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CONNECTIONS BETWEEN EXPONENTIAL STABILITY AND BOUNDEDNESS OF SOLUTIONS OF A COUPLE OF DIFFERENTIAL TIME DEPENDING AND PERIODIC SYSTEMS

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Abstract. Among others, we prove that the vectorial time dependent q-periodic differential system

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{C}^n \qquad (A(t))$$

is uniformly exponentially stable (i.e. all its solutions decay exponentially at infinity) if and only if for each vector $b \in \mathbb{C}^n$, the solution of the Cauchy Problem

$$\dot{y}(t) = A(t)y(t) + e^{i\mu t}b, \quad t \geq 0, \quad b \in \mathbb{C}^n, \quad y(0) = 0$$

is bounded on $\mathbb{R}_+$, uniformly in respect with the parameter μ on the entire real axis. As a consequence, we get that the system $(A(t))$ is uniformly exponentially stable if and only if for each vector $x \in \mathbb{C}^n$, the map

$$t \mapsto \int_0^t | < \Phi(t)\Phi(s)^{-1}(s)x, x > | ds$$

is bounded on $\mathbb{R}_+$. This latter result is a weak version of the Barbashin theorem which seems to be new. Here $\Phi(t)$ is the fundamental matrix associated to the system $(A(t))$.

1. Introduction

Let consider the vectorial time dependent and q-periodic system

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{C}^n \qquad (A(t))$$

and the Cauchy Problem

$$\begin{cases} \dot{y}(t) = A(t)y(t) + e^{i\mu t}f(t), & t \geq 0 \\ y(0) = 0. \end{cases} \qquad (A(t), \mu, f, 0)$$

Here, f is a $\mathbb{C}^n$-valued continuous function defined on $\mathbb{R}_+$.

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By $\mathcal{F}_{\mathbf{\Phi}}(\mathbb{R}_+, \mathbb{C}^n)$ will denote the set of all q-periodic and continuous functions $f_b : \mathbb{R}_+ \to \mathbb{C}^n$, whose restrictions to the interval $[0, q]$ are given by $f_b(s) = h(s)\Phi(s)b$, with $b \in \mathbb{C}^n$. Here $\Phi(\cdot)$ is the fundamental matrix associated to the vectorial homogeneous system $(A(t))$. See the next section for further details. The map $h(\cdot)$ belongs to the set $\{h_1(\cdot), h_2(\cdot)\}$ where h_1 and h_2 are scalar valued functions, defined on the interval $[0, q]$, by:

$$h_1(s) = \begin{cases} s, & s \in [0, \frac{q}{2}); \\ q - s, & s \in [\frac{q}{2}, q] \end{cases} \quad \text{and } h_2(s) = s(q - s).$$

As is known, see [7], the system $(A(t))$ is uniformly exponentially stable if and only if for each real number μ and each function $f_b \in \mathcal{F}_{\mathbf{\Phi}}(\mathbb{R}_+, \mathbb{C}^n)$, the solution y_{μ, f_b} of the Cauchy Problem $(A(t), \mu, f_b, 0)$, is bounded on $\mathbb{R}_+$. In the present note we improve this result showing that it may be preserved if consider only $\mathbb{C}^n$-valued constant functions instead of functions in $\mathcal{F}_{\mathbf{\Phi}}(\mathbb{R}_+, \mathbb{C}^n)$. However, our boundedness assumption is stronger than the given one in [7]. More exactly, we require that the solution of the Cauchy Problem is bounded on $\mathbb{R}_+$ uniform in respect with the parameter μ on the real axis. Our interest to the present result is stimulated by the discrete analogous one proved recently in [3] and [1]. The main ingredient of the proof in [3] is that to use a development in a Fourier series of a smooth, q-periodic, $\mathbb{C}^n$-valued function whose restriction to the set of all positive integer numbers is a given q-periodic sequence (z_n) decaying at zero. Then the assertion is a consequence of a result in [6]. In the discrete case, q is an integer number greater than one. The infinite dimensional version of the described result is also proved in [3]. This was possible because the linear span of the range of the sequence (z_n) is a finite dimensional one. The Fourier method was successfully applied by Lan Thanh Nguyen in his research on higher order differential equations in Hilbert spaces. See [11]. In this paper we use other trick. Namely, we prove that each function in $\mathcal{F}_{\mathbf{\Phi}}(\mathbb{R}_+, \mathbb{C}^n)$ satisfies a Lipschitz condition on $\mathbb{R}_+$ and then, by an well-known classical Fourier theorem ([16], pp. 93), it belongs to the space $AP_1(\mathbb{R}_+, \mathbb{C}^n))$ of all almost periodic functions in the sense of Bohr, whose associated series of the Fourier-Bohr coefficients is absolutely convergent in $\mathbb{C}^n$. Further details about the space $AP_1(\mathbb{R}_+, \mathbb{C}^n))$ may be found, for example, in the recent monograph [8] of Constantin Corduneanu.

Our interest is also stimulated by the possibility to find new real integral characterizations for the exponential stability of the system $(A(t))$. It is widely known that such characterizations are very useful in the control theory and in the stability theory, especial when we want to build Lyapunov functions associated to the system $(A(t))$. We refer only to the theorems of the Datko type and to the theorems of Barbashin type which are briefly presented in as follows. The theorem of Datko states that a (non necessarily periodic) system $(A(t))$ is uniformly exponentially stable if and only if each trajectory $t \mapsto \Phi(t)(\Phi)^{-1}(s)b : [s, \infty) \to \mathbb{C}^n$ of the evolution family $\mathcal{U} = \{\Phi(t)(\Phi)^{-1}(s)\}_{t \geq s}$ belongs to $L^p([s, \infty))$ and for each $b \in \mathbb{C}^n$, the map $s \mapsto \|\Phi(\cdot)\Phi^{-1}(s)b\|_p$, is bounded on $\mathbb{R}_+$, for some (and then for all) $p \geq 1$. Further details refereing to this type of theorems in the general framework of strongly continuous evolution families of bounded linear operators acting on a Banach space may be found in [9]. For further proofs and generalizations of such theorems and for different approaches of this theory we refer to [13],[10], [14], [15], and the references therein. The uniform variant of the theorem of Barbashin states that the system $(A(t))$ is uniformly exponentially stable if and only if the map $t \mapsto \int\limits_0^t ||U(t, s)||^p ds : [0, \infty) := \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $L^\infty(\mathbb{R}_+)$ for some (and then for all) $p \geq 1$. Further details and proofs may be found in [2], where the finite dimensional case is treated, and in [12] where the general framework of strongly continuous evolution families of bounded linear operators acting on a Banach space is analyzed. The strong variant of the theorem of Barbashin asserts that the evolution family $\{U(t, s)\}$ of bounded linear operators acting on a Banach space X is uniformly exponentially stable if for each $x \in X$ and some $p \geq 1$, the following estimation holds true.

$$\sup_{t \in \mathbb{R}_+} \int_0^t ||U(t, s)x||^p ds := M_p(x) < \infty. \qquad (SBA)$$

Surprisingly, the proof of the strong variant of the theorem of Barbashin seems to be more difficult and as long we can see, it is still an open problem for strongly continuous evolution families acting on an arbitrary Banach space. In this direction, some progress has made in [5], where the dual family of $\mathcal{U}$ was involved, and the estimation like

(SBA), is related to the strong operator topology in $\mathcal{L}(X^*)$. In the discrete and periodic case the strong variant of the Barbashin problem is completely solved in [1] and [3]. In this paper we clarify such result in the continuous case of the finite dimensional and time dependent and periodic systems.

The paper is organized as follows. Second section contains the necessary definitions and preliminary results for that the paper to be self-contained. In the third section we state and prove the announced results and establish its natural consequences.

2. Notations and preliminary results

Let X be a Banach space and let $\mathcal{L}(X)$ be the space of all bounded linear operators acting on X. The norm in X and in $\mathcal{L}(X)$ is denoted by the same symbol, namely $\|\cdot\|$.
A family $\mathcal{U} = \{U(t,s) : t \geq s\} \subset \mathcal{L}(X)$ is called evolution family if

- $U(t,t) = I$ and
- $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r \geq 0$,

where I denote the identity operator on $\mathcal{L}(X)$.
An evolution family $\mathcal{U}$ is called strongly continuous if for each $x \in X$ the map

$$(t,s) \mapsto U(t,s)x : \{(t,s) \in \mathbb{R}^2 : t \geq s \geq 0\} \to X$$

is continuous. Such a family is called q-periodic (with some $q > 0$) if

$$U(t+q, s+q) = U(t,s), \text{ for all pairs } (t,s) \text{ with } t \geq s \geq 0.$$

Clearly, a q-periodic evolution family also satisfies

$$U(pq+\rho, pq+u) = U(q,u), \quad \forall p \in \mathbb{N}, \quad \forall \rho \geq u \in \mathbb{R}_+$$

and

$$U(pq, rq) = U((p-r)q, 0) = U(q,0)^{p-r}, \quad \forall p \in \mathbb{N}, r \in \mathbb{N}, p \geq r.$$

The family $\mathcal{U}$ is called uniformly exponential stable if there exist two positive constants N and ν such that

$$\|U(t,s)\| \leq Ne^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0.$$

In the next proposition we collect some equivalent characterizations for the exponential stability of a q-periodic evolution family.

Proposition 2.1. *Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a strongly continuous and q-periodic evolution family acting on the Banach space X. The following four statements are equivalent:*

(1) The family $\mathcal{U}$ is uniformly exponentially stable.
(2) There exist two positive constants N and ν such that

$$\|U(t,0)\| \leq Ne^{-\nu t}, \text{ for all } t \geq 0.$$

(3) The spectral radius of $U(q,0)$ is less than one, i.e.

$$r(U(q,0)) := \sup\{|\lambda|, \lambda \in \sigma(U(q,0))\} = \lim_{n\to\infty} \|U(q,0)^n\|^{\frac{1}{n}} < 1.$$

(4) For each $\mu \in \mathbb{R}$, one has

$$\sup_{\nu\geq 1} \left\| \sum_{k=1}^{\nu} e^{-i\mu k} U(q,0)^k \right\| := L(\mu) < \infty.$$

Proof. The proof of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious. The proof of (4) $\Rightarrow$ (1) can be found in [6].

The result of this section is based on the next technical lemma.

Lemma 2.2. *Let us consider the functions h_1, $h_2 : [0,q] \to \mathbb{C}$, defined by:*

$$h_1(s) = \begin{cases} s & , \quad s \in [0, \frac{q}{2}) \\ q-s & , \quad s \in [\frac{q}{2}, q] \end{cases} \quad \text{and} \quad h_2(s) = s(q-s), \quad s \in [0,q].$$

Denote $H_1(\mu) := \int\limits_0^q h_1(s)e^{-i\mu s}ds$ and $H_2(\mu) := \int\limits_0^q h_2(s)e^{-i\mu s}ds$. Then, $H_1(\mu) = 0$ if and only if $\mu \in \mathcal{A} := \{\frac{4k\pi}{q} : k \in \mathbb{Z}\backslash\{0\}\}$. Moreover, $H_2(\mu) = \frac{-2q^3}{(4k\pi)^2} \neq 0$ for all $\mu \in \mathcal{A}$.

Proof. For $\mu \neq 0$, a simple computation shows that

$$\int\limits_0^q h_1(s)e^{-i\mu s}ds = \frac{1}{(i\mu)^2}\left(1 - 2e^{-\frac{i\mu q}{2}} + e^{-i\mu q}\right).$$

The first assertion is now clear. Let $\mu_k := \frac{4k\pi}{q}$. Then

$$\int\limits_0^q h_2(s)e^{-i\mu_k s}ds = q\int\limits_0^q se^{-i\mu_k s}ds - \int\limits_0^q s^2 e^{-i\mu_k s}ds$$

$$= q\left(-\frac{se^{-i\mu_k s}}{i\mu_k}\Big|_0^q + \int_0^q \frac{e^{-i\mu_k s}}{i\mu_k}ds\right) - \left(-\frac{s^2e^{-i\mu_k s}}{i\mu_k}\Big|_0^q + 2\int_0^q \frac{se^{-i\mu_k s}}{i\mu_k}ds\right)$$

$$\begin{aligned}
&= -\tfrac{qe^{-i\mu_k s}}{(i\mu_k)^2}\Big|_0^q - 2\left[-\tfrac{se^{-i\mu_k s}}{(i\mu_k)^2}\Big|_0^q + \int_0^q \tfrac{e^{-i\mu_k s}}{(i\mu_k)^2}ds\right]\\
&= -\tfrac{qe^{-i\mu_k q}}{(i\mu_k)^2} + \tfrac{q}{(i\mu_k)^2} - 2\left[-\tfrac{qe^{-i\mu_k q}}{(i\mu_k)^2} - \tfrac{e^{-i\mu_k s}}{(i\mu_k)^3}\Big|_0^q\right]\\
&= \tfrac{2q}{(i\mu_k)^2} = -\tfrac{2q^3}{(4k\pi)^2}.
\end{aligned}$$

Let $P_q^0(\mathbb{R}_+, X)$ be the set of all continuous X-valued functions f defined on $\mathbb{R}_+$, with $f(0) = 0$ and $f(t+q) = f(t)$ for $t \in \mathbb{R}_+$. Next theorem is essentially contained in article [4] of the second named author of this paper, but we include here by sake of completeness and because there the proof is not complete.

Theorem 2.3. *Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a strongly continuous and q-periodic evolution family on the Banach space X. If for each $\mu \in \mathbb{R}$ and each $f \in P_q^0(\mathbb{R}_+, X)$, one has*

$$\sup_{t>0} \|\int_0^t e^{-i\mu s}U(t,s)f(s)ds\| := K(\mu, f) < \infty, \tag{2.1}$$

then $\mathcal{U}$ is uniformly exponentially stable.

Proof. Let $V = U(q,0), x \in X, x \neq 0, n \in \mathbb{N}$ and $g \in P_q^0(\mathbb{R}_+, X)$ given on $[0,q]$ by:

$$g(s) = h_1(s)U(s,0)x, \quad s \in [0,q].$$

From (2.1), for $t = (n+1)q$ and denoting

$$T_k(g) = \int_{qk}^{q(k+1)} U((n+1)q, s)e^{-i\mu s}g(s)ds$$

results

$$\sup_{n\in\mathbb{N}} \left\|\sum_{k=0}^{n} T_k(g)\right\| := L(\mu, g) < \infty. \tag{2.2}$$

Now, for each $k = 0, 1, \ldots,$ we have:

$$
\begin{aligned}
T_k(g) &= \int\limits_{qk}^{q(k+1)} U((n+1)q,(k+1)q)U((k+1)q,s)e^{-i\mu s}g(s)ds \\
&= \int\limits_{qk}^{q(k+1)} U(q(n-k),0)U((k+1)q,s)e^{-i\mu s}g(s)ds \\
&= V^{n-k}\int\limits_{qk}^{q(k+1)} U((k+1)q,s)e^{-i\mu s}g(s)ds \\
&= V^{n-k}\int\limits_{0}^{q} U((k+1)q,kq+u)e^{-i\mu(kq+u)}g(kq+u)du \\
&= V^{n-k}e^{-i\mu kq}\int\limits_{0}^{q} U(q,u)e^{-i\mu u}g(u)du \\
&= V^{n-k}e^{-i\mu kq}\int\limits_{0}^{q} e^{-i\mu u}h_1(u)U(q,u)U(u,0)xdu
\end{aligned}
$$

$$
\begin{aligned}
&= V^{n-k}e^{-i\mu kq}\int\limits_{0}^{q} e^{-i\mu u}h_1(u)U(q,0)xdu \\
&= e^{-i\mu kq}\left(\int\limits_{0}^{q} e^{-i\mu u}h_1(u)du\right)V^{n-k+1}x \\
&= H_1(\mu)e^{-i\mu(n+1)q}e^{i\mu(n-k+1)q}V^{n-k+1}x.
\end{aligned}
$$

Using Lemma 2.2, we may write

$$
e^{i\mu(n-k+1)q}V^{n-k+1}x = \frac{1}{H_1(\mu)}e^{-i\mu(n+1)q}T_k(g), \mu \notin \mathcal{A}.
$$

Set

$$
\widetilde{g}(s) = h_2(s)U(s,0)x, \quad s \in [0,q].
$$

Puting $\widetilde{g}$ instead of g, we get

$$
e^{i\mu(n-k+1)q}V^{n-k+1}x = \begin{cases} \frac{1}{H_1(\mu)}e^{-i\mu(n+1)q}T_k(g), \mu \notin \mathcal{A} \\ \frac{1}{H_2(\mu)}e^{-i\mu(n+1)q}T_k(\widetilde{g}), \mu \in \mathcal{A}. \end{cases}
$$

Then

$$
\left\|\sum_{k=0}^{n} e^{i\mu(n-k+1)q}V^{n-k+1}x\right\| \le \begin{cases} \frac{1}{|H_1(\mu)|}L(\mu,g) & ,\mu \notin \mathcal{A} \\ \frac{1}{|H_2(\mu)|}L(\mu,\widetilde{g}) & ,\mu \in \mathcal{A}. \end{cases}
$$

Finaly, we obtain:

$$
\sup_{n\in\mathbb{N}}\left\|\sum_{j=1}^{n+1} e^{i\mu jq}V^j\right\| < \infty.
$$

The assertion follows now from Proposition 2.1.

$\square$

3. Connections between the uniform exponential stability and boundedness

Let us consider the vectorial time dependent and q-periodic system

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R}, x(t) \in \mathbb{C}^n \tag{$A(t)$}$$

and the vectorial Cauchy Problem

$$\begin{cases} \dot{y}(t) = A(t)y(t) + e^{i\mu t}b, & t \geq 0, b \in \mathbb{C}^n; \\ y(0) = 0. \end{cases} \tag{$A(t), \mu, b, 0$}$$

By $\Phi(\cdot)$ we denote the fundamental matrix associated to the system $(A(t))$. As is well-known $\Phi(\cdot)$ is the unique solution of the Cauchy Problem

$$\begin{cases} \dot{X}(t) = A(t)X(t), & t \in \mathbb{R} \\ X(0) = I. \end{cases} . \tag{$A(t), I$}$$

Throughout in the paper, we assume that the map

$$t \mapsto A(t) : \mathbb{R} \to \mathcal{M}(n, \mathbb{C})$$

is continuous and q-periodic. The matrix $\Phi(t)$ is an invertible one and its inverse is the unique solution of the Cauchy Problem

$$\begin{cases} \dot{X}(t) = -X(t)A(t), & t \in \mathbb{R} \\ X(0) = I. \end{cases}$$

Set $U(t,s) := \Phi(t)\Phi^{-1}(s)$, for all $t \geq s$. Obviously, the family $\mathcal{U} = \{U(t,s) : t \geq s\}$ has the following properties:

1. $U(t,t) = I$ for all $t \in \mathbb{R}$.
2. $U(t,s)U(s,r) = U(t,r)$, for all $t \geq s \geq r$.
3. The map $(t,s) \mapsto U(t,s) : \{(t,s) \in \mathbb{R}^2 : t \geq s\} \to \mathcal{L}(\mathbb{C}^n)$ is continuous.
4. $U(t+q, s+q) = U(t,s)$, for all $t, s \in \mathbb{R}$.
5. $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$, for all $t, s \in \mathbb{R}$.
6. $\frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s)$, for all $t, s \in \mathbb{R}$.
7. There exist two real constants ω and $M \geq 1$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \text{ for all } t \geq s.$$

We also consider:

$$\begin{cases} \dot{X}(t) = A(t)X(t) + e^{i\mu t}I, \quad t \geq 0; \\ X(0) = 0. \end{cases} \qquad (A(t), \mu, I, 0)$$

The solution of $(A(t), \mu, I, 0)$ is given by

$$\Phi_\mu(t) = \int_0^t U(t,s)e^{i\mu s}ds.$$

First result of this section connects the uniform exponential stability of the system $(A(t))$ with the boundedness of all solutions of the Cauchy problems $(A(t), \mu, b, 0)$, with $\mu \in \mathbb{R}$ and $b \in \mathbb{C}^n$.

Theorem 3.1. *The following two statements hold true:*

(1) If the system $(A(t))$ is uniformly exponentially stable (i.e if the spectral radius of $U(q, 0)$ is less then one) then for each real number μ and each vector $b \in \mathbb{C}^n$, the solution of the Cauchy Problem $(A(t), \mu, b, 0)$ is bounded on $\mathbb{R}_+$.

(2) Conversely, if for each real number μ and each vector $b \in \mathbb{C}^n$ the solution of the Cauchy Problem $(A(t), \mu, b, 0)$ is bounded on $\mathbb{R}_+$ and if the matrix $\Phi_\mu(q)$ is invertible, then the system $(A(t))$ is uniformly exponentially stable.

Proof. **(1)** Let ν be the integer part of $\frac{t}{q}$ and let $r := (t - q\nu) \in [0, q)$. Then

$$\begin{aligned} \Phi_\mu(t)b &= \int_0^{q\nu+r} U(t,s)e^{i\mu s}bds \\ &= \int_0^{q\nu} U(t,s)e^{i\mu s}bds + \int_{q\nu}^{q\nu+r} U(t,s)e^{i\mu s}bds \\ &= \sum_{k=0}^{\nu-1} \int_{qk}^{q(k+1)} U(q\nu + r, s)e^{i\mu s}bds + \int_{q\nu}^{q\nu+r} U(t,s)e^{i\mu s}bds. \end{aligned}$$

On the other hand

$$\begin{aligned}
\sum_{k=0}^{\nu-1}\int_{qk}^{q(k+1)} U(q\nu+r,s)e^{i\mu s}bds &= \sum_{k=0}^{\nu-1}\int_{qk}^{q(k+1)} U(q\nu+r,q\nu)U(q\nu,s)e^{i\mu s}bds \\
&= \sum_{k=0}^{\nu-1}\int_{qk}^{q(k+1)} U(r,0)U(q\nu,s)e^{i\mu s}bds \\
&= U(r,0)\sum_{k=0}^{\nu-1}\int_{0}^{q} U(q\nu,qk+\tau)e^{i\mu(qk+\tau)}bd\tau \\
&= U(r,0)\sum_{k=0}^{\nu-1} e^{i\mu qk}\int_{0}^{q} U(q(\nu-k),\tau)e^{i\mu\tau}bd\tau \\
&= U(r,0)\sum_{k=0}^{\nu-1} e^{i\mu qk}U(q,0)^{\nu-k}\int_{0}^{q} U(q,\tau)e^{i\mu\tau}bd\tau.
\end{aligned}$$

Therefore,

$$\int_0^t U(t,s)e^{i\mu s}bds = I_1 + I_2$$

where

$$I_1 = U(r,0)\sum_{k=0}^{\nu-1} e^{i\mu qk}U(q,0)^{\nu-k}\Phi_\mu(k)b \quad \text{and} \quad I_2 = \int_{q\nu}^{q\nu+r} U(t,s)e^{i\mu s}bds.$$

The boundedness of I_1 follows because the spectral radius of $U(q,0)$ is less than 1. The family $\mathcal{U}$ has exponential growth and $0 \le t-s \le r \le q$ hence we have:

$$\|I_2\| = \left\| \int_{q\nu}^{q\nu+r} U(t,s)e^{i\mu s}bds \right\| \le Me^{q\omega}\|b\|.$$

(2) Using the invertibility of the matrices $U(r,0)$ and $\Phi_\mu(q)$ and the identity

$$\Phi_\mu(t) = U(r,0)\sum_{k=0}^{\nu-1} e^{i\mu qk}U(q,0)^{\nu-k}\Phi_\mu(q) + \int_{q\nu}^{q\nu+r} U(t,s)e^{i\mu s}ds$$

we get

$$U(r,0)^{-1}(\Phi_\mu(t))\Phi_\mu^{-1}(q) = \sum_{k=0}^{\nu-1} e^{i\mu qk}U(q,0)^{\nu-k} + U(r,0)^{-1}\left(\int_{q\nu}^{q\nu+r} U(t,s)e^{i\mu s}ds\right)\Phi_\mu^{-1}(q).$$

Thus

$$\left\|\sum_{k=0}^{\nu-1} e^{i\mu qk}U(q,0)^{\nu-k}\right\| \leq \|U(r,0)^{-1}\|\|\Phi_\mu^{-1}(q)\| \times \left[\|\Phi_\mu(t)\| + Me^{q\omega}\right].$$

The assertion follows from Proposition 2.1.

The second result of this section may be read as follows:

Theorem 3.2. *The system $(A(t))$ is uniformly exponentially stable if and only if for each $b \in \mathbb{C}^n$, the solution of $(A(t), \mu, b, 0)$ is bounded on $\mathbb{R}_+$ uniformly in respect to the parameter μ on $\mathbb{R}$, i.e.*

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t \Phi(t)\Phi^{-1}(s)e^{i\mu s}bds\right\| := K(b) < \infty. \tag{3.1}$$

Before giving the proof, we state a new lemma. Recall that $\mathcal{F}_\Phi(\mathbb{R}_+, \mathbb{C}^n)$ is the set of all $\mathbb{C}^n$-valued continuous and q-periodic functions defined on $\mathbb{R}_+$, given on $[0, q]$ by $f_b(s) = h_i(s)\Phi(s)b$, for $i \in \{1, 2\}$ and $b \in \mathbb{C}^n$, where h_1 and h_2 was introduced in the previous section.

Lemma 3.3. *If $\sup_{t\geq s}\|\Phi(t)\Phi^{-1}(s)\| = M < \infty$ then each function $f_b \in \mathcal{F}_\Phi(\mathbb{R}_+, X)$ satisfies a Lipschitz condition on $\mathbb{R}_+$.*

Proof. Let $b \in \mathbb{C}^n$ and $i \in \{1, 2\}$ and let $f_b \in \mathcal{F}_\Phi(\mathbb{R}_+, X)$. Clearly, h_i is bounded

$$\max_{s\in[0,q]} h_i(s) = h, \text{ where } h := \max\left\{\frac{q}{2}, \frac{q^2}{4}\right\}.$$

Integrating the equality $\frac{\partial}{\partial t}\Phi(t) = A(t)\Phi(t)$ between the positive numbers t_1 and t_2, with $t_2 \geq t_1$, we get:

$$\|\Phi(t_2)-\Phi(t_1)\| = \left\|\int_{t_1}^{t_2} A(s)\Phi(s)ds\right\| \leq \int_{t_1}^{t_2} \|A(s)\Phi(s)\|dt \leq M\|A(\cdot)\|_\infty|t_2-t_1|.$$

The last inequality holds true also for $t_2 \le t_1$. Moreover, one has

$$\begin{array}{rcl} \|f_b(t_2) - f_b(t_1)\| & = & \|\Phi(t_2)h_i(t_2)b - \Phi(t_1)h_i(t_1)b\| \\ & \le & \|\Phi(t_2)h_i(t_2)b - \Phi(t_2)h_i(t_1)b\| \\ & & +\|\Phi(t_2)h_i(t_1)b - \Phi(t_1)h_i(t_1)b\| \\ & \le & \|\Phi(t_2)\||h_i(t_2) - h_i(t_1)|\|b\| \\ & & +|h_i(t_1)|\|\Phi(t_2) - \Phi(t_1)\|\|b\| \\ & \le & M\left(\widetilde{h} + h\|A(\cdot)\|_\infty\right)\|b\||t_2 - t_1|. \end{array}$$

In the previous estimation was used the following inequality

$$|h_i(t_2) - h_i(t_1)| \le \widetilde{h}|t_2 - t_1|, \quad \widetilde{h} := \max\{1, q\}$$

which is an easy consequence of the Lagrange's Mean Value Theorem.

Proof of Theorem 3.2.

We establish that (2.1) is a consequence of (3.1). It is known (see [16], pp. 93) that each q-periodic function f satisfying a Lipschitz condition on $\mathbb{R}_+$ belongs to $AP_1(\mathbb{R}_+, \mathbb{C}^n)$, i.e. there exists a sequence $(b_\nu)_{\nu\in\mathbb{Z}}$, with $b_\nu \in \mathbb{C}^n$, such that

$$f(t) = \sum_{\nu=-\infty}^{\infty} e^{2\pi i\nu\frac{t}{q}} b_\nu \quad \text{and} \quad \sum_{\nu=-\infty}^{\infty} \|b_\nu\| < \infty,$$

where b_ν are the Fourier-Bohr coefficients of f given by

$$b_\nu = \frac{1}{q}\int_0^q e^{-2\pi i\nu\frac{t}{q}} f(t)dt, \text{ for } \nu \in \mathbb{Z}.$$

Using the Uniform Boundedness Principle and (3.1) we find a positive constant K such that

$$\sup_{\mu\in\mathbb{R}} \sup_{t\ge 0} \left\| \int_0^t \Phi(t)\Phi^{-1}(s)e^{i\mu s} b ds \right\| \le K\|b\|.$$

Let $f \in \mathcal{F}_\Phi(\mathbb{R}_+, \mathbb{C}^n)$ and $\mu \in \mathbb{R}$. We analyze two cases.

Case 1. When $\sup_{t\ge s} \|\Phi(t)\Phi^{-1}(s)\| := M < \infty$.

$$\begin{aligned}\left\|\int_0^t \Phi(t)\Phi^{-1}(s)e^{i\mu s}f(s)ds\right\| &= \left\|\int_0^t \Phi(t)\Phi^{-1}(s)e^{i\mu s}\left(\sum_{\nu=-\infty}^{\infty} e^{2\pi i\nu\frac{s}{q}}b_\nu\right)ds\right\| \\ &\le \sum_{\nu=-\infty}^{\infty}\left\|\int_0^t \Phi(t)\Phi^{-1}(s)e^{i(2\pi\nu\frac{1}{q}+\mu)s}b_\nu ds\right\| \\ &\le \sum_{\nu=-\infty}^{\infty} K\|b_\nu\| < \infty.\end{aligned}$$

The last inequality holds true because the map f belongs to $AP_1(\mathbb{R}_+, \mathbb{C}^n)$. The assertion follows from Theorem 2.3.

Case 2. We assume only the fact that the family $\mathcal{U}$ has an exponential growth. There exist $M \ge 1$ and $\omega > 0$ such that $\|U(t,s)\| \le Me^{\omega(t-s)}$, for all $t \ge s$. Let $V(t,s) := e^{-\omega(t-s)}U(t,s)$. Integrating by parts, we get

$$\int_0^t e^{i\mu s}V(t,s)ds = \int_0^t e^{i\mu s}U(t,s)ds - \omega e^{-\omega t}\int_0^t e^{\omega s}\int_0^s e^{i\mu\tau}U(t,\tau)d\tau ds.$$

Passing to the norms in the both sides, we get

$$\begin{aligned}\left\|\int_0^t e^{i\mu s}V(t,s)ds\right\| &\le \left\|\int_0^t e^{i\mu s}U(t,s)ds\right\| + \omega e^{-\omega t}\int_0^t e^{\omega s}\left\|\int_0^s e^{i\mu\tau}U(t,\tau)d\tau\right\| ds \\ &\le K + K(1-e^{-\omega t}) \le 2K.\end{aligned}$$

As is stated above may find two positive constants N and ν such that

$$\|U(t,s)\| = e^{\omega(t-s)}\|V(t,s)\| \le Ne^{(\omega-\nu)(t-s)}, \quad \forall t \ge s.$$

The assertion is obtained by repeating this reasoning for a certain number of times.

Theorem 3.2 yields the following weak version of the Barbashin theorem. This result seems to be a new one and it opens the problem if similar result could be preserved in infinite dimensional spaces.

Corollary 3.4. *The time dependent q-periodic system $(A(t))$ is uniformly exponentially stable if and only if for each $x \in \mathbb{C}^n$, have that*

$$\sup_{t\ge 0}\int_0^t |<\Phi(t)\Phi^{-1}(s)x, x>|ds < \infty.$$

Proof. The polarization identity

$$\langle \Phi(t)\Phi^{-1}(s)x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle \Phi(t)\Phi^{-1}(s)(x+i^k y, x+i^k y)\rangle$$

and the assumption yield

$$\sup_{t\geq 0}\int_0^t |\langle \Phi(t)\Phi^{-1}(s)x, y\rangle| ds := K(x,y) < \infty, \quad \forall x \in \mathbb{C}^n, \forall y \in \mathbb{C}^n.$$

Therefore

$$\begin{aligned} \left|\left\langle \int_0^t e^{i\mu s}\Phi(t)\Phi^{-1}(s)x ds, y\right\rangle\right| &= \left|\int_0^t e^{i\mu s} < \Phi(t)\Phi^{-1}(s)x, y > ds\right| \\ &\leq \int_0^t | < \Phi(t)\Phi^{-1}(s)x, y > | ds \leq K(x,y). \end{aligned}$$

Using the Uniform Boundedness Principle we may find a positive constant K_1 such that

$$\left|\left\langle \int_0^t e^{i\mu s}\Phi(t)\Phi^{-1}(s)x ds, y\right\rangle\right| \leq K_1\|x\|\|y\|.$$

Then $\left\|\int_0^t e^{i\mu s}\Phi(t)\Phi^{-1}(s) ds\right\| \leq K_1$. The assertion is a consequence of Theorem 3.2.

Corollary 3.5. *The time dependent q-periodic system $(A(t))$ is uniformly exponentially stable if and only if for each $b \in \mathbb{C}^n$, one has*

$$\sup_{t\geq 0}\int_0^t \|\Phi(t)\Phi^{-1}(s)b\| ds := K(b) < \infty. \tag{3.2}$$

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