 3 sets $a_v$ (for example $a_0, a_1$ and $a_2$) then for arbitrary $v < n$ $c \subset a_v$. Namely, $|ca_v| \equiv l-2$ can not be true, because in this case $|a_0a_v| \equiv l-1$, similarly $|ca_v| = l-2$ can not hold, since its consequence would be $a_v \supseteq ca_v, a_v \supseteq a_1 - c$ and $a_v \supseteq a_2 - c$ because of $|ca_v| = |a_1a_v| = |a_2a_v| = l-1$, that is $|a_v| \equiv l+1$, which is impossible. This completes the proof in Case 2, since here $|A^1| = n + l - 1 > n$.

In Case 2 equality can hold if and only if every set of $l-1$ is included at most by two $a_v$, and $A$ consists of all sets $a_v$ satisfying (4). This falls under Case 1, where equality holds.

Case 3: $2l-k < m$ and $g \geq 1$.

We use induction over $m$, and we apply Cases 1 and 2.

Here $1 < g < l \equiv m$, thus $m \equiv 3$. First we consider $m=3$. Here $l=3$, thus $n=1$, $g=2, k=1$ or $2$ ($k=3$ is impossible, since we should then have $2l-k = m$, and this is Case 1). Since $|A^2| = 3$ and $\frac{5}{2} = 1, \frac{4}{2} = \frac{3}{2}$, in both cases strict inequality holds.

Suppose that $m > 3$ and for $m-1$ Theorem 2 is true. We prove the theorem for $m$. Denote by $s_v$ the sum of the elements of $a_v$. We can clearly assume that our system is such that $|A^a|$ is minimal and amongst all such systems $\sum_{v=0}^{n-1} s_v$ is minimal. Denote now by $A$ the system $A = \{a_v : v < n\}$.
We separate in Case 3 two subcases.

Case 3a. Suppose that whenever

\[ m - 1 \in a_v \in A \quad \text{and} \quad \lambda \in [0, m) - a_v \]

then

\[ a_v - \{m - 1\} + \{\lambda\} \in A. \]

We may assume that for some \( n_0 \equiv n \), \( m - 1 \in a_v \), \( v \equiv n_0 \), and \( m - 1 \not\in a_v \) \((n_0 \equiv v < n)\). If \( n = 1 \), this is Case 1, because we can choose \( m = l \) and thus \( 2l - k \equiv m \) holds. Let be \( n > 1 \). If \( n_0 = 0 \), then the theorem holds by our induction hypothesis. Suppose that \( n_0 \equiv 2 \). Let be \( \mu \equiv v < n_0 \). Then \( |a_{\mu} + a_v| \equiv 2l - k < m \), and there exists an element \( \lambda \in [0, m) - a_{\mu} - a_v \). Put \( b_{\mu} = a_{\mu} - \{m - 1\} \) \((\mu \equiv n_0)\). Here \( b_{\mu} + \{\lambda\} \in A \), \( \|b_{\mu} + \{\lambda\}\| = \|(b_{\mu} + \{\lambda\})b_v\| = \|(b_{\mu} + \{\lambda\})a_v\| \equiv k \), and therefore \( l - 1 \equiv k \) and \( B = \{b_0, \ldots, b_{n_0}\} \in S(k, l - 1, m - 1) \). If \( n_0 = 1 \), since \( n > 1 \), then \( m - 1 \not\in a_1 \) and \( |a_0a_1| \equiv l - 1 \). Thus also \( l - 1 \equiv k \) and \( B = \{b_0\} \in S(k, l - 1, m - 1) \). We can use our induction hypothesis, if \( n_0 \equiv 1 \) and \( g - 1 > 1 \), since both \( g - 1 + k \equiv l - 1 \) (because of \( g + k \equiv l \)) and \( g - 1 < l - 1 \) (because of \( g < l \)) hold, and \( l - 1 \equiv k \equiv 1 \). Therefore we have in this case

\[
\left( \frac{2(l - 1) - k}{g - 1} \right)^{n_0} \leq |B^{g - 1}| = p.
\]

(5)

We can not use the induction hypothesis, when \( g - 1 = 1 \) that is \( g = 2 \). However, (5) holds, because we can apply Theorem 2 for \( k, l - 1 \) and \( g - 1 = 1 \) (Case 2).

On the other hand \( C = \{a_{n_0}, \ldots, a_n\} \in S(k, l, m - 1) \). We can use the induction hypothesis, if \( l \equiv m - 1 \):

\[
(n - n_0) \left( \frac{2l - k}{g} \right)^{2l - k} \leq |C^g| = r.
\]

(6)

If \( l = m \), then this is Case 1, because \( 2l - k \equiv m \). Trivially

\[
\left( \frac{2l - k}{g} \right)^{2l - k} \leq \left( \frac{2(l - 1) - k}{g - 1} \right)^{2(l - 1) - k} \leq \left( \frac{2(l - 1) - k}{l - 1} \right)^{2(l - 1) - k}
\]

(7)

since \( l > g \) and \( g + k - l \equiv 0 \).

Adding (5) and (6), applying (7) we get

\[
n \left( \frac{2l - k}{g} \right)^{2l - k} \leq p + r.
\]

(8)
Denote by $d_v (v < p)$ elements of $B^{q-1}$, and by $c_v (v < r)$ elements of $C^q$. Let $e_v = d_v + \{m - 1\} (v < p)$. Then obviously $|e_v| = q$, $e_v \not\subset e_v (\mu < v < n)$. Moreover for every $v < p$ there exists an index $\mu < n_0$ such that $d_v \subset b_\mu$. Hence $e_v \subset a_\mu$, since $e_v = d_v + \{m - 1\}$ and $a_\mu = b_\mu + \{m - 1\}$. Thus $e_v \subset A^q$, moreover trivially $e_v \not\subset c_v (\mu < p, v < r)$, since $m - 1 \not\subset e_\mu$ and $m - 1 \not\subset c_v$. Consequently $c_0, c_1, \ldots, c_r, e_0, \ldots, e_p$ are distinct elements of $A^q$, that is

$$ p + r \equiv |A^q|,$$

which completes the proof of 3a.

It remains to prove that in Case 3a equality can not hold. If $m = 3$, this is true. Suppose now that $m \geq 3$, and use induction over $m$. Apply the same steps, as in the proof of the inequality. In those cases, where then induction could be used, it can be used here too, that is if $m > 2l - k$, then $m - 1 > 2l - k$. Thus it follows by induction hypothesis that in (5) (and in the theorem) strict inequality holds. Those cases where induction could not be used are settled by Cases 1 and 2. Thus in Case 3a strict inequality always holds.

Case 3b. Suppose that there are $a \in A$ and $\lambda \in [0, m) - a$ such that $m - 1 \in a$ and $a - \{m - 1\} + \{\lambda\} \notin A$. Then $\lambda < m - 1$.

We may assume that the sets are labelled in such a way, that the following relations hold:

$$m - 1 \in a_v, \quad \lambda \in a_v, \quad b_v = a_v - \{m - 1\} + \{\lambda\} \in A \quad (v \leq n_0),$$

$$m - 1 \in a_v, \quad \lambda \in a_v, \quad c_v = a_v - \{m - 1\} + \{\lambda\} \in A \quad (n_0 \leq v < n_1),$$

$$m - 1 \in a_v, \quad \lambda \in a_v \quad (n_1 \leq v < n_2),$$

$$m - 1 \notin a_v \quad (n_2 \leq v < n).$$

Here $1 \leq n_0 \leq n_1 \leq n_2 \leq n$. Put $b_v = a_v (n_0 \leq v < n)$. We have now to prove that $B = \{b_0, \ldots, \hat{b}_n\} \subseteq S(k, l, m)$.

Let be $\mu < v < n$. We must prove that $b_\mu \neq b_v$ and $|b_\mu b_v| \equiv k$.

For $\mu < v < n_0$ or $n_0 \leq \mu < v$ these are obvious. Now let be $\mu < n_0 \equiv v$. Then $b_\mu \in A$, $b_v \subset a_v \in A$, and hence $b_\mu \neq b_v$.

If $n_0 \equiv v < n_1$, then $c_v \in A$; and there are $k$ distinct common elements of $a_\mu$ and $c_v$. $\lambda$ and $m - 1$ are not among these, therefore they are common elements also of $b_\mu$ and $b_v = a_v$.

If $n_1 \equiv v < n_2$, then $|a_\mu a_v| \equiv k$, but $\lambda \notin a_\mu a_v$. If instead of $a_\mu$ we take $b_\mu$, then out of the common elements at most one is lost: $|b_\mu a_v| = |b_\mu b_v| \equiv k - 1$, but $\lambda$, which is common element, does not belong to these $k - 1$ elements. Thus $|b_\mu b_v| \equiv k$.

Finally, if $n_2 \equiv v < n$, then $b_\mu$ and $a_v$ have $k$ common elements. $m - 1$ does not belong to them, since $m - 1 \notin a_v$. Therefore the same $k$ elements are also common elements of $b_\mu$ and $b_v$. Thus $B \subseteq S(k, l, m)$ is proved.

Now we must show, that $|A^q| \equiv |B^q|$. Let $c$ be such a set that $|c| = q$, $c \in B^q$ but $c \not\subset A^q$. Then $c \subset b_v$ for some $v < n$, because of $c \subset B^q$. Obviously $v < n_0$, because if $n_0 \equiv v < n_0$, then $b_v = a_v$ and $c \subset A^q$.

$\lambda \in c$, because $c \subset b_v$ for some $v < n_0$, and $c \subset a_v = b_v + \{m - 1\} - \{\lambda\}$.

On the other hand $m - 1 \notin c$, because of $m - 1 \notin b_v (v < n_0)$.
Let be \( d = c - \{ \lambda \} + \{ m - 1 \} \). Here \( d \subseteq a_v \), that is \( d \in A^v \), since \( c \subseteq b_v \) and \( b_v = a_v - \{ \lambda \} + \{ m - 1 \} \). However, \( d \notin B^v \). If \( d \subseteq b_v \) would hold for some \( v < n \), then obviously \( n_0 \leq v < n_2 \) because in the cases \( v < n_0 \) and \( n_2 \leq v < n \), \( m - 1 \notin b_v \) holds. If \( n_0 \leq v < n_1 \), then \( c \subseteq c_v = a_v - \{ m - 1 \} + \{ \lambda \} \) holds (for such \( v \), for which \( d \subseteq b_v \) and since \( c_v \subseteq A \), follows \( c \subseteq A^v \), which contradicts our supposition. However, if \( d \subseteq b_v \) holds for \( n_1 \leq v < n_2 \), then \( c \subseteq a_v \) because of \( \lambda \in b_v = a_v \), \( m - 1 \notin b_v = a_v \), and this also is a contradiction.

Hereby we associated a set \( d \) to every set \( c \), which is an element of \( B^v \), but is not one of \( A^v \) (to distinct sets \( c \) correspond distinct sets \( d \)) in such a way, that set \( d \) is an element of \( A^v \), but is not one of \( B^v \). From this follows
\[
|A^v| \leq |B^v|.
\]

Since for fixed \( n \) we supposed \( A \) to be the system, for which \( |A^v| \) is minimal, in (9) only equality can hold. However we have
\[
f(b_0, ..., b_n) - f(a_0, ..., a_n) = n_0[-(m - 1) + \lambda] < 0,
\]
which contradicts the maximum property of \( A \). This shows that Case 3b can not occur.

Remarks.
1. In this proof I used the sequence of ideas contained in the proof of Erdős—Chao Ko—Rado’s Theorem 1 (11).
2. We showed also, that equality can hold in Case 1, and here only if \( m = 2l - k \), and \( A \) contains every subset of cardinal number \( l \), or in the trivial case \( g = l \).

The following are all consequences of Theorem 2.

3. Theorem 1 of Erdős—Chao Ko—Rado [1]. If \( 1 \leq l \leq \frac{1}{2} m \) and
\[
A = \{ a_0, ..., a_n \} \subseteq S(1, l, m), \text{ then } n \leq \binom{m - 1}{l - 1}.
\]

Proof. Let \( b_v = (0, m) - a_v \) and \( B = \{ b_0, ..., b_n \} \). Then \( |b_v| = m - l \leq l \), \( |b_v.b_v| = |(0, m) - (a_v + a_v)| \geq m - 2l + 1 \), because \( |a_v + a_v| \leq 2l - 1 \) (\( \mu < v < n \)). We use now Theorem 2 for \( m - 2l + 1 \), \( m - l \) and \( l \) in place of \( k \), \( l \) and \( g \). We can apply the theorem, since \( 1 \leq l \leq m - l \), \( 1 \equiv m - 2l + 1 \leq m - l \), and \( l + (m - 2l + 1) \equiv m - l \). Thus
\[
\binom{m - 1}{l} \leq |B^v|.
\]

Let be \( c \subseteq B^v \). Then there exists a number \( \mu < n \) such that \( c \subseteq b_\mu \). For this \( \{ 0, m \} - b_\mu = c a_\mu = \emptyset \). Thus \( c \notin A \). Consequently
\[
|B^v| + |A| = |B^v| + n \leq \binom{m}{l}.
\]

From this, applying (10),
\[
n \leq \binom{m - 1}{l - 1}.
\]
REMARKS.
1. [1] contains this theorem in a more general form which follows from the form proved here by a simple step, shown in [1].

2. If $2l-k=m$, $|a_v|=l$ ($v<n$), and $\{a_0, ..., a_n\} \in S(k, l, m)$, then trivially $n \leq \binom{m}{l}$, and this estimate is the best possible. If $2l-k<m$, then according to Theorems 1 and 2 of ERDős—CHAO KO—RADO [1], the estimate $n \leq \binom{m-k}{l-k}$ holds in most cases. The estimate is however not true for every case: In [1] an interesting example is cited. Further simple example:

2a. Let $k = l-1$, $|a_v|=l$. Then either $n \leq l+1$ or $n > l+1$.

Consider in the latter case the subsets of $a_0$ having $l-1$ elements. The number of these is $l$, and one of these is included in $a_v$ ($1 \leq v < n$). Thus there exists a set $c \subseteq a_0$ such that $|c| = l-1$, and there exist two sets, for example $a_1$ and $a_2$ for which $c \subseteq a_1$, $c \subseteq a_2$. We showed that if there is a set $c$ for which $|c| = l-1$ and which is included at least in 3 sets $a_v$, then for every $v<n$ $c \subseteq a_v$. As a consequence $n \leq m-l+1$, because there can exist at most as many sets $a_v$ as the number of distinct elements which are not contained in set $c$. That is $n \leq \max(l+1, m-l+1)$, and there is always a system satisfying the equality.

2b. Let $m = 2l-k+1$, $|a_v|=l$ ($v<n$) and $k>1$. Use Theorem 2 for $g = l-k+1$:

$$n \frac{l}{l-k+1} \leq |\{a_0, ..., \hat{a_n}\}^{l-k+1}| = p.$$  \hfill (11)

If $c \in \{a_0, ..., \hat{a_n}\}^{l-k+1}$ then $|[0, m)-c| = l$. Moreover, since $c \subseteq a_v$ for some $v<n$, $|a_v([0, m)-c)| = |a_v-c| = k+1$. Thus $[0, m)-c \notin A$ and the elements of $A$ and the complementary sets of the elements of $A^{l-k+1}$ are distinct, therefore

$$n + p \leq \binom{2l-k+1}{l}.$$

Hence applying (11)

$$n \leq \binom{2l-k}{l} = \binom{m-1}{l}.$$  

Here equality holds if and only if $A$ is the system of all subsets of $[0, m-1)$ having $l$ elements. Namely, according to Remark 2 of Theorem 2 equality in (11) can hold only if the number of elements is $2l-k$ and $A$ is as specified. In this case equality is trivial.

**Theorem 4.** Let $2 \leq k \leq m$. If $A = \{a_0, ..., \hat{a_n}\}$ is a system such that $a_\mu \neq a_v$, $|a_\mu a_v| \equiv k$, $a_v \subset [0, m)$ ($\mu < v < n$), then either

(a) $k+m = 2v \quad n \leq \sum_{i=v}^{m} \binom{m}{i},$

or

(b) $k+m = 2v-1 \quad n \leq \binom{m-1}{v-1} + \sum_{i=v}^{m} \binom{m}{i}.$
Moreover there exists a unique maximal system of sets \( a \subset [0, m) \) and \( |a| \equiv v \) in case (a), and in case (b) a system of sets of the same property and additionally of all the sets satisfying the conditions

\[
a \subset [0, m-1) \quad \text{and} \quad |a| = v - 1.
\]

**Proof.** 1. If \( 1 \leq k \leq l \equiv m \) and \( A = \{a_0, \ldots, a_n\} \in S(k, l, m) \), then \( n \leq \binom{m}{l-k} \).

From Theorem 2 for \( l-k = g \) follows \( n \leq |A^{l-k}| \). However, \( |A^{l-k}| \leq \binom{m}{l-k} \), that is \( n \leq \binom{m}{l-k} \).

If in addition \( l-k < \frac{1}{2}(m-1) \) then \( n \leq \binom{m}{l-k} \leq \binom{m}{l-k+1} \).

2. If \( l < \frac{1}{2}(m+k-1) \) and \( A \) is an arbitrary system satisfying the conditions of Theorem 4, then

\[
|a| + |A_{m-l+k-1}| \leq \binom{m}{m} \leq \binom{m}{l-k+1} = \binom{m}{l-k+1}.
\]

and equality can hold only if \( |a| = 0 \) and \( A_{m-l+k-1} \) consists of all the sets \( a \) such that \( |a| = m-l+k-1 \).

Proof of (12): If \( l < \frac{1}{2}(m+k-1) \), then \( l-k < \frac{1}{2}(m-1) \) thus by 1:

\[
|a| \leq \binom{m}{l-k+1}.
\]

If \( |a| = 0 \), (12) is true. If \( 0 < |a| < \binom{m}{l-k+1} \), we shall show, that

\[
|A_{m-l+k-1}| \leq \binom{m}{m-l+k-1} - |a|.
\]

Let \( c \in (A_i)^{l-k+1} \). Then there exists a number \( v \) such that \( c \subset a_v \), \( |a_v| = l \) and \( a_v \in A \), thus \( |a_v([0, m)-c]| = |a_v-c| = k-1 < k \), and hence \( [0, m)-c \in A \). Since \( |[0, m)-c| = m-l+k-1 \), there are \((A_i)^{l-k+1}\)

sets of cardinal number \( m-l+k-1 \), which can not be elements of \( A \) and \( A_{m-l+k-1} \) respectively. We have

\[
|A_{m-l+k-1}| \leq \binom{m}{m-l+k-1} - |(A_i)^{l-k+1}|.
\]

To complete our proof we must show, that \((A_i)^{l-k+1}| > |A_i|\). This trivially follows from Theorem 2. We can use the theorem because of \( k \geq 2 \), \((l-k+1)+k \equiv l \) and \( l > l-k+1 \) and thus the coefficient (2) is larger than 1. Equality can hold only in the case \( |A_i| = 0 \).

3. \( |A_i| = 0 \) (\( \mu < k \)), thus we have to determine the maximum of \( |A| = |A_1| + \ldots + |A_m| \). By 2 the pairs \( |A_1| + |A_{m-1}| \), \( |A_{k+1}| + |A_{m-2}| \), \ldots are maximal, if the first term is 0. The last pair is \( |A_{\frac{1}{2}(m+k-2)}| + |A_{\frac{1}{2}(m+k)}| \) and here also \( l = \frac{1}{2}(m+k-2) \leq \frac{1}{2}(m+k-1) \). The maximum of \( |A_m| \) is 1. This completes the proof in the case (a).

Similarly in case (b) for the maximal system \( |A_i| = 0 \) (\( \mu < \frac{1}{2}(m+k-1) \)) and \( |A_m| = 1 \). Only the term \( |A_{\frac{1}{2}(m+k-1)}| \) remains. In the Remark 2b we have shown that
\[ |A_{\frac{1}{2}(m+k-1)}| \leq \binom{m-1}{\frac{1}{2}(m+k-1)} = \binom{m-1}{v-1}, \text{ and equality can hold only if } A_{\frac{1}{2}(m+k-1)} \]
is the system of all the sets satisfying the conditions

\[ a \subset [0, m] \quad |a| = \frac{1}{2}(m+k-1). \]

This system trivially satisfies the conditions. Therefore this is the maximal system as stated.

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Bibliography