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On vertex independence number of uniform hypergraphs

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Abstract. Let H be an r-uniform hypergraph with $r \geq 2$ and let $\alpha(H)$ be its vertex independence number. In the paper bounds of $\alpha(H)$ are given for different uniform hypergraphs: if H has no isolated vertex, then in terms of the degrees, and for triangle-free linear H in terms of the order and average degree.

1 Introduction to independence in graphs

Let n be a positive integer. A *graph* G on vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is a pair (V, E), where the edge set E is a subset of $V \times V$. n is the *order* of G and $|E|$ is the *size* of G.

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Let $v \in V$ and $N(v)$ be the *neighborhood* of v, namely, the set of vertices x so that there is an edge which contains both v and x. Let U be a subset of V, then the *subgraph* of G induced by U is defined as a graph on vertex set U and edge set $E_U = \{(u, v) | u \in u \text{ and } v \in U\}$.

The *degree* $d(v)$ of a vertex $v \in V$ is the number of edges that contains v. Let $d(G)$ be the *average degree* of G, then $nd(G) = \sum_{v \in V} d(v) = 2|E|$ for any graph G. Let $\delta(G)$ be the *minimal degree*, $\Delta(G)$ the *maximal degree* of G A graph G is *regular*, if $\Delta(G) = \delta(G)$, and it is *semi-regular*, if $\Delta(G) - \delta(G) = 1$.

Three vertices v_1, v_2, v_3 form *a triangle* in G if there are distinct verticess $e_1, v_2, v_3 \in F$ such that $\{v_i, v_{i+1}\} \subseteq E$, where the indices are taken mod 3. If G does not contain a triangle, then it is *trianglefree*.

A subset $U \subseteq V$ of vertices in a graph G is called a *vertex independent set* if no two vertices in U are adjacent. The maximum-size vertex independent set is called *maximum vertex independent set*. The size of the maximum vertex independent set is called *vertex independence number* and is denoted by $\alpha(G)$. The problem of finding a vertex maximum independent set and vertex independence number are NP-hard optimization problems [73, 167].

A *maximal vertex independent set* is a vertex independent set such that adding any other vertex to the set forces the set to contain an edge. The problem of finding a maximal vertex independent set can be solved in polynomial time (see e.g. the algorithms due to Tarjan and Trojanowski [155], Karp and Widgerson [101], further the improved algorithms due to Luby [128] and Alon [9].

There are exponential time exact (as Alon [9]) and polynomial time approximate algorithms (as Boppana and Haldórsson [30], Agnarsson, Haldórsson, and Losievskaja [4, 5], Losievskaja [126]) determining $\alpha(G)$. Also there are known algorithms producing the list of all maximum independent sets of graphs (see e.g. Johnson and Yannakakis [93], Lawler, Lenstra, Rinnooy Kan [121]).

An *independent edge set* of a graph G is a subset of the edges such that no two edges in the subset share a vertex of G [166]. An independent edge set of maximum size is called a *maximum independent edge set*, and an independent edge set that cannot be expanded to another independent edge set by addition of any other edge in the graph is called a *maximal independent edge set*. The size of the largest independent edge set (i.e., of any maximum independent edge set) in a graph is known as its *edge independence number* (or *matching number*), and is denoted by $\nu(G)$. The determination of $\nu(G)$ is an easy task for bipartite graphs [49, 50], but it is a polynomially solvable problem for general graphs too [10, 101, 161, 162].

Let $G = (V, E)$ be an n-order graph. The classical Turán theorem [159] gives a simple lower bound for $\alpha(G)$.

Theorem 1 (Turán [159]) *If* $n \geq 1$ *and* G *is an* n*-order graph, then*

$$\alpha(G) \geq \frac{n}{d(G)+1}\,. \tag{1}$$

This result was strengthened independently in 1979 by Caro and in 1981 by Wei.

Theorem 2 (Caro [36], Wei, [165]) *If* $G(V, E)$ *is a graph, then*

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}\,. \tag{2}$$

Proof. See [36, 165]. □

A nice probabilistic proof of the result can be found in the paper of Alon and Spencer [11]. Since the function $\frac{1}{x+1}$ is convex, $\sum_{v \in V} \frac{1}{d(v)+1} \geq \frac{n}{d+1}$ [170].

Since this bound is the best-possible only for graphs which are unions of cliques, additional structural assumptions excluding these graphs allow improvement of 2 [80, 81]. A natural candidate for such assumptions is connectivity. In 2013 Angel, Campigotto, and Laforest [14] improved (2) for some connected graphs. For locally sparse graphs Ajtai, Erdős, Komlós and Szemerédi improved Turán's bound greatly.

Theorem 3 (Ajtai, Erdős, Komlós and Szemerédi [6, 7, 8]) *If* G *is an* n*-order triangle-free graph with average degree* d*, then*

$$\alpha(G) \geq \frac{cn \ln d}{d+1}\,. \tag{3}$$

Proof. See [6, 7, 8]. □

They conjectured that $c = 1 - o(1)$ when d tends to ∞. Griggs [72] improved that c can be $\frac{5}{12}$. Shearer [152] finally proved $c = 1 - o(1)$, thus confirming the conjecture. In 1994 Selkow improved the bound due to Caro and Wei supposing that the degrees of the neighbors of the vertices are also known.

Theorem 4 (Selkow [150]) *If* $G(V, E)$ *is a graph, then*

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1} \left(1 + \max\left(0, \frac{d(v)}{d(v)+1} - \sum_{u \in N(v)} \frac{1}{d(u)+1}\right)\right). \tag{4}$$

Proof. See [150]. □

The bound of Selkow is equal to Caro–Wei bound for regular graph and always less then twice the Caro–Wei bound. A recent review on lower bounds for 3-order graphs was published by Henning and Yeo [89].

Let j and k be a positive integers. A subset $I \subseteq V(G)$ is a *vertex-k-independent set* of G, if every vertex in I has at most $k-1$ neighbors in I. The *vertex-k-independence number* $\alpha_k(G)$ of G is the cardinality of the largest vertex-k-independent set of G.

A subset $D \subseteq V(G)$ is a *vertex-j-dominating set* of G, if every vertex of D has at least $j-1$ neighbors in D. The *vertex-k-independence number* $\gamma_j(G)$ of G is the cardinality of the largest vertex-j-dominating set of G.

In 1991 Caro and Tuza [38] extended theorem of Turán to the estimation of the maximal size of k-independent sets. Thiele [156] in 1999, Csaba, Pick, and Shokoufandeh [44] in 2012 improved the bound due to Caro and Tuza. In 2008 Favoron, Hansberg and Volkmann [54] analyzed k-domination and minimum degree in graphs. Harant, Rautenbach, and Schiermeier [81, 83, 84, 85] proved different lower bounds on vertex independent number.

In 2012 Chellali and Rad [42] published a paper on k-independence critical graphs. In 2013 Caro and Hansberg [37] proposed a new approach to k-independence of graphs. Recently Chellali, Favaron, Hansberg, and Volkmann [41] published a review on k-independence.

Last year Hansberg and Pepper [79] investigated the connection between $\alpha_k(G)$ and $\gamma_j(G)$. They proved the following theorems.

Theorem 5 (Hansberg, Pepper [79]) *If Let G be an n-order graph, j, k and m be positive integers such that $m = j + k - 1$ and let H_m and G_m denote, respectively, the subgraphs induced by the vertices of degree at least m and the vertices of degree at least m. Then*

$$\alpha_k(H_m) + \gamma_j(G_m) \leq n \tag{5}$$

and

$$\alpha_k(G) + \gamma_j(G) \leq n(G_m). \tag{6}$$

Proof. See [79]. □

Theorem 6 (Hansberg, Pepper [79]) *Let G be a connected n-order graph with maximum degree Δ and minimum degree $\delta \geq 1$. Then*

$$\alpha_k(G) + \gamma_j(G) = n(G) \quad \textit{and} \quad \alpha_{k'}(G) + \gamma_{j'}(G) = n(G) \tag{7}$$

for every pair of integers j, k *and* j', k' *such that* $j+k-1 = \delta$ *and* $j'+k'-1 = \Delta$ *if and only ig* G *is regular.*

Proof. See [79]. □

Theorem 7 (Hansberg, Pepper [79]) *For any graph* G *the following two statements are equivalent:*

$$\gamma(G) + \alpha_\delta(G) = n(G) \tag{8}$$

and

$$G \quad \textit{is regular or} \quad \gamma(G) + \gamma_2(G) = n(G). \tag{9}$$

Proof. See [79]. □

Spencer [153] also published some extension of Turán theorem.

In 2014 Henning, Löwenstein, Southey and Yeo [87] proved the following theorem, which is an improvement of the result due to Fajtlowicz [53].

Theorem 8 (Henning et al. [87]) *If* G *is a graph of order* n *and* p *is an integer, such that for every clique* X *in* G *there exists a vertex* $x \in X$ *such, that* $d(x) < p - |X|$, *then* $\alpha(G) \geq 2n/p$.

There are results on the independence number of random graphs (e.g. Balogh, Morris, Samotij [18] and Frieze [60], Henning, Löwenstein, Southey and Yeo [87], on the weighted independence number (see e.g. Halldórsson [75], Kako, Ono, Hirata, and Halldórsson [98], further Sakai, Mitsunori, and Yamazaki [149]), and on the enumeration of maximum independent sets (see e.g. Gaspers, Kratsch, and Liedloff [69].

Let $G(n,p) = (V,E)$ the random graph with vertex set $V = \{v_1, \ldots, v_n\}$, p, $\alpha(G_{n,p})$ denote the independence number of $G_{n,p}$. In 1990 Frieze [60] proved, that if $d = np$ and $\epsilon > 0$ is fixed, then with probability going to 1 as $n \to \infty$

$$\left| \alpha(G_n, p) - \frac{2n(\ln d - \ln\ln d - \ln 2 + 1)}{d} \right| \leq \frac{\epsilon n}{d}, \tag{10}$$

provided $d_\epsilon \leq d = o(n)$, where d_ϵ is some fixed constant and p is the join probability for each edge to be included in E.

In 1983 Shearer proved the following lower bound.

Theorem 9 (Shearer [152]) *If* G *is triangle-free, then*

$$\alpha(G) \geq nf(d), \tag{11}$$

where

$$f(x) = \frac{x \ln x - x + 1}{(x-1)^2}\,, \tag{12}$$

$f(0) = 1$ *and* $f(1) = \frac{1}{2}$.

According to the proof of Shearer for $0 < x < \infty$ hold $0 < f(d) < 1$, $f'd) < 0$ and $f''(d) < 0$. Further $f(x)$ satisfies the differential equation

$$(x+1)f(x) = (x+1)d^2 f'(x). \tag{13}$$

It is easy to see that

$$\lim_{x\to\infty} \frac{f(x)}{x} = \frac{\ln x}{x}\,. \tag{14}$$

In 1995 Füredi [62] determined the number of different vertex maximal independent set in path graphs.

It is known [22] a minimum covering set of G is also a maximum vertex independent set of G. Therefore we are interested in the results on dominating sets (see e.g. [41, 54, 79, 82, 143].

The structure of the paper is as follows. After this introduction in Section 2 we present a review of results connected with th vertex and edge independence number of hypergraphs, then in Section 3 a lower bound of $\alpha(H)$ is presented for n-order r-uniform hipergraphs with average degree $d(H)$, and finally in Section 4 a similar bound is proved for hypergraphs not containing isolated vertex.

2 Introduction to independence in hypergraphs

Let $n \geq 1$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a finite set called *vertex set*. A *hypergraph* H on vertex set W is a pair (W, F), where the edge set F is a family of the elements of W. We always assume that distinct edges are distinct as subsets. If each edge in F contains exactly $r \geq 2$ vertices, then H is r-*uniform*. So any graph G is a 2-uniform hypergraph.

Let $w \in W$ and $N(w)$ be the *neighborhood* of w, namely, the set of vertices x so that there is an edge which contains both w and x. Let U be a subset of W. The *sub-hypergraph* of H induced by U is defined as a hypergraph on vertex set U with edge set $F_U = \{f \in F : f \subseteq U\}$.

The *degree* $d(w)$ of a vertex $w \in W$ is the number of edges that contain w. Let $d(H) = d$ be the *average degree* of an r-uniform H, then $nd = \sum_{w\in W} d(w) = r|F|$.

For the simplicity we usually omit G and H as arguments of $d(H)$ and similar notations.

A hypergraph H is *linear*, if any two edges of H have at most one vertex in common. Note that a graph G is always linear. Three vertices w_1, w_2, w_3 *form a triangle* in H, if there are distinct edges $f_1, f_2, f_3 \in F$ such that $\{f_i, f_{i+1}\} \subseteq F$, where the indices are taken mod 3.

A subset $U \subseteq W$ of vertices in a hypergraph H is called a *vertex independent set* if no two vertices in U are adjacent. The maximum-size vertex independent set of H is called *maximum vertex independent set*. The size of the maximum vertex independent set is called *vertex independence number* and is denoted by $\alpha(H)$. The problem of finding a maximum vertex -independent set and vertex independence number are NP-hard optimization problems [73, 167].

There are exponential time exact (as Alon [9], Tarjan and Trojanowski [155]) and polynomial time approximate algorithms (as Boppana and Haldórsson [30], Agnarsson, Haldórsson, and Losievskaja [4, 5], Losievskaja [126]). Also there are known algorithms producing the list of all maximum independent sets of graphs (see e.g. Johnson and Yannakakis [93], Lawler, Lenstra, Rinnooy Kan [121]) and hypergraphs (see e.g. Kelsen [107]).

A *maximal vertex independent set* is a vertex independent set such that adding any other vertex to the set forces the set to contain an edge. The problem of finding a maximal vertex independent set can be solved in polynomial time (see e.g. the algorithms due to Tarjan and Trojanowski [155], Karp and Widgerson [101], further the improved algorithms due to Luby [128] and Noga [9]).

In 2012 Dutta, Mubayi, and Subramanian [48] gave new lower bond for the vertex independence number of sparse hypergraphs.

In 2013 Eustis devoted a PhD dissertation to the problems of hypergraph independence numbers [51, 52].

An *independent edge set* of a hypergraph H is a subset of the edges such that no two edges in the subset share a vertex of H [136]. An independent edge set of maximum size is called a *maximum independent edge set*, and an independent edge set that cannot be expanded to another independent edge set by addition of any other edge in the hypergraph is called a *maximal independent edge set*. The size of the largest independent edge set (i.e., of any maximum independent edge set) in a hypergraph is known as its *edge independence number* (or *matching number*), and is denoted by $\nu(H)$. The determination of $\nu(H)$ is an easy task for bipartite graphs [49, 50], but it is a polynomially solvable problem for general graphs too [10].

There are many results on the characterization of hypergraph score sequences and on their reconstruction (see e.g. [20, 110, 140, 171, 139, 164, 172]), on the enumeration of different hypergraphs (see e.g. [21, 47, 138, 144, 145]) and directed hypergraphs (see e.g. [15]).

An r-uniform hypergraph with n vertices is called *complete*, if its set of edges has the cardinality $\binom{n}{2}$. The *complement* of an r-uniform hypergraph H is $\overline{H} = (W, \overline{F})$, if $|F \cup \overline{F}| = \binom{n}{2}$ and $|F \cap \overline{F}| = 0$.

A set $P \subseteq W$ is called an *edge cover* of H, if for any non-isolated vertex $x \in W$ there exists an edge $f_i \in P$ that $x \in f_i$. The cardinality of a minimum set which is an edge covering of H is called the *edge covering number* of H, and is denoted by $\nu(H)$.

The following lemma, proved in [97], gives a relation between the edge covering number and the edge independence number in an r-uniform hypergraph H without isolated vertices.

Lemma 10 (Jucovič, Olejník [97]) *For an r-uniform n-order hypergraph H with n without isolated vertices the following inequalities hold:*

$$\alpha(H) \leq n - (kr - 1)\nu(H), \tag{15}$$

$$\alpha(H) + (r - 1)\nu(H) \leq n. \tag{16}$$

$$\nu(H) + (r - 1)r - 1\nu(H) \geq n, \tag{17}$$

Proof. See [97]. $\square$

This lemma generalizes the relations published by Gallai [67] in 1959. In 1991 Tuza [160] extended Gallai's inequalty for uniform hypergraphs.

In 1989 Olejník proved the following three theorems characterizing $\alpha(H)$ and $\nu(H)$.

Theorem 11 (Olejník [136]) *For an r-uniform n-order hypergraph $H = (W, F)$ with n and its complement $\overline{H} = (W, \overline{F})$*

$$\left\lfloor \frac{n}{r} \right\rfloor \leq \nu(H) + \nu(\overline{H}) \leq 2 \left\lfloor \frac{n}{r} \right\rfloor \tag{18}$$

and

$$0 \leq \nu(H)\nu(\overline{H}) \leq \left\lfloor \frac{n}{r} \right\rfloor^2. \tag{19}$$

Proof. See [136]. $\square$

This bounds are direct generalizations of the bounds published by Chartrand and Schuster in 1974 [40].

Theorem 12 (Olejník [136]) *For an* r*-uniform* n*-order hypergraph* $H = (W, F)$ *and its complement* $\overline{H} = (W, \overline{F})$*, where neither* H *nor* $\overline{F}$ *have isolated vertices,*

$$\left\lfloor \frac{n}{r} \right\rfloor \leq \nu(H) + \nu(\overline{H}) \leq 2 \left\lfloor \frac{n}{r} \right\rfloor \tag{20}$$

and

$$0 \leq \nu(H)\nu(\overline{H}) \leq \left\lfloor \frac{n}{r} \right\rfloor^2. \tag{21}$$

Proof. See [136]. □

This result is an extension of the work of R. Laskar and B. Auerbach published in 1978 [120].

Theorem 13 (Olejník [136]) *For an* r*-uniform* n*-order hypergraph* $H = (W, F)$ *and its complement* $\overline{H}, \overline{F}$*, where neither* H *nor* $\overline{H}$ *have isolated vertices and* $n \neq 2r$

$$2 \left\lfloor \frac{n}{r} \right\rfloor \leq \alpha H + \alpha\overline{H} \leq 2n - (r-1) \left\lfloor \frac{n}{r} \right\rfloor - r + 1 \tag{22}$$

and

$$\left\lfloor \frac{n}{r} \right\rfloor^2 \leq \alpha(H)\alpha(\overline{H}) \leq \frac{1}{4}\left(2n - (r-1)\left\lfloor \frac{n}{r} \right\rfloor - k + 1\right)^2. \tag{23}$$

Proof. See [136]. □

In 1993 Gallo, Longo, Nguyen, and Pallottino [68] studied the applications of directed hypergraphs. In 2004 Vietri [163] wrote on the complexity of the arc-coloring of directed hypergraphs. In 2003 Frank, Király and Király [55] analized the orientation of directed hypergraphs.

Let

$$B(p, q) = \int_0^1 (1-t)^{p-1} t^{q-1} \, dt \tag{24}$$

denote the beta-function with $p,\ q > 0$. Set constants $0 < a \leq 1$, $0 < b \leq 1$, and $B = B(a, 1-b)$, and let

$$f_r(x) = \frac{1}{B}\int_0^1 \frac{1-t)^a}{(t^b[1+(x-1)t]} dr. \tag{25}$$

In 2004 Zhou and Li [170] proved the following theorem on sparse hypergraphs.

Theorem 14 (Zhou, Li [170]) *Let* H *be a triangle-free,* r*-uniform* $(r \geq 2)$ n*-order linear hypergraph with average degree d. Then its strong vertex independence number* $\alpha_s(G)$ *is at least* $nf_r(d)$*.*

Proof. See [170]. □

In 2004 Greenhill, Ruciński, and Wormald [71] analyzed random hypergraph processes with degree restrictions. In 2008 Plociennik [141] proposed an approximation algorithm for the vertex maximum independence set problem of uniform random hypergraphs. M. Halldórsson, and Losievskaja [4, 5] used semidefinit programming to find maximum vertex independent set of hypergraphs.

Shearer's result ([152], further (11) and (12)) was generalized in [170] with the function $g_r(x)$ satisfying

$$(r-1)^2 x(x-1)g_r'(x) + [(r-1)x+1]g_r(x) = 1 \tag{26}$$

for r-uniform, triangle-free linear hypergraphs, with sparse neighborhood and in [125] with the function $g_{r,m}(x)$ satisfying

$$(r-1)^2 x(x-m)g_{r,m}'(x) + [(r-1)x+1]g_{r,m}(x) = 1 \tag{27}$$

for r-uniform, triangle-free, and double linear hypergraphs, in which each subhypergraph induced by a neighborhood, has maximum degree less than m. A linear hypergraph is called *double linear* if for any non-adjacent distinct vertices w and z, each edge containing w has at most one neighbor of z. From the uniqueness of solutions of the differential equations, we see that $g_2(x) = g(x)$ and $g_{r,1}(x) = g_r(x)$. It is shown [125] that $g_{2,m}(x) \sim \frac{\log x}{x}$, and for $g_{r,m}(x) \sim \frac{c}{d^{1/(r-1)}}$ for $r \geq 3$, where $c = c(r,m) > 0$ is a constant without knowing exact values.

Independent sets and numbers are studied in many papers (see e.g. the papers of Abraham [1], Alon, Uri and Azar [12], Berger and Ziv [23], Bollobás, Daykin and Erdős [27], Bonato, Brown, Mitsche and Pralat [28, 29], Bordewich, Dyer and Karpiński [31], Boros, Gurvich, Elbassioni, Gurvich and Khachiyan [32, 33], Borowiecki and Michalak [34], Cutler and Radcliffe [45], Greenhill [70], Halldórson and Losievskaja [76], Hofmeister and Lehman [90], Johnson and Yannakakis [93], Khachiyan, Boros, Gurvich, and Elbassioni [108], Lepin [122], Li and Zhang [125], Losievskaja [126], Shachnai and Srinivasan [151], Tarjan and Trojanowski [155], Yuster [168]).

Since independence number and matching number are closely connected, we are interested in the results on maximum matching algorithms too (see e.g. [25, 26, 46, 47, 49, 50, 56, 57, 61, 65, 66, 77, 78, 86, 88, 89, 91, 92, 100, 104, 105, 109, 112, 113, 118, 119, 127, 131, 132, 133, 135, 137, 142, 146, 147, 148, 154, 157, 158, 169]).

Minimum dominating set of H and maximum vertex independent set of H are connected concepts, therefore we are interested in the results on dominating sets of hypergraphs (see e.g. [2, 96]).

Further connected problems are also often analyzed (see e.g. e.g. in the papers of Agnarsson, Egilssson, and Halldórson [3], Alon, Frankl, Huan, Rödl, Ruciński [10], Alon and Yuster [13], Baranyai [19], Balogh, Butterfield, Hu and Lenz [17], Bertram-Kretzberg and Letzman [24], Bujtás and Tuza [35], Cockayne, Hedetniemi, and Laskar [43], Frank, Király and Király [55], Frankl and Rödl [58, 59], Füredi, Ruszinkó, and Selver [63, 64], Hán, Person and Schacht [78], Henning and Yeo [89], Huang, Loh and Sudakov [92], Johnson and Yannakakis [93], Johnston and Lu [94, 95], Jucovič and Olejník [97], Karoński and Luczak [99], Katona [102, 103], Keevash and Sudakov [106], Kelsen [107], Kohayakawa, Rödl, Skokan [111], Krivelevich [115], Kühn and Loose [117], Kostochka, Mubayi, Verstraëte [114], Krivelevich, Nathaniel, and Sudakov [116], Li, Rousseau and Zang [123, 124], Luczak and Szymańska [129, 134], Szymańska [154], Treglown and Zhao [157, 158], Tuza [160], Yuster [169]).

Although hypergraphs are less often used in the practice than the graphs, they also have different applications in the practice.

For example Bailey, Manoukian, Ramamohanaro [16], further Gunopolus, Khardon, Mannila and Toivonen [74] reported on the applications in data mining, Gallo, Longo, Nguyen, and Pallottino [68], further and Maier [130] in relational databases.

In 2000 Carr, Lancia, Istrail, and Genomics [39] reported on Branch-and-Cut algorithms for vertex independent set problem and on their application to solve problems connected with protein structure alignment.

In this paper, we obtain $\alpha(H) \geq \sum_{v\in V} \frac{1-1/r}{d(v)^{1/(r-1)}}$ for any r-uniform hypergraph H without the condition of being triangle-free. The algorithm is naive: it deletes a vertex of maximum degree repeatedly. In order to get a large independent set, a commonly used algorithm is to find a suitable vertex v, then delete v and its neighbors, and then do the iterations. Deleting all neighbors seems to be of no use for hypergraphs as in [125, 170]. After deleting a vertex v, we delete only one vertex other than v from each edge containing v. Our new function $f_r(x)$ satisfies

$$[(r-1)x^2 - x]f_r'(x) + (x+1)f_r(x) = 1. \tag{28}$$

Then $f_r(x) \sim \frac{c}{x^{1/(r-1)}}$ as $x \to \infty$. We do not know the exact value of $c = c(r)$. However, when we run the algorithm, we note that for a vertex v, we delete $1 + d(v)$ vertices instead of deleting $1 + (r-1)d(v)$ vertices as in [125, 170]. So if c is the constant such that $g_r(x) \sim \frac{c}{x^{1/(r-1)}}$ as $x \to \infty$, then the new constant seems to be $(r-1)c$, namely, $f_r(x) \sim \frac{(r-1)c}{x^{1/(r-1)}}$.

3 Bound for uniform hypergraphs without isolated vertex

The following Theorem 15 is a corollary of Theorem 18, but it has an easy probabilistic proof.

Theorem 15 *Let $H = (V, E)$ be an r-uniform hypergraph of order n and average degree $d \geq 1$, then*

$$\alpha(H) \geq \left(1 - \frac{1}{r}\right) \frac{n}{d^{1/(r-1)}}. \tag{29}$$

Proof. Define a random subset $U \subseteq V$ by $\mathtt{Pr}(v \in U) = p$ for some $0 \leq p \leq 1$ with all these events being mutually independent over $v \in V$.

Let $X(U)$ be the number of vertices in U and let $Y(U)$ be the number of edges in the subgraph induced by U. Note that for one of the edges of H, the probability that all of its vertices belong to U is p^r. By linearity of expectation, we have

$$E(X - Y) = E(X) - E(Y) = np - \frac{nd}{r}p^r. \tag{30}$$

Thus there exists a set U satisfying

$$X(U) - Y(U) \geq E(X) - E(Y). \tag{31}$$

Note that U is not that we require, since the sub-hypergraph of H induced by U may have edges. However, if we delete one vertex from each edge contained in U, then at most $Y(U)$ vertices are deleted, we thus obtain a new set with at least $E(X) - E(Y)$ vertices and whose induced sub-hypergraph has no edges. The desired lower bound follows by taking $p = \frac{1}{d^{1/(r-1)}}$. □

For hypergraphs that are not regular, Theorem 18 is stronger than Theorem 15. We need two lemmas for the proof of Theorem 18.

Lemma 16 *Let $r \geq 2$ be an integer and define*

$$h_r(x) = \begin{cases} 1 - x/r & \text{if } 0 \leq x < 1 \\ \frac{1-1/r}{x^{1/(r-1)}} & \text{if } x \geq 1, \end{cases} \tag{32}$$

then $h_r(x)$ is positive, decreasing and convex. Furthermore, for $x \geq 1$, the function $h_r(x)$ satisfies that $(r-1)x\,h'(x) + h_r(x) = 0$.

Proof. It is easy to see that $h_r(x)$ is positive and

$$h'_r(x) = \begin{cases} -1/r & \text{if } \ 0 \le x < 1 \\ \frac{-1/r}{x^{r/(r-1)}} & \text{if } \ x \ge 1. \end{cases} \tag{33}$$

So $h'_r(x)$ is continuous, negative and increasing, thus $h_r(x)$ is decreasing and convex. The fact that $h_r(x)$ satisfies the mentioned differential equation is straightforward. □

Let $\Delta = \Delta(H)$ denote the maximal degree in H and define

$$S(G) = \sum_{x \in V} h(d(x)), \quad S(H) = \sum_{x \in W} h(d(x)). \tag{34}$$

Lemma 17 *If* $\Delta(H) \ge 1$, $w \in W$, $d(w) = \Delta(H)$, *and* $H_1 = H - \{w\}$, *then* $S(H_1) \ge S(G)$.

Proof. For each $x \in V \setminus \{v\}$, denote by n_x the number of edges of H that contain both x and v. Then $n_x = 0$ if x and v are not adjacent, and $n_x \ge 1$ otherwise. It is easy to see

$$\sum_{x \in V \setminus \{v\}} n_x = (r-1)\Delta \tag{35}$$

since H is r-uniform. On the other hand, we have

$$S(H_1) = S(H) - h(\Delta) + \sum_{x \in V \setminus \{v\}} [h(d(x) - n_x) - h(d(x))]. \tag{36}$$

From the fact that $h'(x)$ is negative and increasing, we have

$$h(d(x) - n_x) - h(d(x)) = -h'(\theta_x)n_x \ge -h'(\Delta)n_x, \tag{37}$$

where $\theta_x \in [d(x) - n_x, d(x)]$, thus

$$\begin{aligned} S(H_1) &\ge S(H) - h(\Delta) - h'(\Delta) \sum_{x \in V \setminus \{v\}} n_x \\ &= S(H) - h(\Delta) - (r-1)\Delta h'(\Delta) \\ &= S(H), \end{aligned}$$

proving the claim. □

Theorem 18 *Let* $H = (V, E)$ *be an* r*-uniform hypergraph without isolated vertex, then*

$$\alpha(H) \geq \left(1 - \frac{1}{r}\right) \sum_{v \in V} \frac{1}{d(v)^{1/(r-1)}}. \tag{38}$$

Proof. We write $h_r(x)$ as $h(x)$ for simplicity and define

$$S(H) = \sum_{x \in V} h(d(x)). \tag{39}$$

Repeat the algorithm by deleting the vertex of maximum degree if the degree is at least one, terminate the algorithm if there are no edges. Denote by $H_0 = H, H_1, \ldots, H_\ell$ for the sequence of hypergraphs, where H_ℓ has no edge. We get $S(H_\ell) = n - \ell$ since $h(0) = 1$, where $n - \ell$ is the order of H_ℓ, and $\alpha(H) \geq n - \ell$. So

$$\alpha(H) \geq S(H_\ell) \geq S(H_{\ell-1}) \geq \cdots \geq S(H_0) = S(H), \tag{40}$$

the assertion follows immediately. □

Since the function $\frac{1}{x^{1/(r-1)}}$ is convex, Theorem 15 is truly a corollary of Theorem 18.

Remark. Theorem 18 gives $\alpha(G) \geq \sum_v \frac{1}{2d(v)}$ for a graph G with $\delta(G) \geq 1$, which is weaker than $\alpha(G) \geq \sum_v \frac{1}{d(v)+1}$. However, the later can be proved similarly by replacing the function $h(x)$ with $1/(x+1)$. For details of this algorithm, see Griggs [72].

4 Bound for uniform linear triangle-free hypergraphs

In this section triangle-free hypergraphs are considered. To generalize Shearer's method [152] and to delete less vertices for a hypergraph, we have a definition as follows.

Let $H = (V, E)$ be an r-uniform hypergraph and let v be a vertex of H, denote by $E_v = \{e \in E : v \in e\} = \{e_1, e_2, \ldots, e_{d(v)}\}$ for the set of edges containing v. A *claw* of v is a set of neighbors of v of the form $\{u_1, u_2, \ldots, u_{d(v)}\}$ such that each $u_i \in e_i - v$. For a claw T of v, we write as Q_T, the number of edges that intersect T.

When we run the algorithm in each step, we will delete v and a claw T, so Q_T edges will be deleted. The new function is as follows.

Let $r \geq 2$ be and integer and let $b = \frac{r-2}{r-1}$. Define

$$f_r(x) = \frac{1}{r-1}\int_0^1 \frac{1-t}{t^b[1+((r-1)x-1)t]}dt. \tag{41}$$

Lemma 19 *The function $f_r(x)$ satisfies the differential equation*

$$[(r-1)x^2 - x]f_r'(x) + (x+1)f_r(x) = 1, \tag{42}$$

and it is positive, decreasing and convex.

Proof. By differentiating under the integral and then integrating by parts, we have

$$\begin{aligned}
& [(r-1)x^2 - x]f_r'(x) \\
= \; & -[(r-1)x^2 - x]\int_0^1 \frac{1-t}{t^{1-b}[1+((r-1)x-1)t]^2}dt \\
= \; & x\int_0^1 (1-t)t^{1-b}\frac{d}{dt}\left(\frac{1}{1+[(r-1)x-1]t}\right) \\
= \; & -x\int_0^1 \frac{1}{1+[(r-1)x-1]t}[(1-t)(1-b)t^{-b} - t^{1-b}]dt \\
= \; & -(r-1)(1-b)xf_r(x) + x\int_0^1 \frac{t^{1-b}}{1+[(r-1)x-1]t}dt \\
= \; & -xf_r(x) + \frac{1}{r-1}\int_0^1 \left(\frac{1}{1-t} - \frac{1}{1+[(r-1)x-1]t}\right)(1-t)t^{-b}dt \\
= \; & -xf_r(x) + 1 - f_r(x) \\
= \; & 1 - (x+1)f_r(x)
\end{aligned}$$

which follows by the differential equation. The monotonicity and convexity of $f_r(x)$ can be seen by repeated differentiation under the integral. □

Theorem 20 *Let H be an r-uniform n-order hypergraph with average degree d. If it is triangle-free and linear, then $\alpha(H) \geq nf_r(d)$.*

Proof. We apply induction on $|V|$, the number of vertices of H. The result is trivial for $|V| = 1$, since $f(0) = 1$. Since the case $r = 2$ is exactly what Shearer has given, we suppose that $r \geq 3$.

For each $v \in H$, let $T = \{u_1, u_2, \dots, u_{d(v)}\}$ be a claw of v. Since H is r-uniform, linear and triangle-free, we have

$$Q_T = d(v) + \sum_{i=1}^{d(v)} (d(u_i) - 1) = \sum_{i=1}^{d(v)} d(u_i). \tag{43}$$

Let $\mathcal{T}_v$ be the set of all claws of v, then $|\mathcal{T}_v| = (r-1)^{d(v)}$. Therefore

$$\sum_{T \in \mathcal{T}_v} Q_T = \sum_{T \in \mathcal{T}_v} \sum_{i=1}^{d(v)} d(u_i) = \sum_{u \in n(v)} (r-1)^{d(v)-1} d(u), \tag{44}$$

and

$$\frac{1}{|\mathcal{T}_v|} \sum_{T \in \mathcal{T}_v} Q_T = \sum_{u \in n(v)} \frac{d(u)}{r-1}. \tag{45}$$

We write $f(x)$ for $f_r(x)$ and set

$$R_T(v) = 1 - (d(v)+1) f(d) + (d d(v) + d - r Q_T) f'(d). \tag{46}$$

Then the average of $R_T(v)$ among $T \in \mathcal{T}_v$ is

$$\frac{1}{|\mathcal{T}_v|} \sum_{T \in \mathcal{T}_v} R_T(v) = 1 - (d(v)+1) f(d) + (d d(v) + d) f'(d) - r \sum_{u \in n(v)} \frac{d(u)}{r-1} f'(d). \tag{47}$$

Note that

$$\frac{1}{n} \sum_{v \in V} \sum_{u \in N(v)} \frac{d(u)}{r-1} = \frac{1}{n} \sum_{v \in V} d^2(v) \geq d^2 \tag{48}$$

as x^2 is a convex function. Since $f'(x) < 0$, we have

$$\frac{1}{n} \sum_{v \in V} \frac{1}{|\mathcal{T}_v|} \sum_{T \in \mathcal{T}_v} R_T(v) \geq 1 - (d+1) f(d) + (d^2 + d - r d^2) f'(d) = 0. \tag{49}$$

Hence there exists a vertex, say v, and a claw of v, say $T = \{u_1, u_2, \dots, u_{d(v)}\}$, such that $R(v) \geq 0$. Now by deleting v and $u_1, u_2, \dots, u_{d(v)}$, we obtain a new hypergraph H' with $n - d(v) - 1$ vertices and $\frac{nd}{r} - Q_T$ edges. For an edge e containing v, it contains $r \geq 3$ vertices, and we delete exactly two vertices from e, so H' has some vertices. Note that the average degree $\bar{d}$ of H' is $\frac{nd - rQ_T}{n - d(v) - 1}$. By induction hypothesis, we have

$$\alpha(H) \geq (n - d(v) - 1) f(\bar{d}) = (n - d(v) - 1) f\left(\frac{nd - rQ_T}{n - d(v) - 1} \right). \tag{50}$$

Combining the facts that $\alpha(H) \geq 1 + \alpha(H')$ and $f(x) \geq f(d) + f'(d)(x-d)$ for all $x \geq 0$ as $f(x)$ is convex, we obtain

$$\begin{aligned}\alpha(H) &\geq 1 + (n - d(v) - 1) f\left(\frac{nd - rQ_T}{n - d(v) - 1}\right)\\ &\geq 1 + (n - d(v) - 1) f(d) + (dd(v) + d - rQ_T) f'(d)\\ &= nf(d) + R(v) \geq nf(d)\end{aligned}$$

completing the proof. □

We now get an asymptotic form of $f_r(x)$ as $\frac{c}{x^{1/(r-1)}}$ without knowing exact expression of $c = c(r)$ in hope of improving the old constant based on analysis of the algorithm as mentioned.

Lemma 21 *Let* $r \geq 3$ *be an integer. Then*

$$\lim x \to \infty f_r(x) = \frac{c}{x^{1/(r-1)}}, \tag{51}$$

where $c = c(r)$ *is a positive constant.*

Proof. Recall that a first order linear differential equation $\frac{dy}{dx} = p(x)y + q(x)$ has the unique solution of the form

$$y = e^{\phi(x)}\left(y_0 + \int_{x_0}^{x} q(t)e^{-\phi(t)}dt\right) \tag{52}$$

satisfying $y_0 = y(x_0)$, where $\phi(x) = \int_{x_0}^{x} p(t)dt$. From the differential equation that $f_r(x)$ satisfies, we set

$$p(x) = -\frac{x+1}{(r-1)x^2 - x}, \quad \text{and} \quad q(x) = \frac{1}{(r-1)x^2 - x}. \tag{53}$$

For $x_0 = 2$,

$$\phi(x) = -\int_2^x \frac{t+1}{(r-1)t^2 - t}dt = \ln \frac{c_1 x}{[(r-1)x - 1]^{\frac{r}{r-1}}} \tag{54}$$

Hence

$$e^{\phi(x)} = \frac{c_1 x}{[(r-1)x - 1]^{\frac{r}{r-1}}} \sim \frac{c_2}{x^{1/(r-1)}}. \tag{55}$$

Then we have

$$q(t)e^{-\phi(t)} \sim \frac{1}{c_2(r-1)} x^{1/(r-1)-2}, \tag{56}$$

implying that $c_3 = \int_2^\infty q(t)e^{-\phi(t)}dt < \infty$, and $\int_2^x q(t)e^{-\phi(t)}dt = c_3 + o(1)$ as $x \to \infty$. Therefore,

$$f_r(x) = e^{\phi(x)}\left(y_0 + c_3 + o(1)\right) \sim \frac{c}{x^{1/(r-1)}}, \tag{57}$$

where $c = c_2(y_0 + c_3)$ and $y_0 = f_r(2)$ are positive constants. □

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