Cell mapping methods for investigating micro-chaos

Gergely Gyebrószki†, Gábor Csernák‡

Abstract

Micro-chaos is a phenomenon when small amplitude chaotic oscillations are caused by digital effects (sampling, round-off and processing delay). In previous works, various digitally controlled unstable linear mechanical systems were analysed and the corresponding micro-chaos maps were derived. It was proven [1, 2], that several chaotic attractors coexist in the state space of these maps, and the distance of the farthest attractor from the desired state can be quite large. This is why the phenomenon could be the source of remarkable control error. Cell mapping methods are tools for analysing the global behaviour of nonlinear dynamical systems; determine fixed points, periodic cycles, chaotic attractors and their domain of attraction. This paper shows the application of such methods (Simple Cell Mapping and Generalized Cell Mapping) to investigate micro-chaos maps.

1. The micro-chaos map

1.1. Mechanical model

Consider an inverted pendulum with damping, friction, and digitally implemented PD-control with zero order hold as shown in Figure 1.

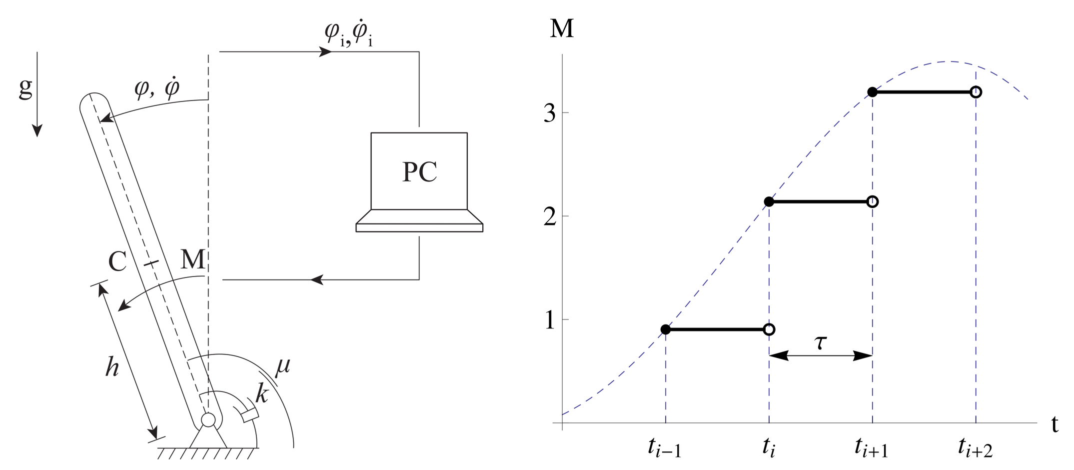

Figure 1: The digitally controlled inverted pendulum and the control torque with respect to time.

The linearized equation of motion of this system is

$$\ddot{\varphi}(t) + 2\beta\dot{\varphi}(t) - \alpha^2\varphi(t) = -P\,\varphi_i - D\,\dot{\varphi}_i - \mathrm{sgn}(\dot{\varphi}(t))\,\mu_0, \qquad t \in [i\tau, (i+1)\tau) \quad (1)$$

where μ_0 is the coefficient of friction and τ is the sampling time. The following notations are introduced: $\hat{\alpha} = \alpha\tau$, $\hat{\beta} = \beta\tau$, $\hat{\gamma}^2 = \hat{\alpha}^2 + \hat{\beta}^2$, $\hat{p} = P\tau^2$, $\hat{d} = D\tau$, $x = \varphi/\varphi_{\mathrm{ref}}$,$v = \omega/\omega_{\mathrm{ref}}$, $\hat{\mu} = \mu_0\,\tau^2/\varphi_{\mathrm{ref}}$, $\mathbf{y} = \begin{bmatrix} x & v \end{bmatrix}^T$.

1.2. Dealing with the friction force

The solution of (1) from a given initial condition $\mathbf{y}_i$ with respect to the dimensionless time $s = \frac{t}{\tau} \in [0, 1)$ can be written as:

$$\mathbf{y}(s) = (\mathbf{\Phi}(s) + \boldsymbol{\delta}(s)\,\mathbf{K})\,\mathbf{y}_i + \mathrm{sgn}(y_{i,2})\,\hat{\mu}\,\boldsymbol{\delta}(s). \quad (2)$$

The dimensionless time s_0, when the solution reaches zero velocity can be expressed from the solution. The condition of crossing the switching line of the friction force within the same sampling interval:

$$0 < s_0 < 1, \quad (3)$$

and the condition of sticking:

$$\mathrm{abs}(\hat{p}\,x_i + \hat{d}\,v_i - \hat{\alpha}^2\,x(s_0)) < \hat{\mu}. \quad (4)$$

Note that the calculated control force does not change within the sampling interval (due to the zero order hold).

1.3. Micro-chaos map

When rounding is applied to the calculated control effort (i.e. to the output of the control system), the formulation of the solution is

$$\mathbf{y}_{i+1} = \begin{cases} \mathbf{\Phi}(1)\,\mathbf{y}_i + \boldsymbol{\delta}(1)\,(\mathrm{Int}(\mathbf{K}\,\mathbf{y}_i) + \mathrm{sgn}(y_{i,2})\,\hat{\mu}) & s_0 > 1 \\ \mathbf{\Phi}(s_0)\,\mathbf{y}_i + \boldsymbol{\delta}(s_0)\,(\mathrm{Int}(\mathbf{K}\,\mathbf{y}_i) + \mathrm{sgn}(y_{i,2})\,\hat{\mu}) & 0 < s_0 < 1 \wedge (4) \\ \mathbf{\Phi}(1-s_0)\,\mathbf{y}^* + \boldsymbol{\delta}(1-s_0)\,(\mathrm{Int}(\mathbf{K}\,\mathbf{y}_i) - \mathrm{sgn}(y_{i,2})\,\hat{\mu}) & 0 < s_0 < 1 \wedge \neg(4) \end{cases} \quad (5)$$

where $\mathbf{y}^* = \mathbf{\Phi}(s_0)\,\mathbf{y}_i + \boldsymbol{\delta}(s_0)\,(\mathrm{Int}(\mathbf{K}\,\mathbf{y}_i) + \mathrm{sgn}(y_{i,2})\,\hat{\mu})$, and the reference angle in $\mathbf{y}$ is $\varphi_{\mathrm{ref}} = r_{\mathrm{out}}\tau^2$, the reference angular velocity is $\omega_{\mathrm{ref}} = r_{\mathrm{out}}\tau$, while $r_{\mathrm{out}}\left[\frac{\mathrm{rad}}{\mathrm{s}^2}\right]$ is the resolution of the actuated control effort. Equation (5) is called *micro-chaos map* or *μ-chaos map*. ($\mathrm{Int}(n)$ denotes rounding towards zero.)

2. Cell mapping methods

Cell mapping (CM) methods are tools for the global investigation of the long term behaviour of nonlinear dynamical systems [3]. Using CM methods, periodic and chaotic solutions of the equations of motion can be found, moreover the basin of attraction can also be determined.

2.1. Simple Cell Mapping

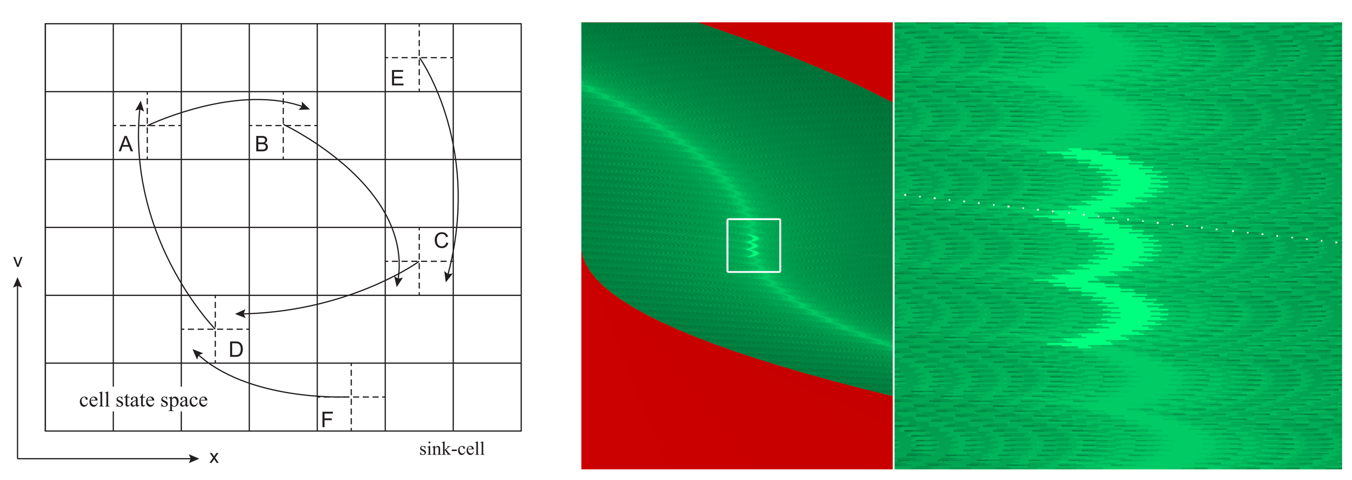

Figure 2: Cell state space with examples: $ABCD$ is a periodic group, E, F are transient cells leading to this group, and an example attractor.

In Simple Cell Mapping (SCM) [3], the state space $\mathbb{R}^N$ of a dynamical system is restricted to a bounded region (Ω), which is divided into *cells*. The region $\mathbb{R}^N \backslash \Omega$ is called *sink-cell*. The state of the examined system is described with the index of the cell corresponding to that state. In SCM only one *image cell* is determined for each cell using the center point of the cell. The image cell corresponding to cell z is denoted by $C(z)$. The mapping $z(n+1) = C(z(n))$ $C : \mathbb{N} \to \mathbb{N}$ is called a SCM. In SCM, two kinds of cells are distinguished

- ▷ *periodic cells*: for which $z = C^m(z)$ is true, for $m \in \mathbb{N}$. In this case z is an m-periodic cell, in an m-periodic group.
- ▷ *transient cells*: which are not periodic cells.

The main procedure of the SCM method is determining the type and properties of every cell. In the context of SCM only periodic motions occur, however:

- ▷ *chaotic behaviour* is represented by periodic groups with relatively high period.
- ▷ *a chaotic attractor* is represented by a set of periodic cells covering a part of the attractor in the state space.

2.2. Generalized Cell Mapping

In Generalized Cell Mapping (GCM) more image cells are determined for each cell. One can determine the state transition matrix

$$\mathbf{P} = p_{ij}, \qquad p_{ij} = \Pr\{z(n+1) = j | z(n) = i\} \quad (6)$$

(p_{ij} is the probability of reaching cell j from cell i in one step) and write the Generalized Cell Mapping

$$\mathbf{z}(n+1) = \mathbf{P}\,\mathbf{z}(n). \quad (7)$$

(7) defines a *Markov-chain*, for which the *persistent* and *transient* states can be determined. *Transient* cells leading to more *persistent groups* form the basin boundary.

3. Results

Utilizing CM methods, the preliminary expected chaotic attractors and their basin of attraction can be found. Considering friction, it can be seen, that the unstable fixed points of the map turn to sticking regions, but some chaotic attractors persist (Fig. 3).

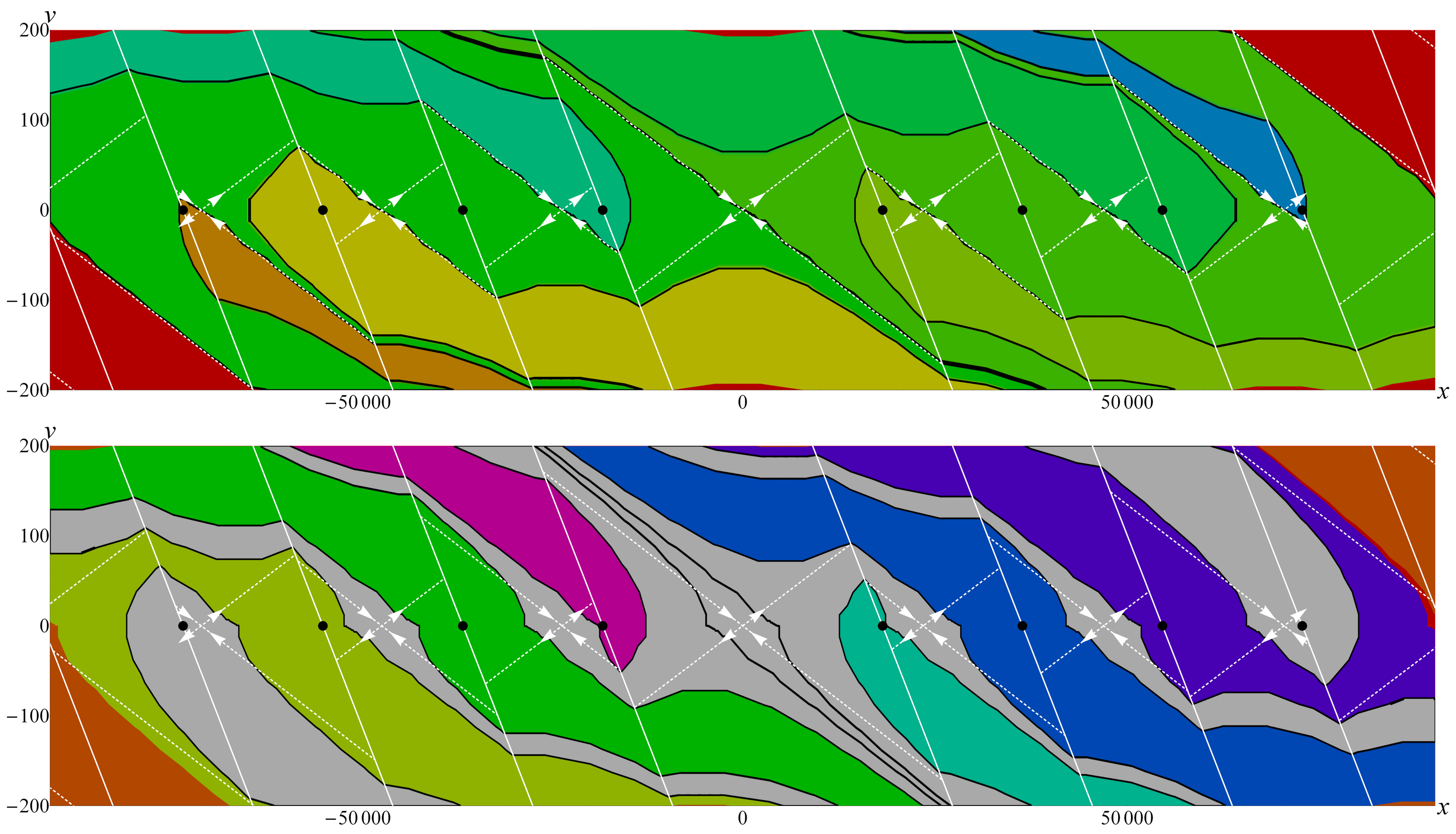

Figure 3: Cell mapping results without (*upper fig.*) and with (*lower fig.*) friction, using realistic parameters. White: switching lines, white-dashed: stable and unstable manifolds of fixed points, black dots: expected location of attractors.

Acknowledgements

This research was supported by the Hungarian National Science Foundation under grant no. OTKA K 83890.

References

[1] G. Csernák, G. Stépán (2010) Digital Control as Source of Chaotic Behavior. *International Journal of Bifurcation and Chaos* **20(5)**: pp. 1365-1378

[2] G. Csernák, G. Stépán (2011) Sampling and Round-off, as Sources of Chaos in PD-controlled Systems. *Proceedings of the 19th Mediterranean Conference on Control and Automation*

[3] J. A. W. van der Spek (1994) Cell Mapping Methods: Modifications and Extensions. *PhD Thesis*, Eindhoven University of Technology, Eindhoven

†Gergely Gyebrószki (gyebro@mm.bme.hu), Department of Applied Mechanics, Budapest University of Technology and Economics, Budapest, Hungary

‡Gábor Csernák (csernak@mm.bme.hu), MTA-BME Research Group on Dynamics of Machines and Vehicles, Budapest, Hungary