List coloring of Latin and Sudoku graphs

Antal Iványi and Zsolt Németh

Department of Computer Algebra of Eötvös Loránd University

tony@compalg.inf.elte.hu

Department of Numerical Analysis of Eötvös Loránd University

birka0@gmail.com

Abstract

In 2001 Galvin [44] proved a generalization of Hall theorem [49]. We improve his result in two directions, giving a weaker precondition and also a stronger consequence. We show some basic properties of the Latin and Sudoku graphs considering the solution of the Latin and Sudoku puzzles as a list coloring problem of graphs [5, 11, 14, 20, 33, 39, 57, 58, 60, 62, 84].

1 Introduction

List coloring of graphs was proposed by Vizing in 1976 [87]. Substantial progress was made by Erdős, Rubin and Taylor in 1979 [31].

Today graph coloring is a popular research topic of combinatorics and computer science. Concrete applications appear in connection with VLSI design [30, 61], networks [27, 38, 66, 73], resource allocation [45], and determination of partially given Latin [34] and Sudoku [17, 18, 19, 37, 40, 63, 64, 65] squares.

Let $m \geq 1$ and $n \geq 2$ be integers and $G = (V, E)$ be a finite simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Assume further that an infinite set $\mathcal{C} = \{c_1, c_2, \ldots\}$ of so called *colors,* and a collection $\mathcal{L} = \{L(v_1), L(v_2), \ldots, L(v_n)\}$ of lists of colors are given, where $L(v_i) \subseteq \mathcal{C}$ for $1 \leq i \leq n$. Here each list $L(v_i)$ is considered as the set of allowed colors for vertex v_i. $\mathcal{L}$ is called *list arrangement.*

For the simplicity and without the loss the generality we suppose that $\mathcal{C}$ is the set of positive integers, and so each list is a finite set of positive integers. If the sizes of the lists have a join upper bound, that is if $|L(v_i)| \leq k$, then the collection $\mathcal{L}$ is called a *k-assignment* on G. Since the components of G are colorable independently, for the simplicity we suppose that G is connected.

A sequence of colors (positive integers) $S = (c_{i_1}, c_{i_2}, \ldots, c_{i_n}) = (i_1, i_2, \ldots, i_n)$ is called *a proper vertex coloring* of G with a given collection of lists $\mathcal{L}$, if for all $1 \leq j < k \leq n$

1. $i_j \in L(v_j)$;
2. if $v_j v_k \in E$, then $i_j \neq i_k$.

Graph G with a given $\mathcal{L}$ is $\mathcal{L}$-*colorable* (list colorable), if there exists a proper *vertex coloring function* $\varphi : V \to \mathcal{C}$ satisfying for all for all $1 \leq j < k \leq n$

1. $\varphi(v_j) \in L(v_j)$;
2. if $v_j v_k \in E$, then $\varphi(v_j) \neq \varphi(v_k)$.

Figure 1 shows a graph G_1 with some list assignment A_1.

Graph G_1 with list assignment A_1 has two proper colorings S_1 and S_2, which are represented in Figure 2.

Figure 3 shows graph G_2 with list assignment A_2, but this graph has no proper list coloring.

Let f a function from V to the positive integers, that is

$$f : V \to \{1, 2, \ldots\},$$

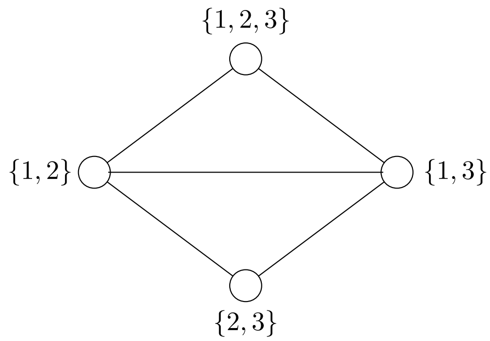

Figure 1: Graph G_1 with list assignment A_1.

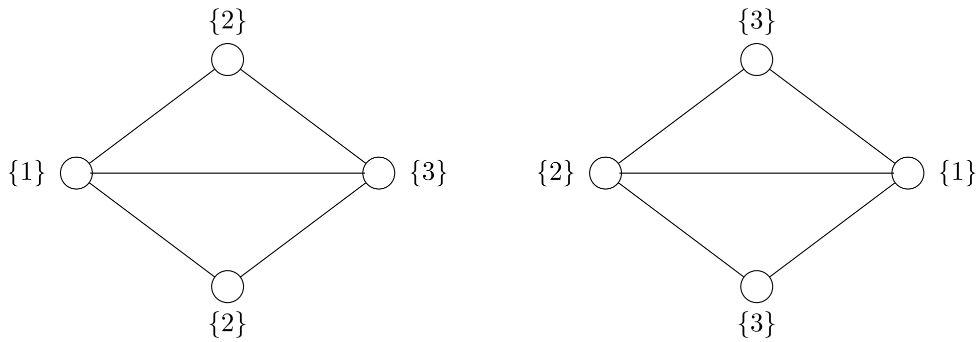

Figure 2: Proper colorings S_1 and S_2 of graph G_1 with list assignment A_1.

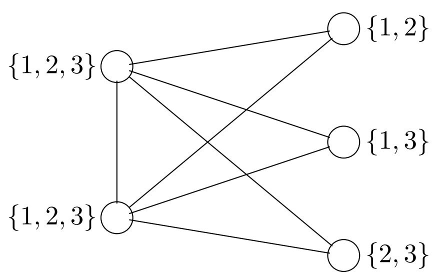

Figure 3: Graph G_2 with list assignment A_2.

then a graph G is called *f-choosable* [31] if there is a proper $\mathcal{L}$-coloring of G whenever $\mathcal{L}$ satisfies

$$|\mathcal{L}| \geq f(v_i) \text{ for all } v_i \in V.$$

For example the graph G_1 represented in Figure 1 is not f-choosable, if f is defined according to Figure 4. In Figure 5 such list assignment A_3 is represented which shows that the graph G_1 is not f-choosable.

The most important graph parameter in list coloring is the *choice number* of a graph G denoted by $\chi_\iota(G)$ and defined as the smallest integer k such that G is $\mathcal{L}$-colorable for every k-assignment $\mathcal{L}$.

The choice number of the graphs represented in Figures 1 and 3 is 3. The $\chi(G)$ chromatic number of these graphs is also 3. It is known that $\chi(G) \leq \chi_\iota(G)$ for all G [31, 87].

The exact characterization of the graphs for which the chromatic number is equal with the choice number is an open problem. The graph G_4 represented in 6 represented with a list assignment A_4 shows $\chi_\iota(G_4) = 3$, while $\chi(G_4) = 2$.

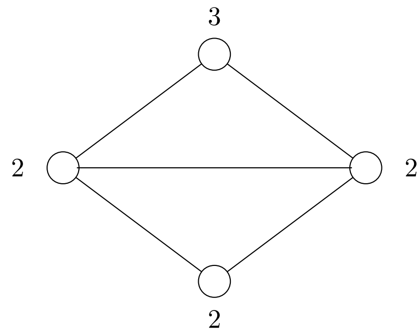

Figure 4: Function f_1 for graph G_1.

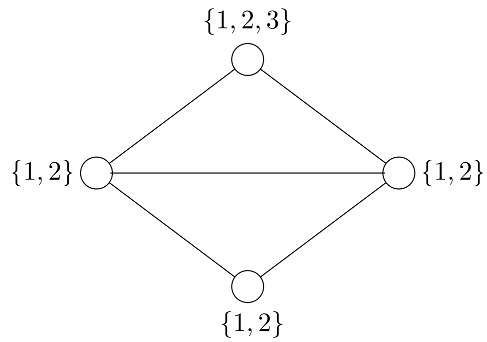

Figure 5: List assignment A_3 showing that graph G_1 is not f_1-choosable.

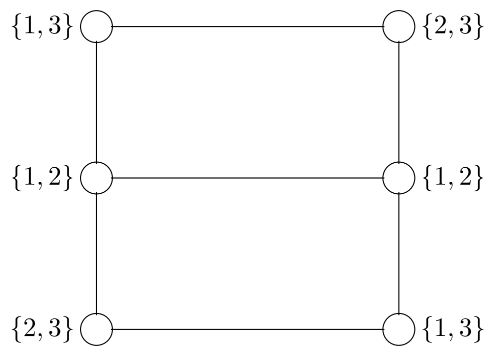

Figure 6: Graph G_4 with list assignment A_4.

2 Extension of Hall theorem

The Hall theorem [49] results the following necessary and sufficient condition of the $\mathcal{L}$-colorability of K_n (the complete graph on n vertices): K_n is $\mathcal{L}$-colorable if and only if the union of any k ($1 \leq k \leq n$) lists contain at least k elements.

The classical Hall theorem has several new proofs [7, 8, 12, 71, 72, 76, 82] and many extensions [1, 2, 3, 4, 13, 25, 24, 26, 31, 35, 41, 42, 47, 48, 50, 52, 54, 55, 56, 69, 70, 74, 89].

In 1998 Fred Galvin [43] proposed the following problem as *Problem 1071* in *The American Mathematical Monthly.*

Let G be a finite, undirected, simple graph with vertex set V. Let $C = \{C_x : x \in V\}$ be a family of sets indexed by the vertices of G. For $X \subseteq V$, let $C_X = \cup_{x \in X} C_x$. A set $X \subseteq V$ is C-colorable if one can assign to each vertex $x \in X$ a "color" $c_x \in C_x$ so that $c_x \neq c_y$ whenever x and y are adjacent in G.

a) Prove that if $|C_X| \geq |X|$ whenever X induces a connected subgraph of G, then V is C-colorable.

b) Prove that if every proper subset of V is C-colorable and $|C_V| \geq |V|$, then V is C-colorable.

c) For every connected graph G, find a family $C = \{C_x : x \in V\}$ showing that the condition $|C_V| \geq |V|$ in part **b)** cannot be weakened to $|C_V| \geq |V| - 1$.

Part **a)** was solved by Stephen C. Locke (Florida Atlantic University, Boca Raton, FL), part **(b)** by Sung Soo Kim (Hanyang University, Ansan, Kyunggi, Korea), and part **c)** by David Callan (University of Wisconsin, Madison, WI) in 2001 [44, 67].

These assertions on list coloring of graphs can be strengthened in two directions:

- there is a weaker sufficient condition than in part **a)**;
- the condition in part **a)** implies a stronger consequence for which the given condition is necessary and sufficient.

Let $\Gamma(x) = \{y : x \text{ and } y \text{ are adjacent in } G\}$ and $\deg(x) = |\Gamma(x)|$. A set $X \subseteq V$ is D-colorable if one can assign to each vertex $x \in X$ a color $c_x \in C_x$ so that $c_x \neq c_y$ whenever x and y are connected with a path in G.

d) Prove that if $\deg(v) > |C_v|$ for each $v \in V$, then G is C-colorable.

e) Prove that G is D-colorable iff $|C_X| \geq |X|$ whenever X induces a connected subgraph of G.

Solution to part **d)**

Let $V = (v_1, v_2, \ldots, v_{|V|})$. Then we can assign colors to vertices e.g. in the increasing order of their indices, since a preceding vertex can exclude at most the colors of $\Gamma(v_i)$ adjacent vertices, but the degree of v_i is greater than the number of its adjacent vertices.

Solution to part **e)**

If G is D-colorable and X induces a connected subgraph of G, then the elements of X are connected with a path in G, therefore $|C_X| \geq |X|$ holds due to the definition of D-colorability.

If $|C_X| \geq |X|$ whenever X induces a connected subgraph S_X of G, then S_X is D-colorable due to Hall's theorem and D-colorability of the connected subgraphs of G implies the D-colorability of G.

We remark that if G is connected, then part **e)** is equivalent with Hall's theorem [49].

3 List coloring of Latin and Sudoku graphs

For a positive integer $n \geq 2$ the n-order Latin graph $\lambda_n = (V_n, E_n)$ corresponds to an n-order Latin square [28, 29] and is the finite simple graph with $V_n = \{v_{ij} : 1 \leq i,\ j \leq n\}$, and $E_n = \{(v_{ab}, v_{cd}) : a = c \text{ and } b \neq d, \text{ or } b = d \text{ and } a \neq c\}$.

Figure 9 shows the $n^2 = 16$ vertices of λ_4. λ_4 has $n \times n \times (n-1) = 48$ edges:

$$E_4 = \{v_{11}v_{12}, v_{11}v_{13}, v_{11}v_{14}, v_{12}v_{13}, v_{12}v_{14}, v_{13}v_{14}, \ldots, v_{43}v_{44}, v_{11}v_{21}, \ldots, v_{34}v_{44}\}.$$

v_{11}	v_{12}	v_{13}	v_{14}
v_{21}	v_{22}	v_{23}	v_{24}
v_{31}	v_{32}	v_{33}	v_{34}
v_{41}	v_{42}	v_{43}	v_{44}

Figure 7: The vertices of λ_4.

For a positive integer $m \geq 2$ the m-order Sudoku graph $\sigma_m = (V_m, E_m)$ [51, 78, 90] corresponds to an m-order Sudoku square [17, 18, 19, 37, 40, 63, 64, 65], which is an m^2-order Latin square divided into $m \times m$ disjoint subsquares of size $m \times m$ called *blocks.* So σ_m is the finite simple graph with $V_m = \{v_{ij} : 1 \leq i,\ j \leq m^2\}$, and $E_m = \{(v_{ab}, v_{cd} : a = c$ and $b \neq d$, or $b = d$ and $a \neq c$ or $\lfloor (a+1)/m \rfloor = \lfloor (c+1)/m \rfloor$, and $\lfloor (b+1)/m \rfloor = \lfloor (d+1)/m \rfloor$ and $(a,b) \neq (c,d)$.

Figure 8 shows the $9^2 = 81$ vertices of σ_3. σ_3 has $m^2 \times m^2 \times (m^2 - 1) = 648$ edges in the rows and columns, plus $m^2 \times m^2 \times (m-1)^2/2 = 162$ edges in the blocks, so $|E_3| = 810$ as follows:

$$E_3 = \{v_{11}v_{12}, \dots v_{18}v_{19}, \dots, v_{91}v_{92}, \dots, v_{98}v_{99},$$

$$v_{11}v_{21}, \dots, v_{81}v_{91}, \dots, v_{19}v_{29}, \dots, v_{89}v_{99},$$

$$v_{11}v_{22}, \dots, v_{23}v_{32}, \dots, v_{77}v_{88}, \dots, v_{89}v_{98}\}.$$

v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}
v_{21}	v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	v_{28}	v_{29}
v_{31}	v_{32}	v_{33}	v_{34}	v_{35}	v_{36}	v_{37}	v_{38}	v_{39}
v_{41}	v_{42}	v_{43}	v_{44}	v_{45}	v_{46}	v_{47}	v_{48}	v_{49}
v_{51}	v_{52}	v_{53}	v_{54}	v_{55}	v_{56}	v_{57}	v_{58}	v_{59}
v_{61}	v_{62}	v_{63}	v_{64}	v_{65}	v_{66}	v_{67}	v_{68}	v_{69}
v_{71}	v_{72}	v_{73}	v_{74}	v_{75}	v_{76}	v_{77}	v_{78}	v_{79}
v_{81}	v_{82}	v_{83}	v_{84}	v_{85}	v_{86}	v_{87}	v_{88}	v_{89}
v_{91}	v_{92}	v_{93}	v_{94}	v_{95}	v_{96}	v_{97}	v_{98}	v_{99}

Figure 8: The vertices of σ_3,

The following algorithm [28, 29]—defined using the pseudocode of [22]—produces a list coloring of an n-order Latin square λ_n for the case $L(v_{ij}) = \{1, 2, \dots, n\}$ for all v_{ij}. The input parameter is n, and the output is a matrix $M_\lambda = [m_{ij}]_{n \times n}$ of the colors of the vertices of the graph λ_n.

```
LATIN(n, M_λ)
for i ← 1 to n
    do for j ← 1 to n
        do M_λ[i, j] ← i + j − 1 mod n
return M_λ
```

The order of the running time of this algorithm corresponds to the number of elements of the matrix M_λ, it is $\Theta(n^2)$.

If the input is $n = 4$, then the output of Latin is the coloring represented in Figure 9.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Figure 9: An $\mathcal{L}$-coloring of λ_4 produced by algorithm Latin.

It is known [79] that λ_4 has 576 different $\mathcal{L}$-colorings, if all vertex v_{ij} have the same color list $L(v_{ij}) = (1, 2, 3, 4)$, and λ_9 has $5\,524\,751\,496\,156\,892\,842\,531\,225\,600 \sim 5.5 \cdot 10^{27}$ different colorings [79, 88], if all color lists are $L_(v_{ij}) = (1, 2, 3, 4, 5, 6, 7, 8, 9)$. These numbers give the number of 4-order, resp. 9-order Latin squares.

The following algorithm [83] produces a list coloring of the m^2-order Sudoku graph σ_m for the case $L(v_{ij}) = \{1, 2, \ldots, m^2\}$ for all v_{ij}. Its basic idea is similar to the idea of algorithm Latin. The input parameter is m, and the output is the matrix $M_\sigma = [m_{ij}]_{n \times n}$.

Sudoku(m, M_σ)
for $i \leftarrow 1$ **to** m^2
  do for $j \leftarrow 1$ **to** m^2
    do $M_\sigma[i, j] \leftarrow i + j - 1 + \lfloor (i-1)/m \rfloor + m \lfloor (i-1)/m \rfloor \bmod m^2$
return M_σ

The running time of this algorithm also corresponds to the number of elements of the matrix M_σ : it is $\Theta(m^4)$.

If the input is $m = 3$, then the output of Sudoku is the coloring represented in Figure 10.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

Figure 10: The $\mathcal{L}$-coloring of σ_3, produced by algorithm Sudoku.

It is known [80] that σ_2 has 288 different $\mathcal{L}$-colorings, if all lists in $\mathcal{L}$ are equal to $\{1, 2, 3, 4\}$. It is also known [36, 77, 80, 85] that σ_3 has $6\,670\,903\,752\,021\,072\,936\,960 \sim 6.7 \cdot 10^{21}$ different $\mathcal{L}$-colorings, if all lists in $\mathcal{L}$ are equal with $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. These numbers give the number of 4-order, resp. 9-order Sudoku squares.

In the typical Latin and Sudoku puzzles the color of some subset of vertices is fixed (these vertex-color pairs are the input data of the puzzle). The aim is to complete the coloring, that is determine the number of possible complete colorings and to determine the proper colorings.

The first complexity result is due to Colbourn who proved in 1984 [21], that the problem of decision whether the coloring of a partially colored Latin graph can be completed is NP-complete.

Using Colburn's result Yato and Seta proved in 2003 [83], that the Sudoku ASP problem is NP-complete: if we know a proper vertex coloring of a partially colored Sudoku graph, then it is NP-complete problem to decide whether the given graph has another coloring or not.

In 1956 Behrens [9] proposed a more general problem, than the coloring of the Sudoku graph. He divided the set of the n^2 vertices of the n-order Latin graph λ_n into n disjunct subsets, and introduced the additional requirement that the elements of each subset have to be colored by different colors.

In 2007 Cameron asked [16] the complexity of the completion of the coloring of such generalized graphs, and Vaughan proved in 2009 [86] the NP-completeness.

Of course there are special graphs which are colorable by polynomial algorithms. E.g. the books written by Bach, Berthier, Erickson, Gordon, Inkala and Stuart [6, 10, 32, 46, 59, 81] and papers written by Crook [23] and Provan [75] contain numerous polynomial algorithms solving special Sudoku puzzles having unique solution.

Acknowledgement. The European Union and the European Social Fund have provided financial support to the project under the grant agreement no. TÁMOP 4.2.1/B-09/1/KMR-2010-0003.

References

[1] **Aharoni, R., Haxell, P.,** Hall's theorem for hypergraphs. *J. Graph Theory* **35** (2) (2000), 83–88.

[2] **Aharoni, R., Berger, E.,** Menger's theorem for infinite graphs. *Invent. Math.* **176** (1) (2009), 1–62.

[3] **Aharoni, R., Berger, E.,** Rainbow matchings in r-partite r-graphs. *Electron. J. Combin.* **16** (1) (2009). Research Paper 119, 9 pp.

[4] **Altinel, T., Cherlin, G., Corredor, L.-J., Nesin, A.,** A Hall theorem for ω-stable groups. *J. London Math. Soc.* **57** (2) (1998), 385–397.

[5] **Asdre, K., Ioannidou, K., Nicolopoulos, S. D.,** The harmoniouos coloring problem is NP-complete for interval and permutation graphs. *Discrete Appl. Math.* **155** (17) (2007), 2377–2382.

[6] **Bach, C.,** *Sudoku-Trick-Kiste.* AEGIS GmbH., Berlin, 2006.

[7] **Bachelis, G. F.,** A short proof of Hall's theorem on SDRs. *Amer. Math. Monthly* **109** (5) (2002), 473–474.

[8] **Balasubramanian, R., Ponnusamy, S.,** An alternate proof of Hall's theorem on a conformal mapping inequality. *Bull. Belg. Math. Soc. Simon Stevin* **3** (2) (1996), 209–213.

[9] **Behrens, W. U.,** Feldversuchsanordnungen mit verbessertem Ausgleich der Bodenunterschiede. *Zeitschrift fr Landwirtschaftliches Versuchs- und Untersuchungswesen* **2** (1956), 176–193.

[10] **Berthier, D.,** *The Hidden Logic of Sudoku.* 2nd edition. Lulu, Morrisville, NC, 2007.

[11] **Bíró, M., Hujter, M., Tuza, Zs.,** Precoloring extension. I. Interval graphs. *Discrete Math.* **100** (1–3) (1992), 267–279.

[12] **Bogomolny, A.,** Marriage Problem, a constructive proof from Interactive Mathematics Miscellany and Puzzles. http://www.cut-the-knot.org/arithmetic/marriage.shtml, Accessed 08 November 2010.

[13] **Bollobás, B., Varopoulus, N. Th.,** Representation of systems of measurable sets, *Math. Proc. Cambridge Philis. Soc.* **78** (2) (1975), 323–325.

[14] **Bollobás, B., Harris, A. J.,** List coloring of graphs. *Graphs Combin.* **1** (1985), 115–127.

[15] **Borgersen, R. D.,** Equivalence of seven major theorems in combinatorics. 2004. http://robertborgersen.info/Presentations/GS-05R-1.pdf. Accessed 08 November 2010.

[16] **Cameron, P. J.,** Problems from CGCS Luminy, May 2007. *European J. Combin.* **31** (2) (2010), 644–648.

[17] **Carmichael, J., Schloeman, K., Ward, M. B.,** Cosets and Cayley-Sudoku tables. *Math. Mag.* **83** (2) (2010), 130–139.

[18] **Chang, Ch.-Ch., Lin, P.-Y., Wang, Z.-H., Li, M.-Ch.,** A sudoku-based secret image sharing scheme with reversibility. *J. Commun.* **5** (1) (2010), 5–12.

[19] **Chien, Y,-F., Hon W.-K.,** Cryptographic and physical zero-knowledge proof: From Sudoku to nonogram. P. Boldi (ed.), *Fun with Algorithms* (5th International Conference, FUN 2010, Ischia, Italy, June 2–4, 2010.) Springer Verlag, Berlin, 2010. Lecture Notes in Computer Science **6099,** pp. 102–112.

[20] **Chierichetti, F., Vattani, A.,** The local nature of list colorings for graphs of high girth. *SIAM J. Comput.* **39** (6) (2010), 2232-2250.

[21] **Colbourn, C.,** The complexity of completing partial Latin squares. *Discrete Appl. Math.* **8** (1984), 25–30.

[22] **Cormen, T. H., Leiserson, C. E., Rivest, R. L.,** *Introduction to Algorithms.* Third edition. The MIT Press, 2009.

[23] **Crook, J. F.,** A pencil-and-paper algorithm for solving Sudoku puzzles. *Notices Amer. Math. Soc.* **56** (4) (2009), 460–468.

[24] **Cropper, M. M.,** Hall's condition yields less for multicolorings. *Australas. J. Combin.* **18** (1998), 263–266.

[25] **Cropper, M. M., Goldwasser, J. L., Hilton, A. J. W., Hoffman, D. G., Johnson, P. D., Jr.,** Extending the disjoint-representatives theorems of Hall, Halmos, and Vaughan to list-multicolorings of graphs. *J. Graph Theory* **33** (4) (2000), 199–219.

[26] **Cropper M., Gyárfás, A., Lehel, J.,** Edge list multicoloring trees: an extension of Hall's theorem, *J. Graph Theory,* **42** (2003), 246–255.

[27] **Czygrinow, A., Hamdckowiak M., Karonski, M.,** Distributed $O(\Delta \log n)$ -edge-coloring algorithm. In *Proceedings of the 9th Annual European Symposium on Algorithms.* Lecture Notes in Comp. Sci. **2161**, Springer-Verlag, London, 2001, pp. 345–355.

[28] **Dénes, J., Keedwell, A. D.,** *Latin Squares and their Applications.* New York-London: Academic Press, Nerw York and London, 1974.

[29] **Dénes, J., Keedwell, A. D.,** *Latin Squares. New Developments in the Theory and Applications.* North-Holland, Amsterdam, 1991.

[30] **Drechsler, N., Günther, W., Drechsler, R.,** Efficient graph coloring by evolutionary algorithms. In *Computational Intelligence* (Dortmund, 1999). Lecture Notes in Comput. Sci., **1625,** Springer, Berlin, 1999, pp. 30–39.

[31] **Erdős, P., Rubin, A. L., Taylor, H.,** Choosability of graphs. In *Proc. West Coast Conf. Comb., Graph Theory Comp.* (Humboldt Stat Univ., Arcata, Calif. 1979). *Congress. Numer.* **26** (1980), 125–157.

[32] **Erickson, M.,** *Pearls of Discrete Mathematics.* Discrete Mathematics and its Applications. CRC Press, Boca Raton, FL, 2010.

[33] **Esperet, L.,** Dynamic list coloring of bipartite graphs. *Discrete Appl. Math* **158** (17) (2010), 1963–1965.

[34] **Euler, R.,** On the completability of incomplete Latin squares. *European J. Combin.* **31** (2010), 535–552.

[35] **Everett, C. J., Whaples, G.,** Representations of sequences of sets. *Amer. J. Math.* **71** (1949), 287–293.

[36] **Felgenhauer, B., Jarvis, F.,** Mathematics of Sudoku I. *Math. Spectrum,* **39** (2005), 15–22.

[37] **Fontana, R., Rapallo, F., Rogantin, M.P.,** Indicator function and sudoku designs. In P. Gibilisco, E. Ricco-magno, M. P. Rogantin, H. P. Wynn (eds.) *Algebraic and Geometric Methods in Statistics*, Cambridge University Press (2010). pp. 203–224.

[38] **Frota,Y., Maculan, N., Noronha, F., Ribeiro, C. C.,** A branch-and-cut algorithm for partition coloring. *Networks* **55** (3) (2010), 194-204.

[39] **Füredi, Z., Gyárfás, A., Sárközy, G. N., Selkow, S.,** Inequalities for the first-fit chromatic number. *J. Graph Theory* **59** (1) (2008), 75–88.

[40] **Gabor, A. F., Woeginger, G. J.,** How *not* to solve a Sudoku. *Operation Research Letters* (accepted). Accessed 08 November 2010.

[41] **Gagola, S. M. III,** Hall's theorem for Moufang loops. *J. Algebra* **323** (12) (2010), 3252–3262.

[42] **Galvin, F.,** The list chromatic index of a bipartite multigraph, *J. Combin. Theory, Ser. B.* **63** (1) (1995), 153–158.

[43] **Galvin, F.,** A Generalization of Hall's Theorem: 10701, *Amer. Math. Monthly,* **105** (10) (1998), 956–956.

[44] **Galvin, F.,** Problems and Solutions: Revivals: A Generalization of Hall's Theorem: 10701, *Amer. Math. Monthly,* **108** (10) (2001), 982–982.

[45] **Gandhi, R., Halldrsson, M., Kortsarz, G., Shachnai, H.,** Improved bounds for sum multicoloring and scheduling dependent jobs with minsum criteria. In *Approximation and Online Algorithms.* Lecture Notes in Comput. Sci., **3351**, Springer, Berlin, 2005, pp. 68–82.

[46] **Gordon, P., Longo, F.** *Mensa Guide to Solving Sudoku.* Sterling, New York, 2006.

[47] **Guterman, A. P.,** Hall theorem on systems of distinct representatives for modules. In *Paul Erdős and his Mathematics* (Budapest, 1999), János Bolyai Math. Soc., Budapest, 1999, pp. 92–95.

[48] **Guterman, A. E., Kreines, E. M. P.,** Hall's theorem on a system of various representatives for modules. (Russian) *Fundam. Prikl. Mat.* **5** (1) (1999), 119–130.

[49] **Hall, P.,** On representatives of subsets. *J. London Math. Soc.* **10** (1935), 26–30.

[50] **Halmos, P. R., Vaughan, H. E.,** The marriage problem, *Amer. J. Math.* **72** (1950), 214–215.

[51] **Herzberg, A. M., Murty, M. R.,** Sudoku squares and chromatic polynomials. *Notices of the AMS* **54** (6) (2007), 708–717.

[52] **Hilton, A. J. W., Johnson, P. D., Jr.,** Extending Hall's theorem. *Topics in Combinatorics and Graph Theory* (Oberwolfach, 1990). Physica, Heidelberg, 1990, pp. 359–371.

[53] **Hilton, A. J. W., Johnson, P. D., Jr.,** The Hall number, the Hall index, and the total Hall number of a graph. *Discrete Appl. Math.* **94** (1999), 227–245.

[54] **Hilton, A. J. W., Johnson, P. D., Jr.,** List multicolorings of graphs with measurable sets. *J. Graph Theory* **54** (3) (2007), 179–193.

[55] **Hladký, J., Král, D., Sereni, J.-S., Stiebitz, M.,** List colorings with measurable sets. *J. Graph Theory* **59** (3) (2008), 229–238.

[56] **Hoffman, D. G., Johnson, P. D., Jr.,** Extending Hall's theorem into list colorings: a partial history. *Int. J. Math. Math. Sci.* 2007, Art. ID 72168, 17 pp.

[57] **Hujter, M., Tuza, Zs.,** Precoloring extension. II. Graphs classes related to bipartite graphs. *Acta Math. Univ. Comenian.* (N.S.) **62** (1) (1993), 1–11.

[58] **Hujter, M., Tuza, Zs.,** Precoloring extension. III. Classes of perfect graphs. *Combin. Probab. Comput.* **5** (1) (1996), 35–56.

[59] **Inkala, A.** *AI Escargot. The Most Difficult Sudoku Puzzle.* Inkala, Helsinki, 2007.

[60] **Iradmusa, M. N.,** A note on partial list coloring. *Australas. J. Combin.* **46** (2010), 19-24.

[61] **Ivković, Z., Sarnath, R., Sunder, S.,** Fully dynamic algorithms for permutation graph coloring. *Int. J. Comput. Math.* **63** (1–2) (1997), 37–55.

[62] **Jamison, R. E.,** Orientable edge colorings of graphs. *Discrete Appl. Math.* (2010). Electronic corrected proof is available. Accessed 08 November 2010.

[63] **Kanaana, I., Ravikumar, B.,** Row-filled completion problem for Sudoku. *Util. Math.* **81** (2010), 65–84.

[64] **Karimi-Dehkordi, Z., Zamanifar, K., Baraani-Dastjerdi, A., Ghasem-Aghaee, N.,** Sudoku using parallel simulated annealing. In Y. Tan et al. (ed.) *Advances in Swarm Intelligence* (First International Conference, ICSI 2010, Beijing, China, June 12–15, 2010, Proceedings, Part II. Lecture Notes in Computer Science, **6146,** Springer Verlag, Berlin, 2010, pp. 461–467.

[65] **Keedwell, A. D.,** Constructions of complete sets of orthogonal diagonal sudoku squares. *Australas. J. Combin.* **47** (2010), 227–238.

[66] **Lee, S. A.,** k-phase oscillator synchronization for graph coloring. *Math. Comput. Sci.* **3** (1) (2010), 61-72.

[67] **Locke, F., Stephen, C., Kim, S. S., Callan, D.,** Problems and Solutions: Solutions: A Generalization of Hall's Theorem: 10701. *Amer. Math. Monthly* **108** (1) (2001), 79–80.

[68] **Longani, V.,** Extension of Hall's theorem and an algorithm for finding the $(1, n)$-complete matching. *Thai J. Math.* **6** (2) (2008), 271–277.

[69] **Lovász, L.,** A generalization of König's theorem, *Acta Math. Acad. Sci. Hungar.* **21** (1970), 443–446.

[70] **Lovász, L., Plummer, M. D.,** *Matching Theory.* Akadémiai Kiadó and Elsevier Science Publisher B.V., 1974, Budapest/Amsterdam.

[71] **Moshonkin, A. G.,** Concerning Hall's theorem. In *Mathematics in St. Petersburg.* Amer. Math. Soc. Transl. Ser. **2,** 174. Amer. Math. Soc., Providence, RI, 1996, pp. 73–77.

[72] **Nambiar, K. K.,** Hall's theorem and compound matrices. *Math. Comput. Modelling* **25** (3) (1997), 23–24.

[73] **Pach, J., Tardos, G.,** Conflict-free colorings of graphs and hypergraphs. *Combin. Probab. Comput.* **18** (5) (2009), 819–834.

[74] **Pinelis, I.,** An extension of Hall's theorem. *Ann. Comb.* **6** (1) (2002), 103–106.

[75] **Provan, J. S.** Sudoku: strategy versus structure. *Amer. Math. Monthly* **116** (8) (2009), 702–707.

[76] **Rizzi, R.,** A short proof of Königs matching theorem. *J. Graph Theory* **33** (2000), 138–139.

[77] **Russell, E., Jarvis, F.** Mathematics of Sudoku II. *Math. Spectrum* **39** (2005), pp. 54–58.

[78] **Sander, T.,** Sudoku graphs are integral. *Electron. J. Combin.* **16** (1) (2009), Note 25, 7 pp.

[79] **Sloane, N. J. A.,** Number of Latin squares of order n. In *The On-Line Encyclopedia of Integer Sequences.* http://www.research.att.com/∼/sequences/A002860. Accessed 08 November 2010.

[80] **Sloane, N. J. A.,** Number of completed Sudokus of size $m^2 \times m^2$. In *The On-Line Encyclopedia of Integer Sequences.* http://www.research.att.com/∼njas/sequences/A107739. Accessed 08 November 2010.

[81] **Stuart, A. C.,** *The Logic of Sudoku.* First edition. Michael Mepham Publishing, London, 2007.

[82] **Sun, Z.-W.,** Hall's theorem revisited (electronic). *Proc. Amer. Math. Soc.* **129** (10) (2001), 3129–3131.

[83] **Yato, T., Seta, T.,** Complexity and completeness of finding another solution and its application to puzzles. *IEICE Trans. Fundamentals* **E86-A** (5) (2003), 1052–1060.

[84] **Tuza, Zs.,** Hall number for list colorings of graphs: extremal results. *Discrete Math.* **310** (3) (2010), 461-470.

[85] **van der Werl, R.,** Mathematics of Sudoku. In *Sudopedia.* http://www.sudopedia.org/. Accessed 08 November 2010.

[86] **Vaughan, E. R.,** The complexity of constructing gerechte designs. *Electron. J. Combin.* **16** (1) (2009), Research Paper 15, 8 pp.

[87] **Vizing, V. G.,** Coloring the vertices of a graph in precribed colors (Russian). *Metody Diskr. Anal. v Teorii Kodov i Shem* **29** (1976), 3–10.

[88] **Wikipedia,** Latin square. http://en.wikipedia.org/wiki/Latin_square. Accessed 08 November 2010.

[89] **Wilton, H.,** Hall's theorem for limit groups. *Geom. Funct. Anal.* **18** (1) (2008), 271–303.

[90] **Yildirim, H., Krishnamoorthy, M. S., Deo, N.,** A study of the Sudoku graph family. In Proceedings of the Thirty-Ninth Southeastern International Conference on Combinatorics, Graph Theory and Computing. *Congr. Numer.* **192** (2008), 85–95.