Reflex delay influenced stability of skateboarding
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Abstract
A simple mechanical model of the skateboarder system is constructed in which a PD controller with time delay is implemented as a model of the rider’s ankle. Equations of motion of this nonholonomic system are derived with the help of the Appell-Gibbs method and are linearized around the straight uniform motion. The linear stability analysis is carried out analytically using the Describing method. Stability charts and the critical time delay are presented for realistic system parameters. The effect of the longitudinal speed on the stability of the uniform motion is also shown.

The model
We constructed the mechanical model (see Figure 1) based on [1, 3]. Here we considered that the front and the rear suspension systems are identical and we consider the balancing effect of the skater as a PD control loop which models the ankle of the rider.

The skateboard is modelled by a massless rod (between the front axle at F and the rear axle at R) which is represented by another massless rod (between the points C and D) with a mass point at C. The free rotation between the skater and the board around the longitudinal axis of the board is constrained by the torque from the PD controller (Mpd = Pd(t − τ) + Dpd(t − τ)), where the reflex time of the skater is considered with the delay τ.

The model consists of one geometric constraint, namely the skateboard always moves in a parallel plane to the ground and there is a relation between the board speed and the steering angle δs such that sin(δs) tan(δs) = tan(δ(t)), where δs is the complement of the angle in the skateboard wheel suspension system. The motion of the skateboard is biased by three kinematic constraints, i.e. the direction of the velocity at point F and R is determined by the force of the board (see Figure 1). We consider that the longitudinal speed of the board is V at any time. Thus, the three constraints are:

\[ \begin{align*}
\dot{\delta}_s &= \sin(\delta(t)) \tan(\delta(t)) = \tan(\delta(t)), \\
\dot{\phi} &= \theta(t) - \delta(t) - \delta_s(t), \\
\dot{\psi} &= \alpha(t) - \delta(t) - \delta_s(t),
\end{align*} \]

The geometric constraint reduces the degrees of freedom of the model to two, so five generalized coordinates are needed \((X, Y, \theta, \phi, \psi)\). See Figure 1. One so-called pseudo velocity \(\sigma^*\) is introduced according to the Appell-Gibbs method, which is a powerful tool for nonholonomic mechanical systems. For the easier handling two dimensionless parameters \(\delta_s = a/\ell \tan(\delta_s)\) and \(\theta_0 = \theta/\ell \tan(\theta_0)\), and three other parameters, the natural angular frequency parameters, are introduced as well: \(\omega^* = \sqrt{\mathcal{M}^2/\mathcal{L}}\) and \(\omega_0 = V^*/\ell^2\).

The effect of the speed on the stability
We investigate the rectilinear motion of the skateboarder system. The delayed differential equations of neutral type (2) can be linearized, and it remains a naturally delayed system. Because \(X, Y, \theta\) are cyclic coordinates, the first two equations of (2) describe the system (2).

During the stability investigation we found saddle-node (SN) bifurcation as well as Hopf bifurcation with the angular frequency \(\omega^*\) (see Figure 4). Such bifurcations occur in case of the simplest human balancing models, e.g. the controlled inverted pendulum (CIP) [3]. Based on our model, the critical time delay is \(\tau_c = 2\Delta\), which is 2¿ times greater than for the PD controlled inverted pendulum (CIP) [3]. The rectilinear motion can be stable if the reflex delay is chosen from the shaded domain of Figure 2 and Figure 3. These two charts are constructed by mean of realistic parameters. The obtained delayed time delays ranges are close to the average reflex delays of humans (see [4]).

In Figure 2, we consider a relatively high stiffness of the skateboard’s suspension. As a result, we obtain that the critical delay increases with the forward speed \(V\) when the skate stands in front of the device of the board \((\alpha > 0, \text{ right paw})\). For \(\alpha < 0\) (left paw), the critical time delay decreases as the skateboard starts moving forward. If the stiffness of the skateboard’s suspension is small enough (see Figure 3) and \(\alpha < 0\) (left paw), the critical delay can reach zero value at a certain speed range, where PD controller can not stabilize. The behavior of the system is even more strange for \(\alpha > 0\) (right paw), namely, there is a speed range where small time delays can also lead to unstable motions.

The effect of the speed can be investigated from another point of view (see Figure 4). The stable domains are represented in the PD plane for a fixed delay and for different speeds. It can be observed that the stable domain shrinks and its location modifies while the speed increases. It means that the skater has to tune the control gains and has more difficult task at higher speeds.

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References

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