Acta Univ. Sapientiae, Mathematica, **4**, 1 (2012) 86–101

Minimal digraphs with given imbalance sequence

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Abstract. Let a and b be integers with $0 \le a \le b$. An (a,b)-*graph* is such digraph D in which any two vertices are connected at least a and at most b arcs. The *imbalance* $a(v)$ of a vertex v in an (a,b)-graph D is defined as $a(v) = d^+(v) - d^-(v)$, where $d^+(v)$ is the outdegree and $d^-(v)$ is the indegree of v. The *imbalance sequence* A of D is formed by listing the imbalances in nondecreasing order. A sequence of integers is (a,b)-*realizable*, if there exists an (a,b)-graph D whose imbalance sequence is A. In this case D is called a *realization* of A. An (a,b)-realization D of A is *connection minimal* if does not exist (a,b')-realization of D with $b' < b$. A digraph D is *cycle minimal* if it is a connected digraph which is either acyclic or has exactly one oriented cycle whose removal disconnects D. In this paper we present algorithms which construct connection minimal and cycle minimal realizations having a given imbalance sequence A.

1 Introduction

Let a, b and n be nonnegative integers with $0 \le a \le b$ and $n \ge 1$. An (a,b)-*graph* is a digraph D in which any two vertices are connected at least a and at most b arcs. If $d^-(v)$ denotes the outdegree and $d^+(v)$ denotes of vertex v in an (a,b)-graph D then the *imbalance* [14] of v is defined as

$$a(v) = d^+(v) - d^-(v).$$

2010 Mathematics Subject Classification: 05C20, 05C07, 60J10, 60J15, 65C05
Key words and phrases: imbalance sequence, arc minimal realization, cycle minimal realization, connection minimal realization

Since loops have no influence on the imbalances therefore for the simplicity we suppose everywhere in this paper that the investigated graphs are loopless.

The *imbalance sequence* of D is formed by listing its imbalances in nondecreasing order (although imbalances can be listed in nonincreasing order as well). The set of distinct imbalances of a digraph is called its *imbalance set*. Mostly the literature on imbalance sequences is concerned with obtaining necessary and sufficient conditions for a sequence of integers to be an imbalance sequence of different digraphs [9, 10, 11, 14, 18, 19, 20, 21], although there are papers on the imbalance sets too [15, 16, 17, 19].

If D is an (a, b)-digraph and A is its imbalance sequence then D is a *realization* of A. If we wish to find a realization of A in any set of directed graphs then

$$\sum_{i=1}^{n} a_i = 0 \tag{1}$$

is a natural necessary condition. If we allow parallel arcs then this simple condition is sufficient to find a realization. If parallel arcs are not allowed then the simple examle $A = [-3, 3]$ shows that (1) is *not sufficient* to find a realization.

Mubayi et al. [14] characterized imbalance sequences of simple digraphs (digraphs without loops and parallel arcs [2, 9, 25]) proving the following necessary and sufficient condition. We remark that simple digraphs are such $(0, 2)$-graphs which do not contain loops and parallel arcs.

Theorem 1 (Mubayi, Will, West, 2001 [14]) *A nondecreasing sequence* $A = [a_1, \dots, a_n]$ *of integers is the imbalance sequence of a simple digraph iff*

$$\sum_{i=1}^{k} a_i \le k(n-k) \tag{2}$$

for $1 \le k \le n$ *with equality when* $k = n$.

Proof. See [14]. □

Mubayi et al. [14] provided a Havel-Hakimi type [3, 4, 7, 8] greedy algorithm Greedy for constructing a simple realization.

The pseudocode of Greedy follows the conventions used in [1].

The *input data* of Greedy are n: the number of elements A ($n \ge 2$); $A = (a_1, \dots, a_n)$: a nondecreasing sequence of integers satisfying (2). Its *output* is M: the $n \times n$ sized incidence matrix of a simple directed graph D

whose imbalance sequence is A. The *working variables* are the cycle variables i, j, k, l, x and y.

```
GREEDY(n, A)
01 for i ← 1 to n                                   // line 01–03: initialization of M
02     for j ← 1 to n
03         M_ij ← 0
04 i = 1                                            // line 04–10: computation of M
05 while a_i > 0
06         Let k = a_i, j_1 < ··· < j_k, further let a_{j_1}, ..., a_{j_k} be
           the k smallest elements among a_{i+1}, ..., a_n, where
           a_x smaller a_y means that a_x < a_y or a_x = a_y and x < y
07         for l ← 1 to k
08             a_l ← a_l + 1
09             M_{i,a_l} ← 1
10         i ← i + 1
11 return M                                         // line 11: return of the result
```

The running time of Greedy is $\Theta(n^2)$ since the lines 1–3 require $\Theta(n^2)$ time, the **while** cycle executes $O(n)$ times and in the cycle line 06 and line 07 require $O(n)$ time.

Kleitman and Wang in 1973 [12] proposed a new version of Havel-Hakimi algorithm, where instead of the recursive choosing the largest remaining element of the investigated degree sequence it is permitted to choose *arbitrary element.* Mubayi et al. [14] point out an interesting difference between the directed and undirected graphs. Let us consider the imbalance sequence $A = [-3, 1, 1, 3]$ of a transitive tournament. Deleting the element 1 and adding 1 to the smallest imbalance leaves us trying to realize $[-2, -1, 3]$, which has no realization among the simple digraphs although it has among the $(0, 2)$-graphs.

Let $\alpha(D)$ denote the number of edges of D. It is easy to see the following assertion.

Lemma 1 *If a directed graph* D *is a realization of a sequence* $A = [a_1, \ldots, a_n]$ *then*

$$\alpha(D) \geq \frac{1}{2} \sum_{i=1}^{n} |a_i|. \tag{3}$$

Proof. Any realization has to contain at least so many outgoing arcs as the sum of the positive elements of A. Since S is realizable for the corresponding set, according to (1) the sum of the absolute values of the negative elements of A equals to the sum of the positive elements, therefore we have to divide the sum in (3) by 2. □

A realization D of A is called *arc minimal* (for a given set of digraphs) if A has no realization (in the given set) containing less arcs than D. Mubayi et al. [14] proved the following characterization of GREEDY.

Lemma 2 (Mubayi et al., 2001 [14]) *If A is realizable for the simple graphs then the realization generated by* GREEDY *contains the minimal number of arcs.*

Proof. See in [14]. □

Lemma 1 and Lemma 2 imply the following assertion.

Theorem 2 *If A is realizable for simple digraphs then the realization generated by* GREEDY *is arc minimal and the number of arcs contained by the realization is given by the lower bound* (3)*.*

Wang [22] gave an asymptotic formula for the number of labeled simple realizations of an imbalance sequence.

In this paper we deal with the more general problem of (a, b)-graphs. The following recent paper characterizes the imbalance sequences of $(0, b)$-graphs.

Theorem 3 (Pirzada, Naikoo, Samee, Iványi, 2010 [19]) *A nondecreasing sequence $A = [a_1, \dots, a_n]$ of integers is the imbalance sequence of a $(0, b)$-graph iff*

$$\sum_{i=1}^{k} a_i \le bk(n-k) \tag{4}$$

for $1 \le k \le n$ with equality when $k = n$.

Proof. See [19]. □

We say that a realization D is *cycle minimal* if D is connected and does not contain a nonempty set of arcs S such that deleting S keeps the digraph connected but preserves imbalances of all vertices. Obviously such a set S, if it exists, must add 0 to the imbalances of vertices incident to it and hence must be a union of oriented cycles. Thus a cycle minimal digraph is either acyclic or has exactly one oriented cycle whose removal disconnects the digraph. For the sake of brevity we shall use the phrase *minimal realization* to refer a cycle minimal realization of A. We denote the set of all minimal realizations of A by $\mathcal{M}(\mathcal{A})$.

A realization D of A is called *connection minimal* (for a given set of directed graphs) if the maximal number of arcs $\gamma(D)$ connecting two different vertices of D is minimal.

The aim of this paper is to construct a connection and a cycle minimal digraph D having a prescribed imbalance sequence A. At first we determine the minimal b which allows to reconstruct the given A. Then we apply a series of arithmetic operations called *contractions* to the imbalance sequence A. This gives us a *chain* $C(A)$ of imbalance sequences. Then by the recursive transformations of $C(A)$ we get a required D.

The structure of the paper is as follows. After the introductory Section 1 in Section 2 we present an algorithm which determines the minimal number of arcs which are necessary between the neighboring vertices to realize a given imbalance sequence then in Section 3 we define a contraction operation and show that the contraction of an imbalance sequence produces another imbalance sequence. Finally in Section 4 we present an algorithm which constructs a connection minimal realization of an imbalance sequence.

2 Computation of the minimal r

According to (1) the sum of elements of any imbalance sequence equals to zero. Let us suppose that according to (1) the sum of the elements of a potential imbalance sequence $P = [p_1, \ldots, p_n]$ is zero and $b = \max(-a_1, a_n)$. Then it is easy to construct such $(0, b)$-digraph D whose imbalance sequence is A connecting the vertices having positive imbalance with the vertices having negative imbalance using the prescribed number of arcs. It is a natural question the value $b_{\min}(P)$ defined as the minimal value of b sufficient for a potential imbalance sequence P to be the imbalance sequence of some $(0, b)$-graph.

$b_{\min}(P)$ has the following natural bounds.

Lemma 3 *If* $A = [a_1, \ldots, a_n]$ *is an imbalance sequence, then*

$$\left\lceil \frac{a_n - a_1}{n} \right\rceil \leq b_{\min} \leq \min(-a_1, a_n). \tag{5}$$

The following algorithm BMIN computes $b_{\min}(A)$ for a sequence $A = [a_1, \ldots, a_n]$ satisfying (4). BMIN is based on Theorem 3, on the bounds given by Lemma 3 and on the logarithmic search algorithm described by D. E. Knuth [13, page 410] and is similar to algorithm MINF-MAXG [6, Section 4.2].

Input. n: the number of elements A ($n \geq 2$);
$A = [a_1, \ldots, a_n]$: a nondecreasing sequence of integers satisfying (4).

Output. $b_{\min}(A)$: the smallest sufficient value of b.

Working variables. k: cycle variable;
l: current value of the lower bound of $b_{\min}(A)$;
u: current value of the upper bound of $b_{\min}(A)$;
S: the current sum of the first k elements of A.

BMIN(n, A)

```
01 l ← ⌈a_n − a_1⌉                    // line 01–02: initialization of l and u
02 u ← min(a_n, −a_i)
03 while l < u      // line 03–14: computation of the minimal necessary b
04        b ← ⌊(l+u)/2⌋
05        S = S ← 0
06        for k ← 1 to n − 1
07            S ← S + a_i
08            if S < bk(n − k)
09               l ← r
10            if l == r + 1
11               b_min ← l + 1
12               return b
13        go to 03
14        u ← b
15 b_min ← l                  // line 15–16: return of the computed minimal b
16 return b_min
```

The next assertion characterizes BMIN.

Lemma 4 *Algorithm* BMIN *computes* $b_{\min}$ *for a sequence* $A = [a_1, \dots, a_n]$ *satisfying* (4) *in* $\Theta(n \log n)$ *time.*

Proof. BMIN computes $b_{\min}$ on the base of Theorem 3 therefore it is correct. Running time of BMIN is $\Theta(n \log n)$ since the **while** cycle executes $\Theta(\log n)$ times and the **for** cycle in it requires $\Theta(n)$ time. □

3 Contraction of an imbalance sequence

Let D be a digraph having n vertices and m arcs. Throughout we assume that the vertices of D are labeled $v_1, \dots, v_n$ according to their imbalances in nondecreasing order while the arcs of D are labeled $e_1, \dots, e_m$ arbitrarily. Let $A = [a_1, \dots, a_n] = [a_{n1}, \dots, a_{nn}]$ be the imbalance sequence of D, where

$a_i = a_{ni}$ is the imbalance of vertex $v_i = v_{ni}$. We define an arithmetic operation, called *contraction* on A as follows.

Let $n \geq 2$, an imbalance sequence $A = [a_1, \ldots, a_n]$ and an ordered pair (a_i, a_j) with $1 \leq i,\ j \leq n$ and $i \neq j$ be given. Then the *contraction* of (a_i, a_j) means that we delete a_i and a_j from A, add a new element $a_i' = a_i + a_j$ and sort nonincreasingly the received sequence using COUNTING-SORT [1] so that the indices of the elements are updated and the updated index of $a_i + a_j$ is denoted by k_i. The new sequence is denoted by $A/(i, j)$.

Note that $j < i$ is permitted. We refer to $A/(i, j)$ as a *minor* of A. Our terminology is inspired by the concept of minor and edge contraction from graph theory [23, 24]. The proof of Theorem 5 explains our choice of terminology.

An imbalance sequence A is a $(0, b)$-*imbalance sequence* if at least one of its realizations is a $(0, b)$-graph. We also observe that if $b' > b$ then a $(0, b)$-imbalance sequence is also a $(0, b')$-imbalance sequence.

The next assertion allows us to construct imbalance sequences and their realizations recursively. It also establishes a relation between the arithmetic operation of contraction discussed above and the edge contraction operation of graphs.

Theorem 4 *If A is a $(0, b)$-imbalance sequence, then all its minors are $(0, 2b)$-imbalance sequences.*

Proof. Let A be the imbalance sequence of a $(0, b)$-graph. Suppose that $B = A/(p, q)$ and let a_p and a_q be both negative with $a_p \leq a_q$. Then $b_l = a_p + a_q$ so that $b_l < a_p$. Thus, for all $k \leq q$, we have

$$\begin{aligned}
\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{k} b_i + (q-k)a_q, && \text{(since all these elements are negative)}\\
&\geq \sum_{i=1}^{q} a_i, && \text{(since } A \text{ is a nondecreasing sequence).}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{q} a_i - (q-k)a_q\\
&\geq \sum_{i=1}^{k} a_i \geq bk(k-n), && \text{(since } A \text{ is an imbalance sequence)}\\
&\geq (2b)k(k-n+1) && \text{(since } n \geq k+2\text{),}
\end{aligned}$$

For $q < k \leq n-1$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &= \sum_{i=1}^{k} a_i \\ &\geq bk(k-n), \qquad \text{(since A is an imbalance sequence)} \\ &\geq (2b)k(k-n+1)\end{aligned}$$

for $n \geq k = 2$ and equality holds when $k = n-1$. Thus in either case B is an imbalance sequence of a $(0, 2b)$-graph by Theorem 3.

By symmetry, we have that Theorem 4 holds if a_p and a_q are both positive.

Now suppose that $a_p \leq 0$ and $a_q \geq 0$ with $|a_p| \geq |a_q|$. If $b_l = a_p + a_q$, then $b_l \leq 0$. For all $k \leq p$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{k} a_i \geq bk(k-n), \qquad \text{(since A is an imbalance sequence)} \\ &\geq (2b)k(k-n+1), \qquad \text{(since } n \geq k+2).\end{aligned}$$

For all $p < k \leq l$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{k} a_i - a_p \geq \sum_{i=1}^{k} a_i \\ &\geq bk(k-n), \qquad \text{(since A is an imbalance sequence)} \\ &\geq (2b)k(k-n+1), \qquad \text{(since } n \geq k+2).\end{aligned}$$

For all $k > l$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{k} a_i + a_q \geq \sum_{i=1}^{k} a_i \\ &\geq bk(k-n), \qquad \text{(since A is an imbalance sequence)} \\ &\geq (2b)k(k-n+1).\end{aligned}$$

The last inequality holds if $n \geq k = 2$, with equality when $k = n-1$. Thus once again B is an imbalance sequence of a $(0, b)$-graph by Theorem 3. By symmetry, Theorem 4 holds if $|a_p| \leq |a_q|$. □

Suppose that D' is a cycle minimal realization of $A/(1, n)$. Then the following algorithm algorithm VERTEX constructs D'', a cycle minimal realization of A.

Input parameters of Vertex are $n \geq 2$: the number of elements of A; $b \geq 1$: the connection parameter of D'; $A = [a_1, \ldots, a_n]$: the imbalance sequence; $A' = A/(1,n) = [a'_1,\ \ldots,\ a'_{n-1}]$; k: index of the element $a'_1 = a_1 + a_n$ in the minor A'; D': a $(0,b)$-graph, which is a cycle minimal realization of $A/(1,n)$ (D' is given by an $(n-1) \times (n-1)$ sized incidence matrix $\mathcal{X} = [x_{i,j}]$).

The output of Vertex is D'': a $(0,q)$-graph, which is a cycle minimal realization of A, where $q = \max(1, b, a_n)$.

```
Vertex(n, A, k, X)
01 read n                          // line 01–04: read of the input data
02 for i ← 1 to n − 1
03     for j ← 1 to n − 1
04         read x_ij
05 for i ← 1 to n − 1              // line 05–08: add an isolated vertex to D'
06     x_in ← 0
07 for i ← 1 to n
08     x_ni ← 0
09 if a_1 = 0 and a_n = 0          // line 09–12: if all a's are equal to zero
10    for i ← 1 to n
11        x_i,i+1 ← 1              // line 12: i + 1 is taken mod n
12        return X
13 x_nk ← a_n                      // line 13: if a_1 < 0
14 return X                        // line 14: return the incidence matrix of D''
```

We now show that Vertex gives a cycle minimal realization of A.

Theorem 5 *The realization* D'' *obtained by* Vertex *is a cycle minimal* $(0, \max(b, 1, a_n))$*-graph. The running time of* Vertex *is* $\Theta(n^2)$.

Proof. If $a_1 = a_n = 0$, then D'' is constructed in lines 09–12 and contains exactly one cycle and is a 1-digraph. If we remove this cycle then remain isolated vertices that is a not connected graph.

If $a_1 < 0$, then due to Theorem 3 $a_n > 0$. In this case D'' is constructed in line 14 connecting the isolated vertex v'_n with the contracted vertex v'_k. So D'' contains a cycle only if the cycle minimal D' contained a cycle, and removing this cycle changes D'' to a not connected graph. In this case D'' is a $\max r, a_n$-graph.

So D'' is a $(0,q)$-graph, where $q = \max(b, 1, a_n)$.

The double cycle in lines 02–04 requires $\Theta(n^2)$ time, and the remaining part of the program requires only $O(n)$ time, so the running time of Vertex is $\Theta(n^2)$. □

4 Construction of a cycle minimal realization

A *chain* of an imbalance sequence $A = A_n = [a_{n1}, \ldots, a_{nn}]$ is a sequence of imbalance sequences $\mathcal{C}(A_n) = [A_n,\ A_{n-1},\ \ldots,\ A_1]$ with $A_n = A$ and $A_{i-1}(A)$ being a minor of $A_i(A)$ for every $1 \le i \le n-1$. *The simple chain* $\mathcal{S}(\mathcal{A})$ of an imbalance sequence A is the sequence of imbalance sequences $[A_n,\ A_{n-1},\ \ldots,\ A_1]$ with $A_n = A$ and $A_{i-1} = [a_{i-1,1}, \ldots, a_{i-1,i-1}]$ being the minor of $A_i = [a_{i,1}, \ldots, a_{i,i}]$ received by the contraction of the first and last element of A_i. It is worth to remark that the simple chain of an imbalance sequence is unique.

Chain is an algorithm for constructing the simple recursion chain $\mathcal{C}(A)$ of A.

The *input data* of Chain are $n \ge 2$: the length of an imbalance sequence $A = [a_{n1}, \ldots, a_{nn}]$; an imbalance sequence A. The *output* of Chain is $\mathcal{C}$: the simple chain of A. *Working variable* is the cycle variable i.

Chain(n, A_n)
01 **read** n // line 01–03: read of the input data
02 **for** $i \leftarrow 1$ **to** n
03 **read** a_{ni}
04 **for** $i \leftarrow n$ **downto** 2 // line 04–05: construction of $\mathcal{C}$
05 delete the first and last elements of A_i, add a new element
$a_{i1} + a_{ii}$, sort nondecreasingly the received sequence and
denote by k_i the index of the new element
06 **return** $\mathcal{C}$ and k // line 06: return of the results

Now, since each contraction in Step 05 of Chain reduces the number of elements of the corresponding imbalance sequence by 1, the last element $A_1(A)$ of the chain contains exactly one element and so the length of the chain is equal to the number of elements n of the imbalance sequence A. Thus for all $1 \le i \le n$ the sequence $A_i(A)$ contains i elements. To every chain of an imbalance sequence A of length n we can associate bijectively a chain of $n-1$ ordered pairs with i element equal to (j, k), where $A_{n-i} = A_{n-i+1}/(j, k)$. That is (v_j, v_k) is contracted to obtain A_{n-i} from A_{n-i+1}. This bijection allows us to represent every chain of imbalance sequences by the sequence of pairs (j, k).

We present a simple algorithm Realization for associating a small cycle minimal realization D'' to any imbalance sequence A.

Input values are $n \ge 2$: the number of elements of A; A: an imbalance sequence; D': a cycle minimal $(0, b)$-graph which is a realization of $A/(1, n)$

and is given by its incidence matrix X.

The *output* of REALIZATION is D'', a $(0, q)$-graph, which is a cycle minimal realization of A; $k = [k_1, \ldots, k_{n-1}]$: the sequence of the updated indices of the elements received by contraction. D'' is represented by its incidence matrix X, and $q = \max(b, 1, a_n)$. *Working variable* is i: cyclic variable.

```
REALIZATION(n, A)
01 read n                         // line 01–03: read of the input data
02 for i ← 1 to n
03     read a_ij
04 CHAIN(n, A)                    // line 04: construction the simple chain C(A)
05 x_11 ← 0                       // line 05: construction of D''_1
06 for i ← 2 to n                 // line 06–07: recursive construction of D''_n
07     VERTEX(i, A_i, k, X_{i−1})
08 return D''_n and k             // line 08: return of the constructed minimal digraph
```

The next assertion shows that REALIZATION is correct and constructs a cycle minimal realization of an imbalance sequence in polynomial time.

Theorem 6 *Let* A *be a* $(0, b)$*-imbalance sequence having* n *entries and let* $C(A) = [(a_1, b_1) \ldots (a_n, b_n)]$ *be a chain of* A*. Then there exists a cycle minimal digraph* D *having* n *vertices such that* D *is reconstructible from* C *and* D *is a* $q = \max(r, 1, a_n)$*-realization of* A*. Moreover, this reconstruction requires* $O(n^2)n$ *time.*

Proof. By VERTEX, the digraph D_n which is the output of REALIZATION, is assured to be a cycle minimal realization of A. Now, VERTEX constructs D_i from D_{i-1} in $O(n)$ time and there are $n - 1$ such constructions. Thus REALIZATION runs in $O(n^2)$ time. □

The following example illustrate the work of algorithms VERTEX, CHAIN and REALIZATION.

Example 1 *Let* $A = [-2, -2, -2, -1, 3, 4]$*. Figure 1 shows a realization of* A*, therefore* A *is an imbalance sequence.*

Figure 1 also shows that this realization is a 1-digraph. Since there are nonzero imbalances therefore all realizations have to contain arcs so this realization is connection minimal. Since all realization of A has to contain at least

$$m_{\min} = \frac{\sum_{i=1}^{n} a_i}{2}$$

arcs, and now $m_{\min} = 7$, so D is also an arc minimal realization.

Vertex/Vertex	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	0	0	0
v_2	0	0	0	0	0	0
v_3	0	0	0	0	0	0
v_4	0	0	0	0	0	0
v_5	1	1	1	0	0	0
v_6	1	1	1	1	0	0

Figure 1: Incidence matrix of a realization of $A = [-2, -2, -2, -1, 3, 4]$

Now we construct a cycle minimal realization of A using Realization. After the reading of the input data in lines 01–03 Chain constructs the simple chain $\mathcal{S} = [A_1, \ldots, A_6]$ and $k = [k_1, k_2, k_3, k_4, k_5] = [1, 1, 2, 3, 4]$, where $A_6 = A = [-2, -2, -2, -1, 3, 4]$, $A5 = [-2, -2, -1, 2, 3]$, $A_4 = [-2, -1, 1, 2]$, $A_3 = [-1, 0, 1]$, $A_2 = [0, 0]$ and $A_1 = [0]$.

After the construction of C Realization sets $x_{11} = 0$ in Step 5 and so it defines $\mathcal{X}_1$, the incidence matrix of D_1 consisting of an isolated vertex v_1. Then it constructs D_2, …, D_6 in lines 06–07 calling Vertex recursively: at first $k_1 = 1$ helps to construct D_2 having the incidence matrix X_2 which is shown in Figure 2.

Vertex/Vertex	v_1	v_2
v_1	0	1
v_2	1	0

Figure 2: Incidence matrix of D_2 (X_2)

Now using $k_2 = 1$ D_3 is constructed. The result is the incidence matrix X_3 shown in Figure 3.

The next step is the construction of D_4 using $k_3 = 2$. Figure 4 shows X_4.

The next step is the construction of D_5 using $k_4 = 3$. Figure 5 shows X_5.

The final step is the construction of D_6 using $k_5 = 4$. Figure 6 shows X_6.

It is worth to remark that D_6 contains 9 arcs while the realization of A whose incidence matrix is shown in Figure 6 contains only the necessary 7 arcs and is also a cycle minimal realization of A.

The graph D'_6 whose incidence matrix X'_6 shown in Figure 7 contains only

Vertex/Vertex	v_1	v_2	v_3
v_1	0	1	0
v_2	1	0	0
v_3	1	0	0

Figure 3: Incidence matrix of D_3 (X_3)

Vertex/Vertex	v_1	v_2	v_3	v_4
v_1	0	1	0	0
v_2	1	0	0	0
v_3	1	0	0	0
v_4	2	0	0	0

Figure 4: Incidence matrix of D_4 (X_4)

Vertex/Vertex	v_1	v_2	v_3	v_4	v_5
v_1	0	1	0	0	0
v_2	1	0	0	0	0
v_3	1	0	0	0	0
v_4	2	0	0	0	0
v_5	3	0	0	0	0

Figure 5: Incidence matrix of D_5 (X_5)

Vertex/Vertex	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	0	0	0	0
v_2	1	0	0	0	0	0
v_3	1	0	0	0	0	0
v_4	2	0	0	0	0	0
v_5	3	0	0	0	0	0
v_6	0	0	0	4	0	0

Figure 6: Incidence matrix of D_6 (X_6)

7 arcs and is also a cycle minimal realization of A.

Vertex/Vertex	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	0	0	0
v_2	0	0	0	0	0	0
v_3	0	0	0	0	0	0
v_4	0	0	0	0	0	0
v_5	1	1	1	0	0	0
v_6	1	1	1	1	0	0

Figure 7: Incidence matrix of D'_6 (X'_6)

Acknowledgements

The authors thank Professor Zoltán Kása (Sapientia Hungarian University of Transylvania) and the unknown referee for their useful corrections.

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Received: August 2, 2012