Fluid vacation model with Markov modulated load and gated discipline

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ABSTRACT
In this paper we analyze a fluid vacation model with gated discipline. The fluid source is modulated by a background continuous-time Markov chain. The fluid is removed during the service period by constant rate.

We adapt the descendant set approach used in polling models to the continuous fluid model. This enables to establish the steady-state relationship on Laplace transform level among the joint distributions of the fluid level and the state of the modulating Markov chain at end of vacation and at start of vacation. The main results of the paper are the steady-state vector LT and mean of the fluid level at arbitrary epoch in terms of the previously determined quantities at the vacation end and vacation start epochs. We present numerical examples to illustrate the numerical solution.

Categories and Subject Descriptors
G.3 [**Mathematics of Computing**]: Probability and Statistics—*Queueing theory*

General Terms
Theory, Performance

Keywords
queueing theory, fluid queue, Markov modulated fluid source, vacation model

1. INTRODUCTION
Vacation model is an extension of the classical queue by adding vacation periods without service. This implies the necessity of the rule controlling the condition, under which the vacation period starts. In the context of vacation model this rule is called as service discipline. The most commonly used service disciplines are the exhaustive, the gated, the non-exhaustive and the limited-K. There is a huge literature on the classical vacation models, see e.g. the survey of Doshi [3] or the book of Takagi[9] and the references herein.

One fundamental property of the vacation models is the stochastic decomposition property [5]. According to this property the steady-state number of customers in the system can be decomposed to the sum of two independent random variables. One of them is the steady-state number of customers in the corresponding queue (having the same model parameters as the vacation model) and the other is the steady-state number of customers presents in the system at arbitrary epoch in the vacation period. Another characteristic difference comparing to the classical queue arises in the stability condition due to introduction of the different service disciplines. The necessary and sufficient stability condition of the classical (e.g. $M/G/1$) queue $\rho < 1$, where ρ is the steady-state mean of the workload arriving to the system during a time unit, becomes in the corresponding vacation model only necessary condition. Depending on the applied service discipline a supplementary condition can be needed for the sufficiency. This is because several disciplines, like e.g. the limited-K one, set a load independent limit on the mean amount of work, which can be completed during a service period. For the stability on queueing models with such disciplines we refer the reader to [4].

Vacation models are general tools in performance modeling of stochastic systems. They are used among others in modeling of modern telecommunication networks, energy saving schemes or machine repair models [10].

In this paper we extend the concept and analysis of vacation model to fluid input for the case when the fluid source is modulated by a background Markov chain. We consider the fluid vacation model with the gated discipline. The major novelty of this model from the point of view of the analysis is the continuous nature of the fluid which means continuous state-space instead of the discrete one of the classical vacation model. This requires different analysis techniques.

The only paper on related fluid queueing model, which the authors know, is the paper from Czerniak and Yechiali [2] on fluid polling model. In their model the load and the fluid service rate of stations are constant and the only stochastic ingredient of the model is the switchover time. In contrast to it we consider a fluid vacation model in which the load has stochastic nature, because the fluid source is modulated by a background Markov chain.

The main idea of the analysis is the extension of the descendant set approach (see in [1]) to the fluid model context. This together with the transient analysis of the input fluid flow enable to establish a relation on the Laplace transform (LT) level for the evolution of the joint fluid level and the state of the background Markov chain between the consecutive vacation end and vacation start epochs. Based on it the steady-state vector LT, the steady-state probability vector and the first two steady-state vector moments of the fluid level are determined at the vacation end and vacation start epochs. Afterwards we derive a relation for the steady-state vector LT and mean of the fluid level at arbitrary epoch in terms of the previously determined quantities at the vacation end and vacation start epochs.

The rest of the paper is organized as follows. In section 2 we present the fluid vacation model and the concept of embedding matrix LTs needed to the extension of the descendant set approach to fluid model. In section 3 we establish the governing equations of the model describing the evolution of the state of the system between the consecutive vacation end and start epochs. The derivation of steady-state results at vacation end and start epochs follows in section 4. Section 5 deals with the analysis of the steady-state vector LT and mean of the fluid level at arbitrary epoch. Section 6 is devoted to the numerical solution. Section 7 closes the paper with our final remarks.

2. MODEL AND NOTATION

2.1 Model description

We consider a fluid vacation model with Markov modulated load and gated discipline. The model has an infinite fluid buffer.

The input fluid flow of the buffer is determined by a modulating CTMC ($\Omega(t)$ for $t \geq 0$) with state space $\mathcal{S} = \{1, \ldots, L\}$ and generator $\mathbf{Q}$. When this Markov chain is in state j ($\Omega(t) = j$) then fluid flows to the buffer at rate r_j for $j \in \{1, \ldots, L\}$. We define the diagonal matrix $[\mathbf{R}] = diag(r_1, \ldots, r_L)$. During the service period the server removes fluid from the fluid buffer at finite rate $d > 0$. Consequently, when the overall Markov chain is in state j ($\Omega(t) = j$) then the fluid level of the buffer during the service period changes at rate $r_j - d$, otherwise during the vacation periods it changes at rate r_j, because there is no service. In the vacation model the length of the service period is determined by the applied discipline. In this work we consider the gated discipline. Under gated discipline only the fluid is removed during the service period, which is present at the station already at time of the start of that service period. The server occasionally takes vacations according to the gated discipline. The consecutive vacation times are independent and identically distributed (i.i.d.). The random variable of the vacation time, its probability distribution function (pdf), its LT and its mean are denoted by $\widetilde{\sigma}$, $\sigma(t) = \frac{d}{dt}Pr(\widetilde{\sigma} < t)$ and $\sigma^*(s) = E(e^{-s\widetilde{\sigma}})$, $\sigma = E(\widetilde{\sigma})$, respectively. We define the cycle time (or simple cycle) as the time between the starts of two consecutive service periods.

We set the following assumptions on the fluid vacation model:

- **A.1** The generator matrix $\mathbf{Q}$ of the modulating CTMC is irreducible.
- **A.2** The fluid rates are positive and finite, i.e. $r_j > 0$ for $j \in \{1, \ldots, L\}$.

Let $\boldsymbol{\pi}$ be the stationary probability vector of the modulating Markov chain. Due to assumption **A.1** the equations

$$\boldsymbol{\pi}\mathbf{Q} = 0,$$
$$\boldsymbol{\pi}\mathbf{e} = 1. \tag{1}$$

uniquely determine $\boldsymbol{\pi}$, where $\mathbf{e}$ is the $L \times 1$ column vector of ones. The stationary fluid flow rate, λ, and the utilization ρ, is given as

$$\lambda = \boldsymbol{\pi}\mathbf{R}\mathbf{e} \tag{2}$$

and

$$\rho = \frac{\lambda}{d}, \tag{3}$$

respectively. The necessary condition of the stability of the fluid vacation model is that mean fluid arrival rate $\lambda = \boldsymbol{\pi}\mathbf{R}\mathbf{e}$ is less than d, which is equivalent with

$$\rho < 1.$$

If the work during a service period were limited, like e.g. in case of a model with time-limited discipline, then further restriction would be needed for the sufficiency. However the model with the gated discipline does not set any load-independent work limit during a service period, therefore the above necessary condition is also a sufficient one for the stability of the system.

For the i,j-th element of the matrix $\mathbf{Z}$ the notation $[\mathbf{Z}]_{ij}$ is used. Similarly $[\mathbf{z}]_j$ denotes the j-th element of vector $\mathbf{z}$. When $\mathbf{X}^*(s)$, $Re(s) \geq 0$ is a matrix LT, $\mathbf{X}^{(k)}$ denotes its k-th ($k \geq 1$) moment, i.e., $\mathbf{X}^{(k)} = (-1)^k \frac{d^k}{ds^k}\mathbf{X}^*(s)|_{s=0}$ and $\mathbf{X}$ denotes its value at $s = 0$, i.e., $\mathbf{X} = \mathbf{X}^*(0)$. Similarly when $\mathbf{x}^*(s)$, $Re(s) \geq 0$ is a vector LT, $\mathbf{x}^{(k)}$ denotes its k-th ($k \geq 1$) moment, i.e., $\mathbf{x}^{(k)} = (-1)^k \frac{d^k}{ds^k}\mathbf{x}^*(s)|_{s=0}$ and $\mathbf{x}$ denotes its value at $s = 0$, i.e., $\mathbf{x} = \mathbf{x}^*(0)$. $\mathbf{x}$ is called also as 0-th moment.

2.2 Embedded matrix LTs

Let $\mathbf{Z}$ be an $L \times L$ rate matrix which has the following properties:

- the diagonal elements are negative ($\mathbf{Z}_{i,i} < 0$) and the other elements are non-negative ($\mathbf{Z}_{i,j} \geq 0$, for $i \neq j$),
- the row sums are zero.

The matrix $e^{\mathbf{Z}t}$, for $t \geq 0$, has a transition probability matrix interpretation, which means that $e^{\mathbf{Z}t}$ is non-negative and stochastic, i.e. $e^{\mathbf{Z}t}\mathbf{e} = \mathbf{e}$. The well-known connection between $\mathbf{Z}$ and $e^{\mathbf{Z}t}$ is given as

$$\mathbf{Z} = \lim_{t \to 0} \frac{e^{\mathbf{Z}t} - \mathbf{I}}{t},$$

where $\mathbf{I}$ is $L \times L$ identity matrix. For $Re(v) \geq 0$ let

$$\mathbf{H}(v) = \mathbf{T}v - \mathbf{Z} \tag{4}$$

be a linear $L \times L$ matrix function of the complex variable v, where $\mathbf{Z}$ is a rate matrix and $\mathbf{T}$ is diagonal and its diagonal elements are positive, i.e. $[\mathbf{T}]_{j,j} > 0$ for $j \in \{1, \ldots, L\}$. That is $\mathbf{Z}$ and $\mathbf{T}$ are real. The matrix function $-\mathbf{H}(v)$ has the following properties:

- **P.1** it is analytic for $Re(v) \geq 0$,
- **P.2** it is a rate matrix when $v = 0$,
- **P.3** the real part of its diagonal elements are negative for $Re(v) \geq 0$, i.e. $(Re(-\mathbf{H}_{j,j}(v)) < 0)$,
- **P.4** it is a diagonal dominant matrix for $Re(v) \geq 0$, i.e. $|Re(-\mathbf{H}_{j,j}(v))| \geq \sum_{k \neq j} | - \mathbf{H}_{j,k}(v))|$.

We define the operator $\mathcal{O}()$ on a complex variable v and on a linear matrix function $\mathbf{G}(v) = \mathbf{G_1}v + \mathbf{G_2}$ as the operator performing the substitution $v \to \mathbf{H}(v)$. That is $\mathcal{O}(v) = \mathbf{H}(v) = \mathbf{T}v - \mathbf{Z}$ and $\mathcal{O}(\mathbf{G}(v)) = \mathbf{G_1}\mathbf{H}(v) + \mathbf{G_2} = \mathbf{G_1}\mathbf{T}v - \mathbf{G_1}\mathbf{Z} + \mathbf{G_2}$, which are linear matrix functions as well. The order of non-commuting matrices are kept according to this definition. The multifold operator $\mathcal{O}^k(\bullet)$ is defined recursively as

$$\mathcal{O}^k(\bullet) = \mathcal{O}(\mathcal{O}^{k-1}(\bullet)), \quad k \geq 1,$$

where $\mathcal{O}^0(\bullet) = \bullet$ by definition. If $p(x) \geq 0$ for $x \geq 0$ and v is the complex argument of the LT $\int_{x=0}^{\infty} p(x)e^{-vx}dx$ then

$$\int_{x=0}^{\infty} p(x)e^{-\mathcal{O}(v)x}dx = \int_{x=0}^{\infty} p(x)e^{-\mathbf{H}(v)x}dx \tag{5}$$

is an $L \times L$ matrix LT.

According to the Gerschgorin Circle Theorem [6] each eigenvalue of $-\mathbf{H}(v)$ is in one of the disks $\{z : |z - (-\mathbf{H}_{j,j}(v))| \leq \sum_{k \neq j} | - \mathbf{H}_{j,k}(v)|\}$ (i.e. disks in complex z-plane with center at $(-\mathbf{H}_{j,j}(v))$ and radius $\sum_{k \neq j} | - \mathbf{H}_{j,k}(v)|$), for $\forall j \in \{1, \ldots, L\}$. This together with properties **P.3** and **P.4** imply that the eigenvalues of $-\mathbf{H}(v)$ have negative or zero real part for $Re(v) \geq 0$ and the eigenvalues of $e^{-\mathbf{H}(v)}$ are inside the unit disk. When $-\mathbf{H}(v) = \mathbf{B}^{-1}\mathbf{\Lambda}\mathbf{B} =$ is the Jordan decomposition of $-\mathbf{H}(v)$ we have

$$\| \int_{x=0}^{\infty} p(x)e^{-\mathbf{H}(v)x}dx\| \leq \|\mathbf{B}^{-1}\| \int_{x=0}^{\infty} p(x)\|e^{\mathbf{\Lambda x}}\| \; dx \; \|\mathbf{B}\| \leq \int_{x=0}^{\infty} p(x)dx \; L^2 \; \|\mathbf{B}^{-1}\| \; \|\mathbf{B}\|$$

Consequently, (5) is finite when $\int_{x=0}^{\infty} p(x)dx$ is finite.

Applying operator $\mathcal{O}()$ k times on v and multiplying by (-1), $-\mathcal{O}^k(v)$, results in a linear matrix function of size $L \times L$ with the following properties:

- $-\mathcal{O}^k(v)$ is analytic for $Re(v) \geq 0$ (due to **P.1** of $-\mathbf{H}(v)$),
- $- \mathcal{O}^k(v)\big|_{v=0}$ is also a rate matrix (**P.2**), since multiplying rate matrix $\mathbf{Z}$ any times by positive diagonal matrices from left results in a rate matrix and the sum of $L \times L$ rate matrices is also an $L \times L$ rate matrix,
- $-\mathcal{O}^k(v)$ has also the properties **P.3** and **P.4**, which together with the argument on the eigenvalues of $-\mathbf{H}(v)$ above have the consequence that the $L \times L$ matrix LT $\int_{x=0}^{\infty} p(x)e^{-\mathcal{O}^k(v)x}dx$ converges for $Re(v) \geq 0$.

It follows from the recursive definition of the multifold operator $\mathcal{O}^k()$ that the matrix LT $\int_{x=0}^{\infty} p(x)e^{-\mathcal{O}^k(v)x}dx$ is created by consecutive embedding of the matrix $\mathbf{H}(v)$ in the previous matrix LT and therefore we call this matrix LT as *embedded matrix LT*.

The order of matrix and scalar $\mathbf{T}v$ in the definition of $\mathbf{H}(v)$ is crucial in order to ensure the validity of the properties **P.2** and **P.4** for the matrix function $-\mathcal{O}^k(v)$. In spite of this in section 7 we will show that the order of matrix and scalar in the definition of $\mathbf{H}(v)$ is interchangeable without effecting the results derived throughout this paper.

2.3 The properties of the limit $\lim_{\ell \to \infty} \mathcal{O}^{\ell}(v)$

Proposition 1. *The following statements hold for* $\lim_{\ell \to \infty} \mathcal{O}^{\ell}(v)$*:*

- *The limit* $\lim_{\ell \to \infty} \mathcal{O}^{(\ell)}(v)$ *is independent of* v *and it is unique.*
- *If* $\mathbf{T} < \mathbf{I}$ *then the limit* $\lim_{\ell \to \infty} \mathcal{O}^{(\ell)}(v)$ *can be expressed as*

$$\lim_{\ell \to \infty} \mathcal{O}^{(\ell)}(v) \quad = \quad (\mathbf{T} - \mathbf{I})^{-1}\mathbf{Z}. \tag{6}$$

Proof. Applying the operator $\mathcal{O}()$ on v gives

$$\mathcal{O}^{(1)}(v) \quad = \quad \mathbf{T}v - \mathbf{Z}. \tag{7}$$

Now applying the operator $\mathcal{O}()$ in (7) $\ell - 1$ times and taking the limit $\ell \to \infty$ leads to

$$\lim_{\ell \to \infty} \mathcal{O}^{(\ell)}(v) \quad = \quad \mathbf{T} \lim_{\ell \to \infty} \mathcal{O}^{(\ell-1)}(v) - \mathbf{Z}. \tag{8}$$

Multiplying (8) by $\mathbf{T}$ and subtracting matrix $\mathbf{Z}$ yields

$$\mathbf{T} \lim_{\ell \to \infty} \mathcal{O}^{(\ell)}(v) - \mathbf{Z} = \mathbf{T} \left(\mathbf{T} \lim_{\ell \to \infty} \mathcal{O}^{(\ell-1)}(v) - \mathbf{Z} \right) - \mathbf{Z}. \tag{9}$$

Multiplying (9) by $\mathbf{T}^n$, for $n \geq 0$, leads to

$$\begin{aligned} &\mathbf{T}^n \left(\mathbf{T} \lim_{\ell \to \infty} \mathcal{O}^{(\ell)}(v) - \mathbf{Z} \right) = \\ &\mathbf{T}^{n+1} \left(\mathbf{T} \lim_{\ell \to \infty} \mathcal{O}^{(\ell-1)}(v) - \mathbf{Z} \right) - \mathbf{T}^n\mathbf{Z}. \end{aligned} \tag{10}$$

Solving (10) by recursive substitution for $n \geq 0$ results in

$$\begin{aligned} &\mathbf{T} \lim_{\ell \to \infty} \mathcal{O}^{(\ell)}(v) - \mathbf{Z} = \\ &\lim_{n \to \infty} \mathbf{T}^n \left(\mathbf{T} \lim_{\ell \to \infty} \mathcal{O}^{(\ell)}(v) - \mathbf{Z} \right) - \sum_{n=0}^{\infty} \mathbf{T}^n\mathbf{Z}. \end{aligned} \tag{11}$$

In the following we utilize that matrix $\mathbf{T}^n$ is diagonal for $n \geq 0$. If there exists row index k, for which $[\mathbf{T}]_{k,k} \geq 1$ holds, then every elements of the k-th row of matrix $\left(-\sum_{n=0}^{\infty}\mathbf{T}^n\mathbf{Z}\right)$ do not have a limit in absolute value. It follows that every elements of the k-th row of also the matrix on the left hand side (l.h.s.) of (11) behave on the same way, i.e. they do not have a limit in absolute value. On the other hand if there exists row index l, for which $[\mathbf{T}]_{l,l} < 1$ holds, then the elements of the l-th row of $\lim_{n\to\infty}\mathbf{T}^n\left(\mathbf{T}\lim_{\ell\to\infty}\mathcal{O}^{(\ell)}(v) - \mathbf{Z}\right)$ vanish and the elements of the l-th row of $\sum_{n=0}^{\infty}\mathbf{T}^n\mathbf{Z}$ are all finite and independent of v. Thus in this case the elements of the l-th row of matrix on the l.h.s. of (11) are all finite and independent of v. The above arguments together imply that for each row of matrix on the l.h.s. of (11) either every elements of it go to infinity (in absolute value) or they are finite, well-determined values being independent of v. This together with (8) gives the first statement of the proposition.

In addition if the condition $\mathbf{T} < \mathbf{I}$ holds then (11) can be further rearranged by using the formula for the sum of geometrical series, which leads to

$$\mathbf{T}\lim_{\ell\to\infty}\mathcal{O}^{(\ell)}(v) - \mathbf{Z} \quad = \quad -(\mathbf{I}-\mathbf{T})^{-1}\mathbf{Z}. \tag{12}$$

This relation can be obtained also from (9) by applying equivalent rearrangements on it, which is allowed when the condition $\mathbf{T} < \mathbf{I}$ holds, since it ensures that every terms in (9) are finite.

Applying (12) in (8) and rearranging it results in the second statement of the proposition. ☐

3. THE GOVERNING EQUATIONS OF THE SYSTEM

3.1 Transient analysis of the arriving fluid

In this section we consider the accumulated fluid during time $t \geq 0$. More precisely we derive the matrix LT of the fluid flowing into the buffer as a function of time, where the rows and columns of the matrix LT represent the initial and the final states of the modulating Markov chain.

Let $Y(t) \in \mathbb{R}^+$ be the accumulated fluid arrived at the buffer until time t and $\mathbf{A}(t,y)$ be the transition density matrix composed by elements

$$\mathbf{A}_{j,k}(t,y) = \frac{\partial}{\partial y}Pr(\Omega(t)=k, Y(t)<y|\Omega(0)=j, Y(0)=0).$$

The fluid level is zero at $t=0$ ($Y(0)=0$) with probability 1. It follows that for $t=0$ the transition density matrix is given as

$$\mathbf{A}(0,y) = \delta(y)\mathbf{I}, \tag{13}$$

where $\delta(y)$ stands for the unit impulse function at y=0. Furthermore the accumulated fluid is greater than zero for $t>0$ ($Y(t)>0$) due to assumption **A.2**, that is

$$\mathbf{A}(t,0) = \mathbf{0}, \quad t>0, \tag{14}$$

where $\mathbf{0}$ stands for the $L \times L$ zero matrix.

We define the following transforms of matrix $\mathbf{A}(t,y)$

$$\tilde{\mathbf{A}}^*(s,y) = \int_{t=0}^{\infty}\mathbf{A}(t,y)e^{-st}dt,$$

$$\mathbf{A}^*(t,v) = \int_{y=0}^{\infty}\mathbf{A}(t,y)e^{-vy}dy,$$

$$\mathbf{A}^{**}(s,v) = \int_{y=0}^{\infty}\mathbf{A}^*(s,y)e^{-vy}\,dy.$$

Proposition 2. *In the fluid vacation model the matrix LT of the fluid flowing into the buffer in interval $(0,t]$ can be expressed as*

$$\mathbf{A}^*(t,v) = e^{-t(\mathbf{R}v-\mathbf{Q})}. \tag{15}$$

PROOF. The Markov process $\{\Omega(t), Y(t)\}$ characterizes a homogenous first order fluid model, whose transient behavior can be described by the forward Kolmogorov equations

$$\frac{\partial}{\partial t}\mathbf{A}(t,y) + \frac{\partial}{\partial y}\mathbf{A}(t,y)\mathbf{R} = \mathbf{A}(t,y)\mathbf{Q}, \tag{16}$$

with initial conditions (13) and (14). Taking the LT of (16) with respect to t yields

$$\tilde{\mathbf{A}}^*(s,y)s - \mathbf{A}(0,y) + \frac{\partial}{\partial y}\tilde{\mathbf{A}}^*(s,y)\mathbf{R} = \tilde{\mathbf{A}}^*(s,y)\mathbf{Q}. \tag{17}$$

Now taking the LT of (17) with respect to y we have

$$\mathbf{A}^{**}(s,v)s - \mathbf{A}^*(0,v) + \left(\mathbf{A}^{**}(s,v)v - \tilde{\mathbf{A}}^*(s,0)\right)\mathbf{R} = \mathbf{A}^{**}(s,v)\mathbf{Q}, \tag{18}$$

where $\mathbf{A}^*(0,v) = \mathbf{I}$ and $\tilde{\mathbf{A}}^*(s,0) = \mathbf{0}$ according to (13) and (14). Applying them in (18) yields

$$\mathbf{A}^{**}(s,v)s - \mathbf{I} + \mathbf{A}^{**}(s,v)v\mathbf{R} = \mathbf{A}^{**}(s,v)\mathbf{Q},$$

from which by rearrangement we get

$$\mathbf{A}^{**}(s,v) = (\mathbf{I}s + \mathbf{R}v - \mathbf{Q})^{-1}. \tag{19}$$

The statement of the theorem comes by taking the inverse Laplace transform of (19) with respect to s and rearranging it. ☐

3.2 The governing equations of the system at vacation start and end epochs

Let $X(t) \in \mathbb{R}^+$ denote the fluid level in the buffer at time t and $t^f(\ell)$ for $\ell \geq 0$ be the time at the end of the vacation in the ℓ-th cycle. We define the $1 \times L$ row vector $\mathbf{f}(\ell,x)$ by its elements as

$$[\mathbf{f}(\ell,x)]_j = \frac{d}{dx}Pr(\Omega(t^f(\ell)) = j, X(t^f(\ell)) < x), \; j \in \Omega,$$

and its LT as

$$\mathbf{f}^*(\ell,v) = \int_{x=0}^{\infty}\mathbf{f}(\ell,x)e^{-vx}dx.$$

We also define the steady-state vector pdf and the steady-state vector LT of the fluid level at end of vacation, the $1 \times L$

row vectors $\mathbf{f}(x)$ and $\mathbf{f}^*(v)$, respectively, as

$$\mathbf{f}(x) = \lim_{\ell\to\infty} \mathbf{f}(\ell, x),$$
$$\mathbf{f}^*(v) = \lim_{\ell\to\infty} \mathbf{f}^*(\ell, v).$$

Analogously let $t^m(\ell)$ be the time at the start of vacation in the ℓ-th cycle for $\ell \geq 1$. The $1 \times L$ row vector $\mathbf{m}(\ell, x)$ is defined by its elements as

$$[\mathbf{m}(\ell, x)]_j = \frac{d}{dx} Pr(\Omega(t^m(\ell)) = j, X(t^m(\ell)) < x),\ j \in \Omega,$$

and its LT is given by

$$\mathbf{m}^*(\ell, v) = \int_{x=0}^{\infty} \mathbf{m}(\ell, x) e^{-vx} dx.$$

Now we also define the steady-state vector pdf and the steady-state vector LT of the fluid level at start of vacation, the $1 \times L$ row vectors $\mathbf{m}(x)$ and $\mathbf{m}^*(v)$, respectively, as

$$\mathbf{m}(x) = \lim_{\ell\to\infty} \mathbf{m}(\ell, x),$$
$$\mathbf{m}^*(v) = \lim_{\ell\to\infty} \mathbf{m}^*(\ell, v).$$

Furthermore we introduce a notation for the LT with respect to the $L \times L$ matrix function $\mathbf{H}(v)$ as follows

$$\mathbf{g}^*(\mathbf{H}(v)) = \int_{x=0}^{\infty} \mathbf{g}(x) e^{-\mathbf{H}(v)x} dx,$$

where $\mathbf{g}()$ is a scalar function or an $1 \times L$ vector function.

Theorem 1. *In the stable fluid vacation model with gated discipline the steady-state joint vector LTs of the fluid level at the end of vacation, $\mathbf{f}^*(v)$ and at the start of vacation, $\mathbf{m}^*(v)$ satisfy*

- *for the transition $\mathbf{f} \to \mathbf{m}$, i.e, the transition from the end of the vacation (=beginning of service) to beginning of vacation (=the end of service)*

$$\mathbf{m}^*(v) = \mathbf{f}^*(\frac{\mathbf{R}v - \mathbf{Q}}{d}), \tag{20}$$

- *and for the transition $\mathbf{m} \to \mathbf{f}$, i.e, the transition from the start of the vacation (=end of service) to the end of vacation (=beginning of service)*

$$\mathbf{f}^*(v) = \mathbf{m}^*(v)\sigma^*(\mathbf{R}v - \mathbf{Q}). \tag{21}$$

Relations (20) *and* (21) *are referred to as the governing equations of the model.*

PROOF. Due to the gated service discipline the fluid level at the buffer at start of vacation equals the level of the fluid arriving during the service duration. When the amount of fluid at the buffer at end of vacation is $\xi > 0$, then the service duration is $\frac{\xi}{d}$ due to the gated discipline. Utilizing all these we can express $[\mathbf{m}(\ell, x)]_k$ as

$$[\mathbf{m}(\ell, x)]_k = \sum_{j=1}^{L} \int_{\xi=0}^{\infty} [\mathbf{f}(\ell-1, \xi)]_j \mathbf{A}_{jk}(\frac{\xi}{d}, x) d\xi.$$

This can be rearranged into vector and matrix form as

$$\mathbf{m}(\ell, x) = \int_{\xi=0}^{\infty} \mathbf{f}(\ell-1, \xi) \mathbf{A}(\frac{\xi}{d}, x) d\xi.$$

The LT of $\mathbf{m}(\ell, x)$ with respect to x can be given by

$$\mathbf{m}^*(\ell, v) = \int_{\xi=0}^{\infty} \mathbf{f}(\ell-1, \xi) \mathbf{A}^*(\frac{\xi}{d}, v) d\xi. \tag{22}$$

Applying (15) in (22) yields

$$\mathbf{m}^*(\ell, v) = \int_{\xi=0}^{\infty} \mathbf{f}(\ell-1, \xi) e^{-\frac{\xi}{d}(\mathbf{R}v - \mathbf{Q})} d\xi. \tag{23}$$

Using that the right hand side of (23) is an LT with respect to ξ we can write

$$\mathbf{m}^*(\ell, v) = \mathbf{f}^*(\ell-1, \frac{\mathbf{R}v - \mathbf{Q}}{d}). \tag{24}$$

The first statement of the theorem comes by taking the limit $\ell \to \infty$ in (24) and using the definitions of $\mathbf{m}^*(v)$ and $\mathbf{f}^*(v)$.

The fluid level at the buffer at end of vacation is the sum of the fluid level at the previous start of vacation and the fluid flowed into the buffer in between. In other words

$$[\mathbf{f}(\ell, x)]_k = \sum_{j=1}^{L} \int_{t=0}^{\infty} \int_{y=0}^{x} [\mathbf{m}(\ell, x-y)]_j \mathbf{A}_{jk}(t, y) \sigma(t) dy dt. \tag{25}$$

Rearranging (25) to matrix form and using the convolution property of LT we get

$$\mathbf{f}^*(\ell, v) = \int_{t=0}^{\infty} \mathbf{m}^*(\ell, v) \mathbf{A}^*(t, v) \sigma(t) dt. \tag{26}$$

Applying (15) in (26) and using the definition of LT with respect to t leads to

$$\mathbf{f}^*(\ell, v) = \mathbf{m}^*(\ell, v)\sigma^*(\mathbf{R}v - \mathbf{Q}). \tag{27}$$

The second statement of the theorem comes by taking the limit $\ell \to \infty$ in (27) and using the definitions of $\mathbf{f}^*(v)$ and $\mathbf{m}^*(v)$. □

4. THE STEADY-STATE FLUID LEVEL AT END AND START OF VACATION

4.1 Evolution of the fluid level between two consecutive service starts

In order to solve the above governing equations of the fluid vacation model we apply the concept similar to the one of descendant set (see in Borst and Boxma [1]). Let

$$\mathbf{H}(v) = \frac{\mathbf{R}v - \mathbf{Q}}{d},$$

that is $\mathbf{T} = \frac{\mathbf{R}}{d}$ and $\mathbf{Z} = \frac{\mathbf{Q}}{d}$ in (4). It can be seen from the proof of theorem 1 that applying $\mathbf{H}(v)$ in the LT of the fluid level at end of vacation represents the LT of the fluid flowed into the buffer during the service of the fluid originally present at the buffer. This is similar to the descendant set of a customer in the regular vacation model, which consists of the group of customers arrived during the service of the original customer. In this sense the substitution $v \to \mathbf{H}(v)$, i.e. the application of the operator $\mathcal{O}()$ on v characterizes the descendant fluid level due to the service.

Corollary 1. *In the stable fluid vacation model with gated discipline a functional equation can be given for the evolution of the fluid level between two consecutive service starts in term of steady-state vector LT as*

$$\mathbf{f}^*(v) = \mathbf{f}^*(\mathcal{O}(v))\sigma^*(\mathbf{R}v - \mathbf{Q}). \tag{28}$$

Proof. Using (20) in (21) gives

$$\mathbf{f}^*(v) = \mathbf{f}^*(\frac{\mathbf{R}v - \mathbf{Q}}{d})\sigma^*(\mathbf{R}v - \mathbf{Q}). \tag{29}$$

The corollary comes by applying the operator $\mathcal{O}()$ in (29). □

The term $\mathbf{f}^*(\mathcal{O}(v))$ expresses the steady-state vector LT of the fluid, which flowed into the buffer during the service period of the previous cycle. Similarly the term $\sigma^*(\mathbf{R}v - \mathbf{Q})$ represents the vector LT of the fluid flowed into the buffer during the vacation period in the previous cycle. The fluid at start of service period one cycle before were removed from the system completely during that service period. Hence the left hand side of (28) represents the vector LT of the total descendant steady-state fluid level originated by the fluid level at the start of service period one cycle before.

4.2 The steady-state fluid level at end and start of vacation

Let the notation $\mathbf{f}^\infty$ be defined as

$$\mathbf{f}^\infty = \mathbf{f}^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(v)) \tag{30}$$

reflecting that $\mathbf{f}^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(v))$ is independent of v according to proposition 1.

The limit $\sigma^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(\mathbf{R}v - \mathbf{Q}))$ is independent of v. This follows from $\sigma^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(\mathbf{R}v - \mathbf{Q})) = \sigma^*(\mathbf{R}\lim_{\ell\to\infty} \mathcal{O}^\ell(v) - \mathbf{Q})$ together with proposition 1. Thus we introduce a notation also for $\sigma^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(\mathbf{R}v - \mathbf{Q}))$ as

$$\sigma^\infty = \sigma^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(\mathbf{R}v - \mathbf{Q})). \tag{31}$$

Proposition 3. *In the stable fluid vacation model with gated discipline the following statements hold for the steady-state joint vector LT of the fluid level:*

- *The steady-state joint vector LT of the fluid level at end of vacation can be determined as*

$$\mathbf{f}^*(v) = \mathbf{f}^\infty \overleftarrow{\prod}_{\ell=0}^{\infty} \sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v - \mathbf{Q})). \tag{32}$$

where $\overleftarrow{\prod}$ *stands for multiplication from left in increasing index order and* $\mathbf{f}^\infty$ *can be obtained from the system of linear equations*

$$\mathbf{f}^\infty = \mathbf{f}^\infty \sigma^\infty. \tag{33}$$

$$\mathbf{f}^\infty \mathbf{e} = \mathbf{e}. \tag{34}$$

- *The steady-state joint vector LT of the fluid level,* $\mathbf{f}^*(v)$*, is independent of the joint distribution of the fluid level and the state of the modulating Markov chain at* $t = 0$.

Proof. Applying ℓ-times the operator $\mathcal{O}()$ on (28) for $l \ge 0$ gives

$$\mathbf{f}^*(\mathcal{O}^\ell(v)) = \mathbf{f}^*(\mathcal{O}^{\ell+1}(v))\sigma^*(\mathcal{O}^\ell(\mathbf{R}v - \mathbf{Q})). \tag{35}$$

Solving (35) by recursive substitution for $l \ge 0$ gives

$$\mathbf{f}^*(v) = \mathbf{f}^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(v)) \overleftarrow{\prod}_{\ell=0}^{\infty} \sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v - \mathbf{Q})). \tag{36}$$

The first statement of the theorem comes by applying the notation (30) in (36).

Now we determine $\mathbf{f}^\infty$ in (32). Taking the limit $\ell \to \infty$ on (35) yields

$$\mathbf{f}^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(v)) =$$
$$\mathbf{f}^*(\lim_{\ell\to\infty} \mathcal{O}^{\ell+1}(v))\ \sigma^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(\mathbf{R}v - \mathbf{Q})). \tag{37}$$

The system of linear equation (33) comes by applying the notations (30) and (31) in (37).

An additional relation is necessary to make the homogenous system of linear equations (33) complete. The vector LT $\mathbf{f}^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(v)) = \mathbf{f}^\infty$ can be interpreted as the vector LT of the fluid after infinite number of cycles in a system without fluid accumulating vacation, which is initiated by the steady state joint distribution vector of the system, whose LT is $\mathbf{f}^*(v)$. This system would become empty, i.e. the fluid level would become zero due to the lack of the fluid accumulating vacation periods. Hence in the stable model the vector LT $\mathbf{f}^\infty$ is a probability vector. From this the normalization condition (34) follows directly, which completes the first statement of the proposition.

Due to the form of (33) the vector LT $\mathbf{f}^*(\lim_{\ell\to\infty} \mathcal{O}^\ell(v)) = \mathbf{f}^\infty$ is determined by σ^∞. This together with the form of (32) implies that $\mathbf{f}^*(v)$ is independent of the initial fluid level of the model and of the initial state of the modulating Markov chain, which gives the second statement of the proposition. □

The matrix $\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v - \mathbf{Q}))$ can be interpreted as the matrix LT of the fluid level, which is the ℓ-th descendant fluid of the amount of fluid flowed in during a vacation period.

That means that the LT of the fluid after ℓ number of cycles in a system without fluid accumulating vacation, which is initiated by the fluid flowed in during a vacation period.

Remark 1. *The stability of the model ensures that the steady-state quantities* $\mathbf{f}^*(v)$ *and* $\mathbf{f}^\infty$ *exist, i.e. these limits converges.*

Remark 2. *The relation (32) expresses the principal contribution components of the steady-state fluid level at end of vacation. One of them is the fluid level after infinite number of cycles in a system without vacation, which is initiated by the steady state joint distribution vector of the system* ($\mathbf{f}^\infty$)*. The other one, represented by* $\overleftarrow{\prod}_{\ell=0}^{\infty}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\, v-\mathbf{Q}))$*, is the sum of more fluid level components, the fluid flowed in during the vacation period in the last cycle* ($\sigma^*(\mathcal{O}^0(\mathbf{R}\, v-\mathbf{Q}))$*, which can be called as* 0*-th descendant fluid), the* 1*-st descendant fluid of the fluid flowed in during the vacation period in second previous cycle* ($\sigma^*(\mathcal{O}^1(\mathbf{R}\, v-\mathbf{Q}))$*, the* 2*-nd descendant fluid of the fluid flowed in during the vacation period in the third previous cycle* ($\sigma^*(\mathcal{O}^2(\mathbf{R}\, v-\mathbf{Q}))$)*, and so on. However the first principal components adds zero fluid level to the steady-state fluid level at end of vacation, therefore it is fully determined by the sum of the descendant fluid levels originated from the vacation periods in the 1st, 2nd, 3rd,... previous cycles.*

The proposition 3 points out also that $\mathbf{f}^*(v)$ is fully determined by the quantities $\overleftarrow{\prod}_{\ell=0}^{\infty}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))$ and σ^∞. Thus the question arises whether it is necessary to compute $\mathbf{f}^\infty$ at all for the determination of $\mathbf{f}^*(v)$. We get the answer for this from the next theorem.

Theorem 2. *In the stable fluid vacation model with gated discipline the following statements hold in connection with the steady-state joint vector LT of the fluid level at end of vacation and at start of vacation:*

- *The rows of the matrix* $\overleftarrow{\prod}_{\ell=0}^{\infty}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))$ *are the same.*
- *The steady-state joint vector LT of the fluid level at end of vacation can be determined as*

$$\mathbf{f}^*(v)=\mathbf{e}_1\left(\overleftarrow{\prod}_{\ell=0}^{\infty}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\right), \tag{38}$$

  where the $1\times L$ *row vector* $\mathbf{e}_1$ *is given as* $(1,0,\ldots,0)$.
- *The steady-state joint vector LT of the fluid level at start of vacation can be determined as*

$$\mathbf{m}^*(v) \quad = \quad \mathbf{e}_1\left(\overleftarrow{\prod}_{\ell=1}^{\infty}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\right). \tag{39}$$

Proof. Applying (24) in (27) and using the operator $\mathcal{O}()$ gives

$$\mathbf{f}^*(\ell,v)=\mathbf{f}^*(\ell-1,\mathcal{O}(v))\ \sigma^*(\mathbf{R}v-\mathbf{Q}). \tag{40}$$

Applying (40) for $\ell,\ell-1,\ldots,1$ and expressing $\mathbf{f}^*(\ell,v)$ recursively yields

$$\mathbf{f}^*(\ell,v)=\mathbf{f}^*(0,\mathcal{O}^\ell(v))\ \overleftarrow{\prod}_{\ell=0}^{\ell-1}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q})). \tag{41}$$

Taking the limit $\ell\to\infty$ on (41) and using the definition of $\mathbf{f}^*(v)$ leads to

$$\mathbf{f}^*(v)=\mathbf{f}^*(0,\lim_{\ell\to\infty}\mathcal{O}^\ell(v))\ \overleftarrow{\prod}_{\ell=0}^{\infty}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q})). \tag{42}$$

The vector $\mathbf{f}^*(v)$ is independent of the initial joint distribution of the fluid level and the state of the modulating CTMC (second statement of proposition 3). However $\mathbf{f}^*(0,\lim_{\ell\to\infty}\mathcal{O}^\ell(v))$ depends on the initial joint distribution of the fluid level and the state of the modulating CTMC definitely. Moreover this initial joint distribution can be chosen arbitrary. It follows that the value on the r.h.s. of (42) is invariant for the vector LT, which is multiplied by $\overleftarrow{\prod}_{\ell=0}^{\infty}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\, v-\mathbf{Q}))$. An immediate consequence of it that the rows of the matrix $\overleftarrow{\prod}_{\ell=0}^{\infty}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))$ are the same, which gives the first statement of the theorem. This together with (32) and the normalization condition (34) implies the second statement of the theorem. The third statements of the theorem comes by applying (38) in (20). □

4.3 The steady-state vector moments of the fluid level at end and start of vacation

Corollary 2. *In the stable fluid vacation model with gated discipline the steady-state probability vector and the first two steady-state vector moments of the fluid level at end of vacation are given as*

$$\mathbf{f}=\ \mathbf{e}_1\left(\ \overleftarrow{\prod}_{\ell=0}^{\infty}\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\Big|_{v=0}\ \right). \tag{43}$$

$$\mathbf{f}^{(1)}=\ \mathbf{e}_1\Bigg(-\sum_{\ell=0}^{\infty}\ \overleftarrow{\prod}_{i=\ell+1}^{\infty}\sigma^*(\mathcal{O}^i(\mathbf{R}\ v-\mathbf{Q})) \tag{44}$$
$$\times\ \frac{d\big(\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\big)}{dv}\overleftarrow{\prod}_{j=0}^{\ell-1}\sigma^*(\mathcal{O}^j(\mathbf{R}\ v-\mathbf{Q}))\Big|_{v=0}\Bigg).$$

$$\mathbf{f}^{(2)}= \mathbf{e}_1\Bigg(\sum_{\ell=0}^{\infty}\overleftarrow{\prod}_{i=\ell+1}^{\infty}\sigma^*(\mathcal{O}^i(\mathbf{R}\ v-\mathbf{Q})) \tag{45}$$

$$\times\frac{d^2\big(\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\big)}{dv^2}\overleftarrow{\prod}_{j=0}^{\ell-1}\sigma^*(\mathcal{O}^j(\mathbf{R}\ v-\mathbf{Q}))$$

$$+2\sum_{\ell=1}^{\infty}\sum_{n=0}^{\ell-1}\overleftarrow{\prod}_{i=\ell+1}^{\infty}\sigma^*(\mathcal{O}^i(\mathbf{R}\ v-\mathbf{Q}))$$

$$\times\frac{d\big(\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\big)}{dv}\overleftarrow{\prod}_{j=n+1}^{\ell-1}\sigma^*(\mathcal{O}^j(\mathbf{R}\ v-\mathbf{Q}))$$

$$\times\frac{d\big(\sigma^*(\mathcal{O}^n(\mathbf{R}\ v-\mathbf{Q}))\big)}{dv}\overleftarrow{\prod}_{k=0}^{n-1}\sigma^*(\mathcal{O}^k(\mathbf{R}\ v-\mathbf{Q}))\bigg|_{v=0}\Bigg).$$

PROOF. The formulas come by setting $v = 0$ as well as by taking the (negative) first and second derivatives of (38) with respect to v and setting $v = 0$. □

Corollary 3. *In the stable fluid vacation model with gated discipline the steady-state probability vector and the first two steady-state vector moments of the fluid level at start of vacation are given as*

$$\mathbf{m}= \mathbf{e}_1\Bigg(\overleftarrow{\prod}_{\ell=1}^{\infty}\ \sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\bigg|_{v=0}\Bigg). \tag{46}$$

$$\mathbf{m}^{(1)}= \mathbf{e}_1\Bigg(-\sum_{\ell=1}^{\infty}\overleftarrow{\prod}_{i=\ell+1}^{\infty}\sigma^*(\mathcal{O}^i(\mathbf{R}\ v-\mathbf{Q})) \tag{47}$$

$$\times\frac{d\big(\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\big)}{dv}\overleftarrow{\prod}_{j=1}^{\ell-1}\sigma^*(\mathcal{O}^j(\mathbf{R}\ v-\mathbf{Q}))\bigg|_{v=0}\Bigg).$$

$$\mathbf{m}^{(2)}= \mathbf{e}_1\Bigg(\sum_{\ell=1}^{\infty}\overleftarrow{\prod}_{i=\ell+1}^{\infty}\sigma^*(\mathcal{O}^i(\mathbf{R}\ v-\mathbf{Q})) \tag{48}$$

$$\times\frac{d^2\big(\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\big)}{dv^2}\overleftarrow{\prod}_{j=1}^{\ell-1}\sigma^*(\mathcal{O}^j(\mathbf{R}\ v-\mathbf{Q}))$$

$$+2\sum_{\ell=2}^{\infty}\sum_{n=1}^{\ell-1}\overleftarrow{\prod}_{i=\ell+1}^{\infty}\sigma^*(\mathcal{O}^i(\mathbf{R}\ v-\mathbf{Q}))$$

$$\times\frac{d\big(\sigma^*(\mathcal{O}^\ell(\mathbf{R}\ v-\mathbf{Q}))\big)}{dv}\overleftarrow{\prod}_{j=n+1}^{\ell-1}\sigma^*(\mathcal{O}^j(\mathbf{R}\ v-\mathbf{Q}))$$

$$\times\frac{d\big(\sigma^*(\mathcal{O}^n(\mathbf{R}\ v-\mathbf{Q}))\big)}{dv}\overleftarrow{\prod}_{k=1}^{n-1}\sigma^*(\mathcal{O}^k(\mathbf{R}\ v-\mathbf{Q}))\bigg|_{v=0}\Bigg).$$

PROOF. The formulas are derived by setting $v = 0$ as well as by taking the (negative) first and second derivatives of (39) with respect to v and setting $v = 0$. □

5. THE STEADY-STATE FLUID LEVEL AT ARBITRARY EPOCH

5.1 Equilibrium relationships

Let $\widetilde{s}(\ell)$ be the service time in the ℓ-th cycle. The steady-state service time and its mean is defined as

$$\widetilde{s} = \lim_{k\to\infty}\frac{\sum_{\ell=1}^{k}\widetilde{s}(\ell)}{k} \qquad \text{and} \qquad s = \lim_{k\to\infty}\frac{E[\sum_{\ell=1}^{k}\widetilde{s}(\ell)]}{k}.$$

respectively.

Similarly let $\widetilde{c}(\ell)$ be the cycle time between two consecutive service starts in the ℓ-th cycle. The steady state cycle time and its mean is defined as

$$\widetilde{c} = \lim_{k\to\infty}\frac{\sum_{\ell=1}^{k}\widetilde{c}(\ell)}{k} \qquad \text{and} \qquad c = \lim_{k\to\infty}\frac{E[\sum_{\ell=1}^{k}\widetilde{c}(\ell)]}{k}.$$

respectively.

It follows from the definitions of c and s that

$$c = \sigma + s, \tag{49}$$

Let $\Lambda(t)$ be the accumulated fluid flowed into the buffer in interval $(0, t]$. The steady state mean amount of fluid, which flows into the buffer during one cycle, a, is defined as

$$a = \lim_{k\to\infty}\frac{E[\sum_{\ell=1}^{k}\Lambda(t^f(\ell+1))-\Lambda(t^f(\ell))]}{k}.$$

The right hand side of this definition can be rearranged as

$$\lim_{k\to\infty}\frac{E[\sum_{\ell=1}^{k}\Lambda(t^f(\ell+1))-\Lambda(t^f(\ell))]}{E[\sum_{\ell=1}^{k}\widetilde{c}(\ell)]}\lim_{k\to\infty}\frac{E[\sum_{\ell=1}^{k}\widetilde{c}(\ell)]}{k}$$

and thus we get

$$a = \lambda c. \tag{50}$$

Corollary 4. *In the stable fluid vacation model the steady-state mean cycle time can be expressed as*

$$c = \frac{\sigma}{1-\rho}. \tag{51}$$

PROOF. The stable model is in statistical equilibrium, which implies that the mean amount of fluid flowing into the buffer during a cycle equals the mean amount of fluid removed during the same cycle. In other words

$$a = sd. \tag{52}$$

Applying (50) in (52) and expressing s from it leads to

$$s = \frac{\lambda}{d} c. \tag{53}$$

Applying (53) in (49) and changing to the notation of utilizations results in

$$c = \sigma + \rho c. \tag{54}$$

Rearranging (54) gives the statement. □

Remark 3. *The relations (49), (50) and (51) have more general validity scope, since they are valid independently of the used service discipline.*

5.2 The steady-state moments of the service time

Let $1_{(\text{con})}$ denote the indicator of condition "con". The steady state pdf of the service time, $s(t)$, and the corresponding LT, $s^*(v)$, are defined as

$$s(t) = \frac{d}{dt} \lim_{k\to\infty} \frac{E[\sum_{\ell=1}^{k} 1_{(\tilde{s}(\ell)<t)}]}{k},$$
$$s^*(v) = \int_{t=0}^{\infty} s(t) e^{-st} dt.$$

Theorem 3. *In the stable fluid vacation model with gated discipline the steady-state LT of the service time can be expressed as*

$$s^*(v) = \mathbf{f}^*(\frac{v}{d})\mathbf{e}. \tag{55}$$

Proof. When the fluid level at the buffer is x at service start then the service time is $\frac{x}{d}$. Hence the steady-state LT of the service time at station i can be written as

$$s^*(v) = \int_{x=0}^{\infty} \mathbf{f}(x) e^{-v\frac{x}{d}} dx \mathbf{e}. \tag{56}$$

The statement of the theorem comes by rearranging (56). □

Corollary 5. *In the stable fluid vacation model with gated discipline the steady-state moments of the service time are given as*

$$s^{(k)} = \frac{1}{d^k} \mathbf{f}^{(k)} \mathbf{e}, \quad k \geq 1. \tag{57}$$

Proof. The statement comes by taking the k-th derivative of (55) with respect to v at $v = 0$. □

5.3 The steady-state vector LT of the fluid level

The steady-state joint density of the fluid level and the state of the modulating Markov chain at an arbitrary epoch, the $1 \times L$ row vector $\mathbf{q}(x)$ is defined by its j-th element as

$$[\mathbf{q}(x)]_j = \lim_{t\to\infty} \frac{\partial}{\partial x} Pr(\Omega(t) = j, X(t) < x), \; j \in \Omega$$

and its LT with respect to x is given by

$$\mathbf{q}^*(v) = \int_{x=0}^{\infty} \mathbf{q}(x) e^{-vx} dx.$$

Furthermore let $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the $1\times L$ vector with 1 at the j-th position. We define the $1 \times L$ indicator vector $\mathbf{1}_{(\Omega(t))}$ as

$$\mathbf{1}_{(\Omega(t))} = \sum_{j=1}^{L} 1_{(\Omega(t)=j)} \mathbf{e}_j.$$

Theorem 4. *In the stable fluid vacation model with gated discipline the following relation holds for the steady-state vector LT of the fluid level at arbitrary epoch:*

$$\mathbf{q}^*(v)\left(\mathbf{R}v - \mathbf{Q}\right)\left((\mathbf{R} - d\mathbf{I})v - \mathbf{Q}\right) = dv \frac{\mathbf{f}^*(v) - \mathbf{m}^*(v)}{c}. \tag{58}$$

Proof. The fluid level at arbitrary epoch can be expressed by the help of the fluid level at the last service start on LT level by utilizing the transient behavior of the arrived fluid (relation (15)) and taking into account that it can fall either in service or vacation period as well as its position in the actual period. Thus it is enough to average over a cycle for determining the behavior at arbitrary epoch.

$$\mathbf{q}^*(v) = \frac{E[\int_{t=0}^{\tilde{c}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt]}{E[\tilde{c}]} \tag{59}$$
$$= \frac{E[\int_{t=0}^{\tilde{s}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt] + E[\int_{t=\tilde{s}}^{\tilde{c}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt]}{c}.$$

Conditioning on the state of the modulating CTMC and the fluid level at the previous start of the service period the term $E[\int_{t=0}^{\tilde{s}} e^{-X(t)v} 1_{(\Omega(t)=j)} dt]$ can be expressed as

$$E[\int_{t=0}^{\tilde{s}} e^{-X(t)v} 1_{(\Omega(t)=j)} dt] = \int_{\xi=0}^{\infty} \sum_{i=1}^{L} [\mathbf{f}(\xi)]_i$$
$$\times E[\int_{t=0}^{\tilde{s}} e^{-X(t)v} 1_{(\Omega(t)=j)} dt \mid \Omega(0) = i, X(0) = \xi] d\xi. \tag{60}$$

When the fluid level at start of service period is ξ then $\widetilde{s} = \frac{\xi}{d}$. Using it in (60) and rearranging it gives

$$E[\int_{t=0}^{\widetilde{s}} e^{-X(t)v} 1_{(\Omega(t)=j)} dt] = \int_{\xi=0}^{\infty} \sum_{i=1}^{L} [\mathbf{f}(\xi)]_i \times \int_{t=0}^{\frac{\xi}{d}} E[e^{-X(t)v} 1_{(\Omega(t)=j)} \mid \Omega(0) = i, X(0) = \xi] dt \; d\xi. \quad (61)$$

The fluid level at time t in the service time is the sum of the remaining fluid level, $\xi - td$, and the fluid level arrived during t. Taking into account the state change of the modulating CTMC from 0 to t the LT term $E[e^{-X(t)v} 1_{(\Omega(t)=j)} \mid \Omega(0) = i, X(0) = \xi]$ can be given as

$$E[e^{-X(t)v} 1_{(\Omega(t)=j)} \mid \Omega(0) = i, X(0) = \xi] = e^{-(\xi - td)v} [\mathbf{A}^*(t, v)]_{i,j}. \quad (62)$$

Applying (62) in (61) and rearrangement leads to

$$E[\int_{t=0}^{\widetilde{s}} e^{-X(t)v} 1_{(\Omega(t)=j)} dt] = \int_{\xi=0}^{\infty} e^{-\xi v} \sum_{i=1}^{L} [\mathbf{f}(\xi)]_i \times \int_{t=0}^{\frac{\xi}{d}} e^{tdv} [\mathbf{A}^*(t, v)]_{i,j} dt \; d\xi.$$

Changing to vector notation yields

$$E[\int_{t=0}^{\widetilde{s}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt] = \int_{\xi=0}^{\infty} e^{-\xi v} \mathbf{f}(\xi) \int_{t=0}^{\frac{\xi}{d}} e^{tdv} \mathbf{A}^*(t, v) dt d\xi. \quad (63)$$

Applying (15) in (63) and rearrangement gives

$$E[\int_{t=0}^{\widetilde{s}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt] = \int_{\xi=0}^{\infty} e^{-\xi v} \mathbf{f}(\xi) \times \int_{t=0}^{\frac{\xi}{d}} e^{-t((\mathbf{R} - d\mathbf{I})v - \mathbf{Q})} dt d\xi. \quad (64)$$

The internal integral can be evaluated by means of a relation, which can be obtained by the help of the Taylor-expansion of $e^{\mathbf{Z}t}$, and is given by

$$\int_{t=0}^{x} e^{\mathbf{Z}t} dt \mathbf{Z} = (e^{\mathbf{Z}x} - \mathbf{I}). \quad (65)$$

Applying (65) in (64) and rearrangement yields

$$E[\int_{t=0}^{\widetilde{s}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt] \, ((\mathbf{R} - d\mathbf{I})v - \mathbf{Q}) = \int_{\xi=0}^{\infty} e^{-\xi v} \mathbf{f}(\xi) \left(\mathbf{I} - e^{-\frac{\xi}{d}(\mathbf{R} - d\mathbf{I})v - \mathbf{Q})} \right) d\xi.$$

Further rearrangement and applying (20) results in

$$E[\int_{t=0}^{\widetilde{s}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt] \, ((\mathbf{R} - d\mathbf{I})v - \mathbf{Q}) = \mathbf{f}^*(v) - \mathbf{f}^*(\frac{\mathbf{R}v - \mathbf{Q}}{d}) = \mathbf{f}^*(v) - \mathbf{m}^*(v). \quad (66)$$

The relation (66) can be further rearranged as

$$E[\int_{t=0}^{\widetilde{s}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt] \, (\mathbf{R}v - \mathbf{Q}) = \mathbf{f}^*(v) - \mathbf{m}^*(v) + dvE[\int_{t=0}^{\widetilde{s}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt]. \quad (67)$$

Now we consider the term $E[\int_{t=\widetilde{s}}^{\widetilde{c}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt]$. Conditioning on the state of the modulating CTMC and the fluid level at the previous start of vacation period we have

$$\begin{aligned} E[\int_{t=\widetilde{s}}^{\widetilde{c}} e^{-X(t)v} 1_{(\Omega(t)=j)} dt] &= E[\int_{t=0}^{\widetilde{\sigma}} e^{-X(t)v} 1_{(\Omega(t)=j)} dt] \\ &= \int_{\xi=0}^{\infty} \sum_{i=1}^{L} [\mathbf{m}(\xi)]_i \quad (68) \\ &\times E[\int_{t=0}^{\widetilde{\sigma}} e^{-X(t)v} 1_{(\Omega(t)=j)} dt \mid \Omega(0) = i, X(0) = \xi] d\xi \\ &= \int_{\xi=0}^{\infty} \sum_{i=1}^{L} [\mathbf{m}(\xi)]_i \int_{\tau=0}^{\infty} \int_{t=0}^{\tau} \\ &\times E[e^{-X(t)v} 1_{(\Omega(t)=j)} \mid \Omega(0) = i, X(0) = \xi] dt \; \sigma(\tau) \; d\tau \; d\xi. \end{aligned}$$

The fluid level at time t in the vacation time is the sum of the fluid level at start of vacation, ξ, and the fluid level arrived during t. Taking into account the state change of the modulating CTMC from 0 to t the LT term $E[e^{-X(t)v} 1_{(\Omega(t)=j)} \mid \Omega(0) = i, X(0) = \xi]$ can be given as

$$E[e^{-X(t)v} 1_{(\Omega(t)=j)} \mid \Omega(0) = i, X(0) = \xi] = e^{-\xi v} [\mathbf{A}^*(t, v)]_{i,j}. \quad (69)$$

Applying (69) in (68), changing to vector notation and rearrangement yields

$$E[\int_{t=\widetilde{s}}^{\widetilde{c}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt] = \int_{\xi=0}^{\infty}\mathbf{m}(\xi)e^{-\xi v}d\xi\int_{\tau=0}^{\infty}\int_{t=0}^{\tau}\mathbf{A}^*(t,v)dt\ \sigma(\tau)\ d\tau. \quad (70)$$

Using (15) in (70) and rearrangement results in

$$E[\int_{t=\widetilde{s}}^{\widetilde{c}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt] = \mathbf{m}^*(v)\int_{\tau=0}^{\infty}\int_{t=0}^{\tau}e^{-t(\mathbf{R}v-\mathbf{Q})}dt\sigma(\tau)d\tau. \quad (71)$$

Applying again (65), now in (71) gives

$$E[\int_{t=\widetilde{s}}^{\widetilde{c}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt]\,(\mathbf{R}v-\mathbf{Q}) = \mathbf{m}^*(v)\int_{\tau=0}^{\infty}(\mathbf{I}-e^{-\tau(\mathbf{R}v-\mathbf{Q})})\sigma(\tau)d\tau. \quad (72)$$

Rearranging (72) gives a relation for $E[\int_{t=\widetilde{s}}^{\widetilde{c}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt]$ as

$$E[\int_{t=\widetilde{s}}^{\widetilde{c}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt]\,(\mathbf{R}v-\mathbf{Q}) = \mathbf{m}^*(v)\left(\mathbf{I}-\sigma^*(\mathbf{R}v-\mathbf{Q})\right). \quad (73)$$

By applying (67) and (73) in (59) results in

$$\mathbf{q}^*(v)\,(\mathbf{R}v-\mathbf{Q}) = \frac{\mathbf{f}^*(v)}{c} - \frac{\mathbf{m}^*(v)}{c} + \frac{dvE[\int_{t=0}^{\widetilde{s}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt]}{c} + \frac{\mathbf{m}^*(v)\left(\mathbf{I}-\sigma^*(\mathbf{R}v-\mathbf{Q})\right)}{c}. \quad (74)$$

By applying the governing equation (21) in (74) as well as rearranging it leads to

$$\mathbf{q}^*(v)\,(\mathbf{R}v-\mathbf{Q}) = \frac{dvE[\int_{t=0}^{\widetilde{s}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt]}{c}. \quad (75)$$

The theorem comes by combining (75) with (66). □

5.4 The steady-state mean of the fluid level

Let $\mathbf{r}(v)$ is defined as

$$\mathbf{t}(v) = \mathbf{q}^*(v)\,(\mathbf{R}v-\mathbf{Q}). \quad (76)$$

Lemma 5. *In the stable fluid vacation model with gated discipline the steady-state vector mean of the fluid level at arbitrary epoch can be expressed from (76) in terms of* $\mathbf{t}^{(1)}$ *and* $\mathbf{t}^{(2)}\mathbf{e}$ *as*

$$\mathbf{q}^{(1)} = -\frac{1}{2\lambda}\mathbf{t}^{(2)}\mathbf{e}\boldsymbol{\pi} + \left(\mathbf{t}^{(1)}+\boldsymbol{\pi}\mathbf{R}\right)(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\frac{\mathbf{R}\mathbf{e}\boldsymbol{\pi}}{\lambda}-\mathbf{I}\right). \quad (77)$$

Proof. We apply an idea used by Lucantoni in [7] and Neuts in [8], and adopt it to our model. The essence of the idea is to utilize that $(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})$ is nonsingular.

Taking the first two derivatives of (76) at $v=0$ and utilizing $\mathbf{q}=\boldsymbol{\pi}$ yields:

$$\mathbf{q}^{(1)}\mathbf{Q} = -\mathbf{t}^{(1)} - \boldsymbol{\pi}\mathbf{R}, \quad (78)$$

$$\mathbf{q}^{(2)}\mathbf{Q} = -\mathbf{t}^{(2)} - 2\mathbf{q}^{(1)}\mathbf{R}. \quad (79)$$

Adding $\mathbf{q}^{(1)}\mathbf{e}\boldsymbol{\pi}$ to (78) and using $\boldsymbol{\pi}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1} = \boldsymbol{\pi}$ leads to

$$\mathbf{q}^{(1)} = \left(\mathbf{q}^{(1)}\mathbf{e}\right)\boldsymbol{\pi} - \left(\mathbf{t}^{(1)}+\boldsymbol{\pi}\mathbf{R}\right)(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}. \quad (80)$$

Now we determine the unknown term $\left(\mathbf{q}^{(1)}\mathbf{e}\right)$ in (80). Post-multiplying (79) by $\mathbf{e}$ and post-multiplying (80) by $\mathbf{R}\mathbf{e}$ and rearranging leads to

$$\mathbf{q}^{(1)}\mathbf{R}\mathbf{e} = -\frac{1}{2}\mathbf{t}^{(2)}\mathbf{e}, \quad (81)$$

$$\mathbf{q}^{(1)}\mathbf{R}\mathbf{e} = \left(\mathbf{q}^{(1)}\mathbf{e}\right)\boldsymbol{\pi}\mathbf{R}\mathbf{e} - \left(\mathbf{t}^{(1)}+\boldsymbol{\pi}\mathbf{R}\right)(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{R}\mathbf{e}. \quad (82)$$

Subtracting (81) from (82), applying $\boldsymbol{\pi}\mathbf{R}\mathbf{e}=\lambda$ and rearrangement results in:

$$\mathbf{q}^{(1)}\mathbf{e} = -\frac{1}{2\lambda}\mathbf{t}^{(2)}\mathbf{e} + \frac{1}{\lambda}\left(\mathbf{t}^{(1)}+\boldsymbol{\pi}\mathbf{R}\right)(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{R}\mathbf{e}. \quad (83)$$

Applying (83) in (80) leads to

$$\mathbf{q}^{(1)} = -\frac{1}{2\lambda}\mathbf{t}^{(2)}\mathbf{e}\boldsymbol{\pi} + \frac{1}{\lambda}\left(\mathbf{t}^{(1)}+\boldsymbol{\pi}\mathbf{R}\right)(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{R}\mathbf{e}\boldsymbol{\pi} - \left(\mathbf{t}^{(1)}+\boldsymbol{\pi}\mathbf{R}\right)(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}. \quad (84)$$

Rearranging (84) gives the lemma.

□

Let $\mathbf{r}(v)$ is defined as

$$\mathbf{r}(v) = \mathbf{t}(v)\left((\mathbf{R}-d\mathbf{I})v-\mathbf{Q}\right). \quad (85)$$

Lemma 6. *The terms* $\mathbf{t}^{(1)}$ *and* $\mathbf{t}^{(2)}\mathbf{e}$ *can be expressed from (85) in terms of* $\mathbf{r}^{(1)}$, $\mathbf{r}^{(2)}$ *and* $\mathbf{r}^{(3)}$ *as*

$$\mathbf{t}^{(1)} = -\frac{1}{2(\lambda-d)}\mathbf{r}^{(2)}\mathbf{e}\boldsymbol{\pi} - \mathbf{r}^{(1)}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\frac{1}{(\lambda-d)}(\mathbf{R}-d\mathbf{I})\mathbf{e}\boldsymbol{\pi}-\mathbf{I}\right). \quad (86)$$

$$\begin{aligned}\mathbf{t}^{(2)}\mathbf{e} &= -\frac{1}{3(\lambda-d)}\mathbf{r}^{(3)}\mathbf{e} \\ &+\frac{1}{(\lambda-d)}\mathbf{r}^{(2)}\left(\mathbf{I}-\frac{1}{(\lambda-d)}\mathbf{e}\boldsymbol{\pi}(\mathbf{R}-d\mathbf{I})\right) \\ &\times(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}(\mathbf{R}-d\mathbf{I})\mathbf{e} \\ &+\frac{2}{(\lambda-d)}\mathbf{r}^{(1)}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\frac{1}{(\lambda-d)}(\mathbf{R}-d\mathbf{I})\mathbf{e}\boldsymbol{\pi}-\mathbf{I}\right) \\ &\times(\mathbf{R}-d\mathbf{I})(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}(\mathbf{R}-d\mathbf{I})\mathbf{e}. \end{aligned} \tag{87}$$

Proof. Setting $v=0$ in (76) we get

$$\mathbf{t}=-\boldsymbol{\pi}\mathbf{Q}=\mathbf{0}. \tag{88}$$

Taking the first three derivatives of (85) at $v=0$ and applying (88) leads to

$$\mathbf{t}^{(1)}\mathbf{Q}=-\mathbf{r}^{(1)}, \tag{89}$$

$$\mathbf{t}^{(2)}\mathbf{Q}=-\mathbf{r}^{(2)}-2\mathbf{t}^{(1)}(\mathbf{R}-d\mathbf{I}), \tag{90}$$

$$\mathbf{t}^{(3)}\mathbf{Q}=-\mathbf{r}^{(3)}-3\mathbf{t}^{(2)}(\mathbf{R}-d\mathbf{I}). \tag{91}$$

Adding $\mathbf{t}^{(1)}\mathbf{e}\boldsymbol{\pi}$ to (89) and using $\boldsymbol{\pi}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}=\boldsymbol{\pi}$ leads to

$$\mathbf{t}^{(1)}=\left(\mathbf{t}^{(1)}\mathbf{e}\right)\boldsymbol{\pi}-\mathbf{r}^{(1)}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}. \tag{92}$$

Now we determine the unknown term $\left(\mathbf{t}^{(1)}\mathbf{e}\right)$ in (92). Post-multiplying (90) by $\mathbf{e}$ and post-multiplying (92) by $(\mathbf{R}-d\mathbf{I})\mathbf{e}$ and rearranging leads to

$$\mathbf{t}^{(1)}(\mathbf{R}-d\mathbf{I})\mathbf{e} = -\frac{1}{2}\mathbf{r}^{(2)}\mathbf{e}, \tag{93}$$

$$\begin{aligned}\mathbf{t}^{(1)}(\mathbf{R}-d\mathbf{I})\mathbf{e} &= \left(\mathbf{t}^{(1)}\mathbf{e}\right)\boldsymbol{\pi}(\mathbf{R}-d\mathbf{I})\mathbf{e} \\ &-\mathbf{r}^{(1)}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}(\mathbf{R}-d\mathbf{I})\mathbf{e}.\end{aligned} \tag{94}$$

Subtracting (93) from (94), applying $\boldsymbol{\pi}(\mathbf{R}-d\mathbf{I})\mathbf{e}=\lambda-d$ and rearrangement results in:

$$\begin{aligned}\mathbf{t}^{(1)}\mathbf{e} &= -\frac{1}{2(\lambda-d)}\mathbf{r}^{(2)}\mathbf{e} \\ &+\frac{1}{(\lambda-d)}\mathbf{r}^{(1)}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}(\mathbf{R}-d\mathbf{I})\mathbf{e}.\end{aligned} \tag{95}$$

Substituting (95) into (92) results in the first statement.

Adding $\mathbf{t}^{(2)}\mathbf{e}\boldsymbol{\pi}$ to (90) and using $\boldsymbol{\pi}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}=\boldsymbol{\pi}$ leads to

$$\mathbf{t}^{(2)}=\left(\mathbf{t}^{(2)}\mathbf{e}\right)\boldsymbol{\pi}-\left(\mathbf{r}^{(2)}+2\mathbf{t}^{(1)}(\mathbf{R}-d\mathbf{I})\right)(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}. \tag{96}$$

Now we determine the unknown term $\left(\mathbf{t}^{(2)}\mathbf{e}\right)$ in (96). Post-multiplying (91) by $\mathbf{e}$ and post-multiplying (96) by $(\mathbf{R}-d\mathbf{I})\mathbf{e}$ and rearranging leads to

$$\mathbf{t}^{(2)}(\mathbf{R}-d\mathbf{I})\mathbf{e} = -\frac{1}{3}\mathbf{r}^{(3)}\mathbf{e}, \tag{97}$$

$$\begin{aligned}\mathbf{t}^{(2)}(\mathbf{R}-d\mathbf{I})\mathbf{e} &= \left(\mathbf{t}^{(2)}\mathbf{e}\right)\boldsymbol{\pi}(\mathbf{R}-d\mathbf{I})\mathbf{e} \\ &-\left(\mathbf{r}^{(2)}+2\mathbf{t}^{(1)}(\mathbf{R}-d\mathbf{I})\right) \\ &\times(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}(\mathbf{R}-d\mathbf{I})\mathbf{e}.\end{aligned} \tag{98}$$

Subtracting (97) from (98), applying $\boldsymbol{\pi}(\mathbf{R}-d\mathbf{I})\mathbf{e}=\lambda-d$ and rearrangement results in:

$$\begin{aligned}\mathbf{t}^{(2)}\mathbf{e} &= -\frac{1}{3(\lambda-d)}\mathbf{r}^{(3)}\mathbf{e} \\ &+\frac{1}{(\lambda-d)}\left(\mathbf{r}^{(2)}+2\mathbf{t}^{(1)}(\mathbf{R}-d\mathbf{I})\right) \\ &\times(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}(\mathbf{R}-d\mathbf{I})\mathbf{e}.\end{aligned} \tag{99}$$

Applying (86) in (99) and rearrangement results in the second statement. □

Corollary 6. *In the stable fluid vacation model with gated discipline the steady-state vector mean of the fluid level at arbitrary epoch can be determined as*

$$\begin{aligned}\mathbf{q}^{(1)} = &\frac{1}{6\lambda(\lambda-d)}\mathbf{r}^{(3)}\mathbf{e}\boldsymbol{\pi} \\ &-\frac{1}{2(\lambda-d)}\mathbf{r}^{(2)}\frac{1}{\lambda}\left(\mathbf{I}-\frac{1}{(\lambda-d)}\mathbf{e}\boldsymbol{\pi}(\mathbf{R}-d\mathbf{I})\right) \\ &\times(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}(\mathbf{R}-d\mathbf{I})\mathbf{e}\boldsymbol{\pi} \\ &-\frac{1}{2(\lambda-d)}\mathbf{r}^{(2)}\mathbf{e}\boldsymbol{\pi}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\frac{\mathbf{R}\mathbf{e}\boldsymbol{\pi}}{\lambda}-\mathbf{I}\right) \\ &+\mathbf{r}^{(1)}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\frac{1}{(\lambda-d)}(\mathbf{R}-d\mathbf{I})\mathbf{e}\boldsymbol{\pi}-\mathbf{I}\right) \\ &\times\Bigg(\frac{-1}{\lambda(\lambda-d)}(\mathbf{R}-d\mathbf{I})(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}(\mathbf{R}-d\mathbf{I})\mathbf{e}\boldsymbol{\pi} \\ &+(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\frac{\mathbf{R}\mathbf{e}\boldsymbol{\pi}}{\lambda}-\mathbf{I}\right)\Bigg) \\ &+\boldsymbol{\pi}\mathbf{R}(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\frac{\mathbf{R}\mathbf{e}\boldsymbol{\pi}}{\lambda}-\mathbf{I}\right).\end{aligned} \tag{100}$$

where c is given by (51) and $\mathbf{r}^{(1)}$, $\mathbf{r}^{(2)}$ and $\mathbf{r}^{(3)}$ are given by

$$\begin{aligned}\mathbf{r}^{(1)} &= -\frac{d}{c}(\mathbf{f}^*(0)-\mathbf{m}^*(0)), \\ \mathbf{r}^{(2)} &= -\frac{2d}{c}(\mathbf{f}^{(1)}-\mathbf{m}^{(1)}), \\ \mathbf{r}^{(3)} &= -\frac{3d}{c}(\mathbf{f}^{(2)}-\mathbf{m}^{(2)}).\end{aligned}$$

Proof. Applying (76) in (85) gives

$$\mathbf{r}(v)=\mathbf{q}^*(v)(\mathbf{R}v-\mathbf{Q})((\mathbf{R}-d\mathbf{I})v-\mathbf{Q}). \tag{101}$$

$\mathbf{q}^{(1)}$ can be determined from (101) by applying Lemmas 5 and 6. Applying (86) and (87) in (77) gives (100), which is the expression of $\mathbf{q}^{(1)}$ in terms of $\mathbf{r}^{(1)}$, $\mathbf{r}^{(2)}$ and $\mathbf{r}^{(3)}$. The unknowns are obtained from the derivatives of

$$\mathbf{r}(v)=dv\frac{\mathbf{f}^*(v)-\mathbf{m}^*(v)}{c} \tag{102}$$

at $v = 0$, where (102) is obtained from (58) and (101).

□

6. NUMERICAL SOLUTION

In this section we summarize the steps of the numerical solution and provide numerical examples to illustrate the numerical solution and the behavior of the model.

6.1 The steps of the computation

The steps of the computation of steady-state mean $q^{(1)}$ of the model can be given as

1. Calculation of the steady-state probability vectors and the first two steady-state vector moments of the fluid level at end of vacation, $\mathbf{f}$, $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, as well as at start of vacation, $\mathbf{m}$, $\mathbf{m}^{(1)}$, $\mathbf{m}^{(2)}$ by means of the formula (43), (44), (45), (46), (47) and (48)respectively.
2. Computation of $\boldsymbol{\pi}$, λ, ρ and c by applying (1), (2), (3) and (51), respectively.
3. Computation of the steady-state mean $q^{(1)}$ by applying the formula (100).

The computation of the steady-state probability vectors and the steady-state vector moments are kept tractable by assuming a maximum for the index ℓ in (38) and (39). Hence $\mathbf{f}$, $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, $\mathbf{m}$, $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ are computed by iteratively increasing this maximum until the moments converge.

6.2 Numerical examples

In this subsection we provide numerical example for computing the steady-state mean $q^{(1)}$. We also show the dependency of $q^{(1)}$ on ρ for a selected example to illustrate the behavior of the model. In the example we use the following setting for the model parameters:

$$\mathbf{Q} = \begin{pmatrix} -2 & 0.5 & 1.5 \\ 1 & -3 & 2 \\ 0.2 & 0.8 & -1 \end{pmatrix}, \text{ and } \mathbf{R} = \alpha \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where α is the scaling parameter to set the required utilization based on (3) and (2). Furthermore $d = 4$ and the vacation period is selected to be exponentially distributed with parameter $\nu = 0.5$, i.e. $\sigma = \frac{1}{\nu}$. For the above rate matrix, $\mathbf{Q}$, the stationary probability vector is given as $\pi = (0.1628, 0.1977, 0.6395)$.

In Figure 1 we have plotted the dependency of the steady-state partial fluid level on ρ. The curves correspond to the states of the modulating CTMC (phases). As expected the steady-state mean fluid level increases with ρ for every phases.

We have observed that the number of necessary iterations, and hence the running time increases in these examples drastically around $\rho = 0.5$. This is the area where the slope of the curves on the figure increases fast. That means that starting from this load area the level of higher descendant fluids, that is the effect of farer previous vacation periods becomes tangible.

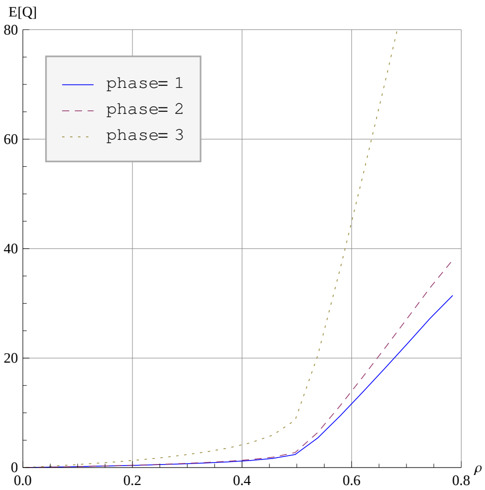

Figure 1: The steady-state partial mean fluid level versus utilization (ρ) for the individual states (phases) of the modulating CTMC.

7. FINAL REMARKS

We remark here that all the results presented in this paper are invariant for the interchange of the order of matrix and scalar in the definition of $\mathbf{H}(v)$. It follows from the uniqueness of $\mathbf{f}^{\infty}$ and $\mathbf{f}^{*}(v)$ (Proposition 3) that applying another order of matrix and scalar in the definition of $\mathbf{H}(v)$, i.e. the order of v and $\mathbf{T}$, leads to the same result. It can be shown that the same is true for all the results derived throughout this paper by using similar arguments in the other derivations.

The presented fluid model is restricted to the gated vacation model. Hence it is an interesting future research topic to extend the presented analysis for the corresponding fluid vacation model with exhaustive discipline or for the corresponding fluid polling models.

Another challenging potential research topic is to extend the presented analysis for the case when the fluid service is also modulated by a Markov chain.

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