Reflex delay influenced stability of skateboarding

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Abstract

A simple mechanical model of the skateboard-skater system is constructed in which a PD controller with time delay is implemented as a model of the rider's ankle. Equations of motion of this nonholonomic system are derived with the help of the Appell-Gibbs method and are linearised around the straight uniform motion. The linear stability analysis is carried out analytically using the D-subdivision method. Stability charts and the critical time delay are presented for realistic system parameters. The effect of the longitudinal speed on the stability of the uniform motion is also shown.

The model

We constructed the mechanical model (see Figure 1) based on [1, 2]. Here we considered that the front and the rear suspensions are similar and we consider the balancing effort of the skater as a PD control loop which models the ankle of the rider.

The skateboard is modelled by a massless rod (between the front axle at F and the rear one at R) while the skater is represented by another massless rod (between the points S and C) with a mass point at C. The free rotation between the skater and the board around the longitudinal axes of the board is antagonized by the torque from the PD controller ($M_{\mathrm{PD}} = P\varphi(t-\tau)+D\dot{\varphi}(t-\tau)$), where the reflex time of the skater is considered with the delay τ.

The model consists one geometric constraint, namely the skateboard always move in a parallel plane to the ground and there is a relation between the lurch of the board β and the steering angle δ_{S}, such that $\sin\beta(t)\tan\kappa = \tan\delta_{\mathrm{S}}(t)$, where κ is the complement of the rake angle in the skateboard wheel suspension system. The motion of the skateboard is blocked by three kinematic constraints, i.e. the direction of the velocity at point F and R is determined by the lurch of the board (see Figure 1). We consider that the longitudinal speed of the board is V at any time. Thus, the three constraints are:

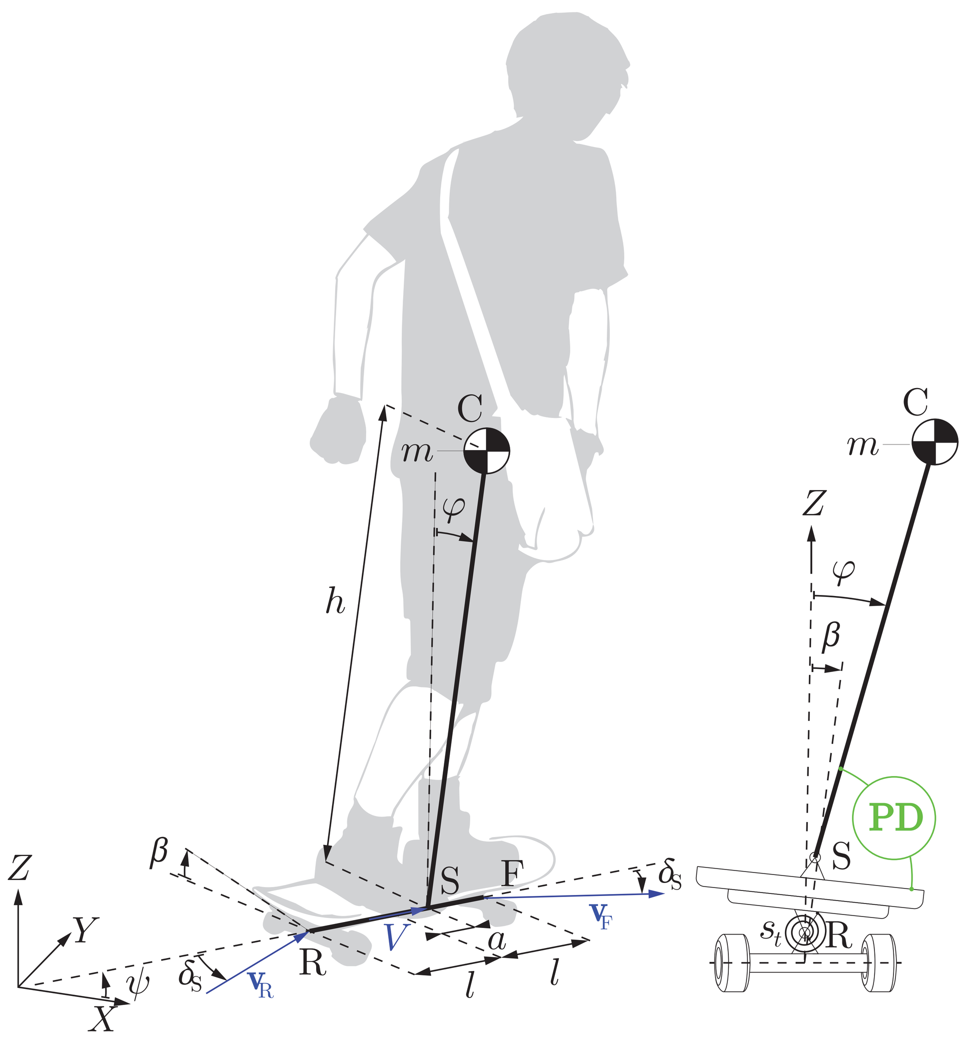

Figure 1: Mechanical model

$$(-\sin\psi+\cos\psi\sin\beta\tan\kappa)\dot{X}+(\cos\psi+\sin\psi\sin\beta\tan\kappa)\dot{Y}+(l-a)\dot{\psi}=0\,, \tag{1a}$$
$$(\sin\psi+\cos\psi\sin\beta\tan\kappa)\dot{X}+(-\cos\psi+\sin\psi\sin\beta\tan\kappa)\dot{Y}+(l+a)\dot{\psi}=0\,, \tag{1b}$$
$$\cos\psi\dot{X}+\sin\psi\dot{Y}=V\,. \tag{1c}$$

The geometric constraint reduces the degrees of freedom of the model by two, so five generalised coordinates are needed (X, Y, ψ, β and φ, see Figure 1). One so-called *pseudo velocity* σ is introduced along the Appell-Gibbs method, what is a powerful method for nonholonomic mechanical systems. For the easier handling two dimensionless parameters $\theta_a := a/l\tan\kappa$ and $\theta_h := h/l\tan\kappa$, and three other parameters, like natural angular frequencies square, are introduced as well: $\alpha_g^2 := g/h$, $\alpha_{s_t}^2 := s_t/(mh^2)$ and $\alpha_V^2 := V^2/h^2$. The lower case p and d are the relative control gains, compared to the spring stiffness of the skateboard suspension ($p := P/s_t$, $d := D/s_t$). With these, we can write the equation of motion:

$$\begin{aligned}\dot{\sigma} =& \frac{1}{2}\alpha_V^2\theta_h^2\sin(2\varphi)\sin^2\left(\operatorname{arcsinh}\left(p\varphi(t-\tau)+d\sigma(t-\tau)\right)\right)+\alpha_g^2\sin\varphi- \\ &-\alpha_{s_t}^2\left(p\varphi(t-\tau)+d\sigma(t-\tau)\right)-\alpha_V^2\theta_h\cos\varphi\sin\left(\operatorname{arcsinh}\left(p\varphi(t-\tau)+d\sigma(t-\tau)\right)\right)- \\ &-\alpha_V\theta_a\frac{p\sigma(t-\tau)+d\dot{\sigma}(t-\tau)}{\sqrt{1+\left(p\varphi(t-\tau)+d\sigma(t-\tau)\right)^2}}\cos\varphi\cos\left(\operatorname{arcsinh}\left(p\varphi(t-\tau)+d\sigma(t-\tau)\right)\right)\,,\end{aligned} \tag{2a}$$

$$\dot{\varphi} = \sigma\,, \tag{2b}$$

$$\frac{\dot{X}}{h} = \alpha_V\left(\cos\psi+\theta_a\sin\psi\sin\left(\operatorname{arcsinh}\left(p\varphi(t-\tau)+d\sigma(t-\tau)\right)\right)\right)\,, \tag{2c}$$

$$\frac{\dot{Y}}{h} = \alpha_V\left(\sin\psi-\theta_a\cos\psi\sin\left(\operatorname{arcsinh}\left(p\varphi(t-\tau)+d\sigma(t-\tau)\right)\right)\right)\,, \tag{2d}$$

$$\dot{\psi} = -\alpha_V\theta_h\sin\left(\operatorname{arcsinh}\left(p\varphi(t-\tau)+d\sigma(t-\tau)\right)\right)\,. \tag{2e}$$

The effect of the speed on the stability

We investigate the rectilinear motion of the skateboard-skater system. The delayed differential equation system of neutral type (2) can be linearised, and it remains a naturally delayed one. Because X, Y and ψ are cyclic coordinates, the first two equations of (2) describe the system uniquely.

During the stability investigation we found *saddle-node* (SN) bifurcation as well as *Hopf* bifurcation with the angular frequency ω (see Figure 4). Such bifurcations occurs in case of the simplest human balancing models, e.g. see the controlled inverted pendulum in [3]. Based on our model, the critical time delay is $\tau_c = 2\alpha_g$, which is $\sqrt{2}$ times greater than for the PD controlled inverted pendulum in [3]. The rectilinear motion can be stable if the reflex delay is chosen from the shaded domains of Figure 2 and Figure 3. These two charts are constructed by means of realistic parameters. The obtained tolerated time delay ranges are close to the average reflex delays of humans (see [4]).

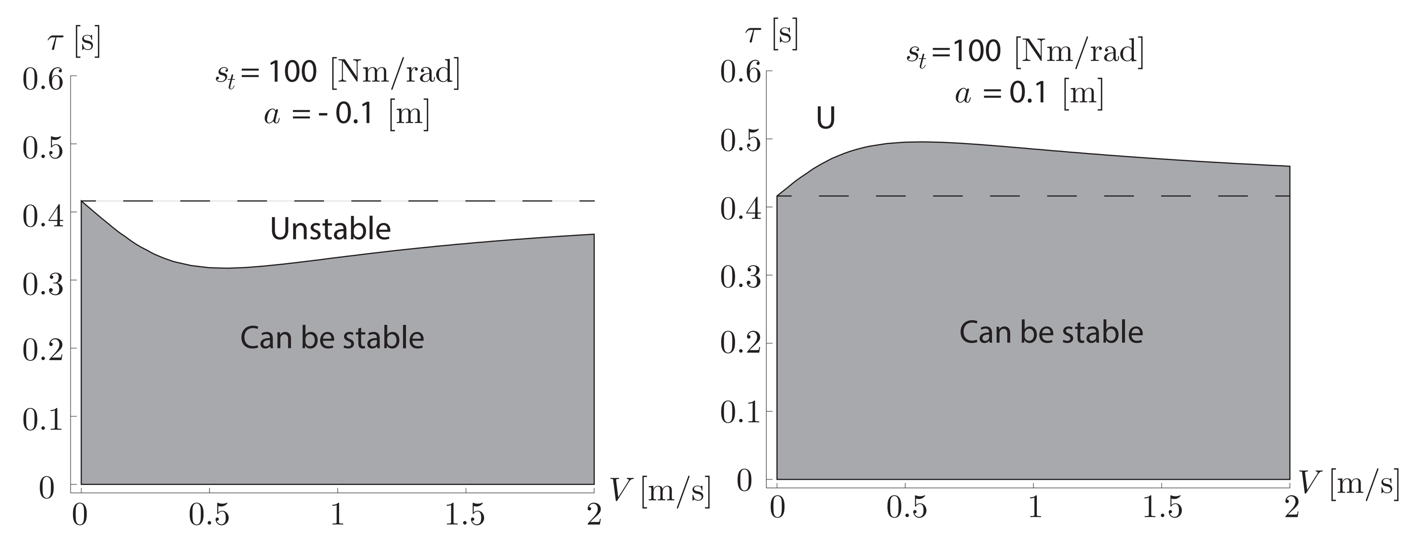

Figure 2: Reflex delay in case of stiff skateboard suspension

In Figure 2, we consider a relatively high stiffness of the skateboard's suspension. As a result, we obtain that the critical delay increases first with the forward speed V when the skater stands in front of the centre of the board ($a > 0$, right panel). For $a < 0$ (left panel), the critical time delay decreases as the skateboard starts moving forward.

If the stiffness of the skateboard's suspension is small enough (see Figure 3) and $a < 0$ (left panel), the critical delay can reach the zero value at a certain speed range, where PD controller can not stabilize. The behaviour of the system is even more strange for $a > 0$ (right panel), namely, there is a speed range where small time delays can also lead to unstable motions.

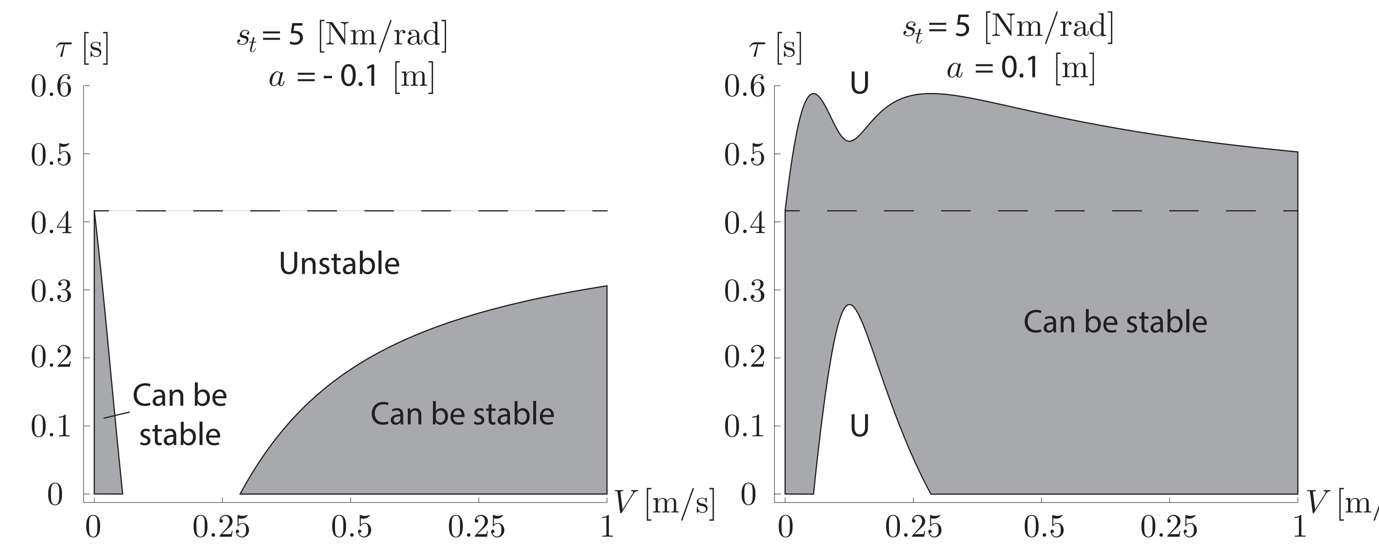

Figure 3: Reflex delay in case of soft skateboard suspension

The effect of the speed can be investigated from another point of view (see in Figure 4). The stable domains are represented in the P-D plane for a fixed delay and for different speeds. It can be observed that the stable domain shrinks and its location modifies while the speed increases. It means that the skater has to tune the control gains and has more difficult task at higher speed.

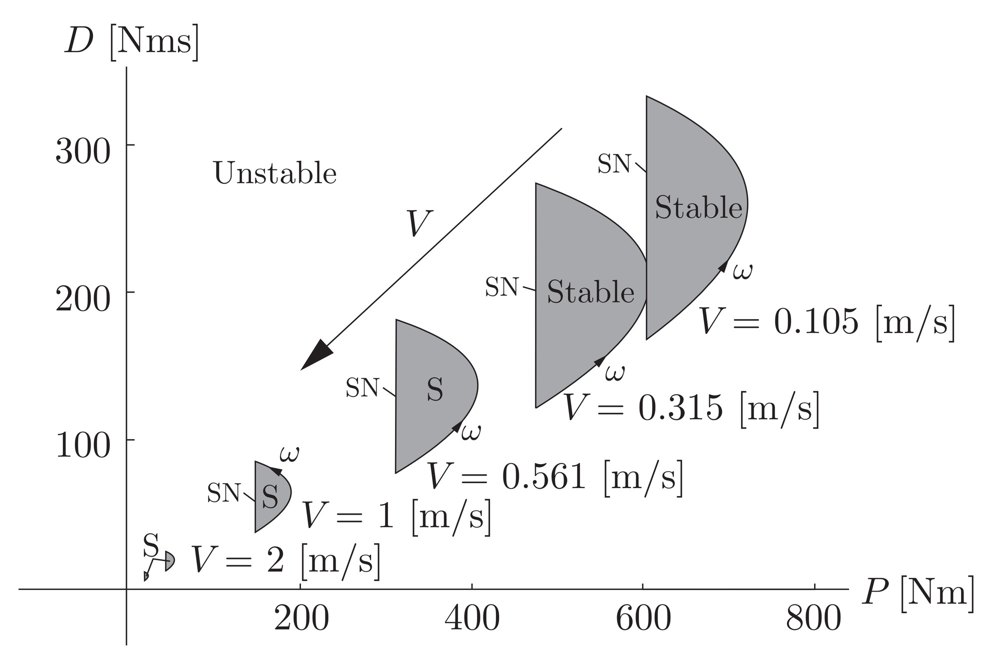

Figure 4: Stable control gains for different velocities

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