Operator splitting for dissipative delay equations

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Operator splitting methods for a special class of nonlinear partial differential equations with delay are investigated.

1 Introduction

Partial differential equations with delay play an important role in modeling physical, chemical, economical, etc. phenomena, since it is quite natural to assume that past occurrences effect the model. For further motivation, notations and definitions, see the monograph by Bátkai and Piazzera [3].

Consider the following abstract delay differential equation on the Hilbert space H

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Bu(t) + \Phi u_t, \qquad t \geq 0,$$

where $u : [-1, \infty) \to H$ is unknown and the *history function* u_t is defined by $u_t(\sigma) := u(t + \sigma)$ for $\sigma \in [-1, 0]$. We impose the initial conditions $u(0) = x$ and $u_0 = f$. In many cases, as explained below, it is easier to solve the equation without delay and the "pure" delay equation separately. In this case, it is natural to apply some operator splitting procedure described below.

Let us fix a time step $h > 0$ and solve first the equation for $t \in [0, h]$

$$\begin{cases} \dfrac{\mathrm{d}v^{(1)}(t)}{\mathrm{d}t} = \Phi v_t^{(1)}, \\ v^{(1)}(0) = x =: x_1, \\ v_0^{(1)} = f =: f_1, \end{cases} \qquad \text{and then solve} \qquad \begin{cases} \dfrac{\mathrm{d}w^{(1)}(t)}{\mathrm{d}t} = Bw^{(1)}(t), \\ w^{(1)}(0) = y_1 := v^{(1)}(h). \end{cases}$$

To initialize the next step we set

$$x_2 := w^{(1)}(h) \qquad f_2 := v_h^{(1)}.$$

We iterate this procedure and, taking $k \in \mathbb{N}$, solve for $t \in [(k-1)h, kh]$

$$\begin{cases} \dfrac{\mathrm{d}v^{(k)}(t)}{\mathrm{d}t} = \Phi v_t^{(k)}, \\ v^{(k)}((k-1)h) = x_k, \\ v_{(k-1)h}^{(k)} = f_k, \end{cases} \qquad \text{and then we solve} \qquad \begin{cases} \dfrac{\mathrm{d}w^{(k)}(t)}{\mathrm{d}t} = Bw^{(k)}(t), \\ w^{(k)}((k-1)h) = y_k := v^{(k)}(kh). \end{cases}$$

Finally, we set

$$x_{k+1} := w^{(k)}(kh) \qquad f_{k+1} := v_{kh}^{(k)}.$$

The *sequentially split solution* at time level $t = kh$ is then $u^{\mathrm{sq}}(kh) := x_{k+1}$.

We shall show that for fixed $t \in [0, \infty)$ and for $h \to 0$ ($t = kh$, so $k \to \infty$) this split solution $u^{\mathrm{sq}}(kh)$ converges to $u(t)$.

This procedure is especially useful, if we can drastically reduce the computational complexity of the problem by the splitting. This is the case for example if $\Phi = \delta(-1)$ is a point delay, when we can integrate the first split equation explicitly, reducing the problem to solving the second equation, a classical partial differential equation. For a different splitting procedure, designed specifically for distributed delays, we refer to Csomós and Nickel [4] and Bátkai, Csomós and Farkas [1].

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2   Convergence results

The convergence of splitting procedures usually requires some further assumptions on the operators involved. These are necessary to ensure consistency and stability of the splitting procedure. In this note, we make the following assumptions.

Assumption 2.1   1. Suppose that B generates a linear contraction semigroup V on H.

2. Suppose that the linear operator

$$\Psi : W^{1,p}([-1,0];Y) \to X$$

is given by the Stieltjes integral

$$\Psi f = \int_{-1}^{0} \mathrm{d}\eta(\sigma)f(\sigma),$$

with $\eta : [1,0] \to \mathcal{L}(X)$ of bounded variation, such that for some $-1 < \sigma_0 < 0$ the function

$$\eta : [\sigma_0, 0] \to \mathcal{L}(X) \quad \text{is Lipschitz continuous.}$$

3. Suppose that for $x \in D(B)$ we have $\eta(s)x \in D(B)$ for all $s \in [-1,0]$. Moreover, suppose that $\eta : [-1,0] \to \mathcal{L}(D(B))$ is of bounded variation such that for the same $-1 < \sigma_0 < 0$ the function

$$\eta : [\sigma_0, 0] \to \mathcal{L}(D(B)) \quad \text{is Lipschitz continuous.}$$

4. Suppose further that $\eta(-1) = 0$ and that $\lim_{\sigma \to -1} \eta(\sigma) \neq 0$.

5. Suppose that the function $g : H \to H$ leaves $D(B)$ invariant and is globally Lipschitz both on X and $D(B)$, with Lipschitz constant β. We set $\Phi = g \circ \Psi$.

With these assumptions, following a long and tedious path, the first order convergence of the splitting procedure can be shown.

Theorem 2.2 *Suppose that the conditions in Assumption 2.1 hold. Then the sequential splitting is of first order. More precisely, let $x \in D(B^2)$ and $f \in W^{1,p_0}([-1,0];D(B)) \cap \mathrm{Lip}([-1,0];H)$ for some $p_0 > 1$ with $f(0) = x$. Then for every $T_m > 0$ there is a constant $C > 0$ such that*

$$\max_{j=1,\ldots,n} \|u^{\mathrm{sq}}(jh) - u(jh)\| \leq Ch\big(1 + \|x\|_B + \|Bx\|_B + \|f\|_{W^{1,1}(D(B))} + \|f\|_{\mathrm{Lip}(H)}\big)$$

for $T_m = nh$, and $n \in \mathbb{N}$ sufficiently large.

The result is illustrated on a reaction-diffusion problem with delay, where we also included a comparison with the implicit Euler scheme. Figure 1 shows the relative global error for various values of the time step h. The slope p of the fitted line in logarithmic scale approximates the order of the corresponding numerical method. One can see that the sequential splitting performs even better than the implicit Euler scheme.
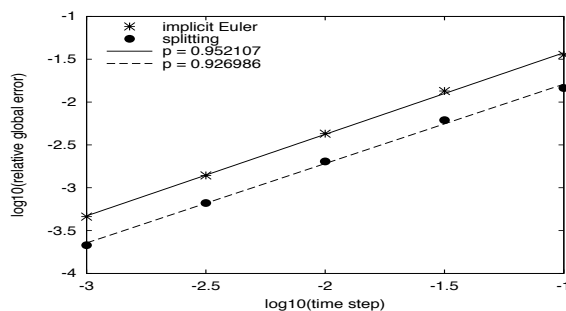

Fig. 1: Order p determined numerically.

The proof uses functional analytic methods and relies heavily on the delay semigroup, see [2].

References

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