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Multiplicity of positive solutions for critical singular elliptic systems with sign-changing weight function*

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Abstract. In this paper, the existence and multiplicity of positive solutions for a critical singular elliptic system with concave and convex nonlinearity and sign-changing weight function, are established. With the help of the Nehari manifold, we prove that the system has at least two positive solutions via variational methods.

Keywords: critical Sobolev exponent; Nehari manifold; concave-convex nonlinearities; elliptic system

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1 Introduction and main result

In this paper, we are concerned with the following critical singular elliptic system

$$\begin{cases} -\Delta u - t\frac{u}{|x|^2} = \lambda f(x)|u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & \text{in } \mathbb{R}^N \setminus \{0\}, \\ -\Delta v - t\frac{v}{|x|^2} = \mu g(x)|v|^{q-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u, v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\lambda, \mu \geq 0$ with $1 < q < 2, \alpha, \beta > 1$ satisfying $\alpha + \beta = 2^*, 2^* = \frac{2N}{N-2}, N \geq 3$ and $0 \leq t < \overline{t} = (\frac{N-2}{2})^2$, $\overline{t}$ is the best constant in the Hardy inequality. The weight functions f, g satisfy the following assumptions:

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(H_1) f, g are measurable functions and locally bounded in $\mathbb{R}^N \setminus \{0\}$, with $0 \not\equiv f_+, g_+ \in C(\mathbb{R}^N \setminus \{0\})$ and

$$f(x) = \begin{cases} O(|x|^b), & \text{as } |x| \to 0, \\ O(|x|^a), & \text{as } |x| \to \infty, \end{cases} \qquad g(x) = \begin{cases} O(|x|^d), & \text{as } |x| \to 0, \\ O(|x|^c), & \text{as } |x| \to \infty, \end{cases}$$

for any a, b, c, d verifying

$$a, c \le \frac{N}{2^*}(q - 2^*) < b, d.$$

(H_2) $\Sigma_f \cap \Sigma_g \neq \phi$, where $\Sigma_f = \{x \in \mathbb{R}^N : f(x) > 0\},\ \Sigma_g = \{x \in \mathbb{R}^N : g(x) > 0\}$.

Similar assumptions have been mentioned in [1-3]. The role of such growth conditions is to get a compactness condition.

Let $E = \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a Hilbert space endowed with norm

$$\|z\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \right)^{\frac{1}{2}},$$

where $z = (u, v) \in E$.

A pair of functions $(u, v) \in E$ is said to be a weak solution of problem (1.1), if

$$\begin{aligned} & \int_{\mathbb{R}^N} (\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 - t\frac{u\varphi_1}{|x|^2} - t\frac{v\varphi_2}{|x|^2}) \\ = & \int_{\mathbb{R}^N} (\lambda f(x)|u|^{q-2}u\varphi_1 + \mu g(x)|v|^{q-2}v\varphi_2) \\ & + \frac{2\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} |u|^{\alpha-2}u|v|^\beta \varphi_1 + \frac{2\beta}{\alpha+\beta} \int_{\mathbb{R}^N} |u|^\alpha |v|^{\beta-2} v\varphi_2 \end{aligned}$$

for all $(\varphi_1, \varphi_2) \in E$.

The corresponding energy functional of problem (1.1) is defined by

$$\begin{aligned} J_{\lambda,\mu}(u, v) \ = \ & \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 - t\frac{u^2}{|x|^2} - t\frac{v^2}{|x|^2}) \\ & - \frac{1}{q} \int_{\mathbb{R}^N} (\lambda f(x)|u|^q + \mu g(x)|v|^q) - \frac{2}{\alpha+\beta} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta. \end{aligned}$$

It follows from $(H1)$-$(H2)$ that the functional $J_{\lambda,\mu}$ is of class $C^1(E, \mathbb{R})$. Moreover, the critical points of $J_{\lambda,\mu}$ are the week solutions of problem (1.1).

Existence and multiplicity of solutions for elliptic problems with concave-convex nonlinearities in bounded domain $\Omega \subset \mathbb{R}^N$ are studied extensively.

Set $\alpha = \beta, \alpha + \beta = p, \lambda = \mu, u = v$, then in Ω the problem (1.1) reads as the scalar elliptic equation

$$\begin{cases} -\Delta u - t\frac{u}{|x|^2} = \lambda f(x)|u|^{q-2}u + |u|^{p-2}u, & \text{in } \Omega \setminus \{0\}, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

In [5], Ambrosetti, Brezis and Cerami considered the problem (1.2) for $t = 0, f(x) \equiv 1$, $2 < p \leq 2^*$ and proved that there exists $\Lambda > 0$ such that problem (1.2) admits at least two positive solutions for $\lambda \in (0, \Lambda)$, has a positive solution for $\lambda = \Lambda$ and no positive solution for $\lambda > \Lambda$. Chen [6] considered the problem (1.2) for $f(x) \equiv 1, p = 2^*$ and proved that there exists $\Lambda > 0$ such that the problem (1.2) admits at least two positive solutions for $\lambda \in (0, \Lambda), 0 \leq t < \bar{t}$. Successively, Tsung-Fang Wu [7] investigated the problem (1.2) for $t = 0, p = 2^*$ with sign-changing weight function f and proved that there exists $\Lambda > 0$ such that the problem (1.2) admits at least two positive solutions for $\lambda \in (0, \Lambda)$.

In whole space, Ambrosetti, Garcia and Peral [4] considered the problem (1.2) for $t = 0, p = 2^*$ and proved the existence of $\Lambda > 0$ such that problem (1.2) admits at least two non-negative solutions for $\lambda \in (0, \Lambda)$, provided that $f \in L^1(\mathbb{R}^N) \cup L^\infty(\mathbb{R}^N)$ and $f_+ \not\equiv 0$. More recently, under the proper hypothesis, Miotto [3] studied the same problem above and obtained the similar results.

In recent years, much attention has been paid to the investigation of the following elliptic system in bounded domain Ω

$$\begin{cases} -\Delta u - t\frac{u}{|x|^2} = \lambda f(x)|u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, & \text{in } \Omega \setminus \{0\}, \\ -\Delta v - t\frac{v}{|x|^2} = \mu g(x)|v|^{q-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & \text{in } \Omega \setminus \{0\}, \\ u, v \in H_0^1(\Omega). \end{cases} \tag{1.3}$$

Alves et [8] studied problem (1.3) with $f(x) = g(x) \equiv 1, t = 0, q = 2$ and proved the existence of least energy solutions for problem (1.3) for $\lambda, \mu \in (0, \lambda_1)$, where λ_1 denoting the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Liu-Han [9] also considered problem (1.3) with $f(x) = g(x) \equiv 1, 0 < t \leq \bar{t} - 1, q = 2$ in bounded domain Ω, and proved that (1.3) admits one positive solution for $\lambda, \mu \in (0, \lambda_1)$. Subsequently, T.S.Hsu [10] considered problem (1.3) with $f(x) = g(x) \equiv 1, 0 \leq t < \bar{t}, 1 < q < 2$, and proved the existence of $\Lambda > 0$ such that problem (1.3) has at least two positive solutions, provided that $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda$.

However, up to now, there are few papers on problem (1.1) in whole space $\mathbb{R}^N$. The purpose to this paper is to investigate the existence and multiplicity of positive solutions of problem (1.1) by using the decomposition of the Nehari manifold.

Inspired by [3] and [10], we have the following result.

Theorem 1 Assume $(H1)$-$(H2)$ hold. Then there exists a positive constant Λ such that problem (1.1) admits at least two positive solutions, provided that $\lambda, \mu > 0$ and $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda$.

2 Notations and preliminaries

Set $\mathcal{D}^{1,2}(\mathbb{R}^N)=\{u\in L^{2^*}(\mathbb{R}^N)|\nabla u\in L^2(\mathbb{R}^N)\}$ with norm $\|u\|=\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{1}{2}}$. For $t\in[0,\bar{t})$, we put

$$S_t=\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\backslash 0}\frac{\int_{\mathbb{R}^N}(|\nabla u|^2-t\frac{u^2}{|x|^2})}{\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{\frac{2}{2^*}}}.$$

Catrina and Wang [11] proved that S_t is achieved by the function

$$U(x)=\frac{1}{\left(|x|^{\frac{\gamma_1}{\sqrt{\bar{t}}}}+|x|^{\frac{\gamma_2}{\sqrt{\bar{t}}}}\right)^{\sqrt{\bar{t}}}},\tag{2.1}$$

where $\gamma_1=\sqrt{\bar{t}}-\sqrt{\bar{t}-t},\gamma_2=\sqrt{\bar{t}}+\sqrt{\bar{t}-t}$.

Moreover, for $\varepsilon>0,U_\varepsilon=\varepsilon^{-\frac{N-2}{2}}U(\frac{x}{\varepsilon})\left(\frac{4N(\bar{t}-t)}{N-2}\right)^{\frac{N-2}{4}}$ satisfies

$$\begin{cases}-\Delta u-t\frac{u}{|x|^2}=|u|^{2^*-2}u & \text{in } \mathbb{R}^N\setminus\{0\},\\ u\to 0 & \text{as } |x|\to\infty.\end{cases}$$

Denote

$$S^t_{\alpha,\beta}=\inf_{u,v\in\mathcal{D}^{1,2}(\mathbb{R}^N)\backslash 0}\frac{\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2-t\frac{u^2}{|x|^2}-t\frac{v^2}{|x|^2})}{\left(\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\right)^{\frac{2}{\alpha+\beta}}}.$$

From [10], we have

$$S^t_{\alpha,\beta}=\left((\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}}+(\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}}\right)S_t.$$

For $z=(u,v)\in E$, let

$$\|z\|_t=\left(\int_{\mathbb{R}^N}|\nabla u|^2+|\nabla v|^2-t\frac{u^2}{|x|^2}-t\frac{v^2}{|x|^2}\right)^{\frac{1}{2}}.$$

By Hardy inequality

$$\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}\le\frac{1}{\bar{t}}\int_{\mathbb{R}^N}|\nabla u|^2 \text{ for all } u\in\mathcal{D}^{1,2}(\mathbb{R}^N),$$

we can derive that $\|\cdot\|_t$ defines an equivalent norm in E.

We define the Palais-Smale (PS) sequence and (PS)-condition in E for $J_{\lambda,\mu}$ as follows.

Definition 2.1 (i) $\{z_n\}$ is a $(PS)_c$-sequence in E for $J_{\lambda,\mu}$, if $J_{\lambda,\mu}(z_n)=c+o(1)$, $J'_{\lambda,\mu}(z_n)=o(1)$ in E^{-1} as $n\to\infty$.

(ii) $J_{\lambda,\mu}$ satisfies the $(PS)_c$-condition in E, if any $(PS)_c$-sequence $\{z_n\}$ in E for $J_{\lambda,\mu}$ has a convergent subsequence.

As consequence of the assumptions $(H1)$-$(H2)$, we have

Lemma 2.1 [3] If $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, then there exists a subsequence $\{u_n\}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x)|u_n|^q = \int_{\mathbb{R}^N} f(x)|u|^q.$$

The same conclusion still holds if f is replaced by g in Lemma 2.1.

Lemma 2.2 If $\{z_n\} \subset E$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$, then $\{z_n\}$ is bounded in E.

Proof. Let $z_n = (u_n, v_n)$. On the contrary, assume that $\|z_n\|_t \to \infty$.

Put

$$\overline{z}_n = (\overline{u}_n, \overline{v}_n) = \frac{z_n}{\|z_n\|_t},$$

then $\{\overline{z}_n\}$ is bounded in E. By passing to a subsequence, we can assume that $\overline{z}_n \rightharpoonup \overline{z} = (\overline{u}, \overline{v})$ in E. So $\overline{u}_n \rightharpoonup \overline{u}, \overline{v}_n \rightharpoonup \overline{v}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. By Lemma 2.1, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (\lambda f(x)|\overline{u}_n|^q + \mu g(x)|\overline{v}_n|^q) = \int_{\mathbb{R}^N} (\lambda f(x)|\overline{u}|^q + \mu g(x)|\overline{v}|^q). \tag{2.2}$$

Since $J_{\lambda,\mu}(z_n) = c + o(1)$, $J'_{\lambda,\mu}(z_n) = o(1)$ and $\|z_n\|_t \to \infty$, then

$$\begin{aligned}
& J_{\lambda,\mu}(z_n) - \frac{1}{2^*} < J'_{\lambda,\mu}(z_n), z_n > \\
= \; & (\frac{1}{2} - \frac{1}{2^*})\|z_n\|_t^2 - (\frac{1}{q} - \frac{1}{2^*}) \int_{\mathbb{R}^N} (\lambda f(x)|u_n|^q + \mu g(x)|v_n|^q) \\
\leq \; & c + o(1)\|z_n\|_t + o(1).
\end{aligned}$$

So

$$\|\overline{z}_n\|_t^2 = \frac{2(2^*-q)}{q(2^*-2)}\|z_n\|_t^{q-2} \int_{\mathbb{R}^N} (\lambda f(x)|\overline{u}_n|^q + \mu g(x)|\overline{v}_n|^q) + o(1).$$

It easily follows from (2.2) and $1 < q < 2$, that $\|\overline{z}_n\|_t \to 0$ as $n \to \infty$, which is a contradiction. □

Lemma 2.3 If $\{z_n\} \subset E$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $z_n \rightharpoonup z$ in E, then $J'_{\lambda,\mu}(z) = 0$ and $J_{\lambda,\mu}(z) \geq -C_0(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}})$ for some positive constant C_0 only depending on f, g, N, q, t.

Proof. If $J_{\lambda,\mu}(z_n) = c + o(1)$, $J'_{\lambda,\mu}(z_n) = o(1)$ in E^{-1} as $n \to \infty$ and $z_n \rightharpoonup z$ in E, it is standard that $J'_{\lambda,\mu}(z) = 0$. Let $z = (u, v)$, we have

$$J_{\lambda,\mu}(z) = (\frac{1}{2} - \frac{1}{2^*})\|z\|_t^2 - (\frac{1}{q} - \frac{1}{2^*}) \int_{\mathbb{R}^N} \lambda f(x)|u|^q + \mu g(x)|v|^q.$$

From the assumptions $(H1)$-$(H2)$, there exist $C_f, C_g > 0$ such that

$$\left|\int_{\mathbb{R}^N} f(x)|u|^q\right| \leq C_f\|u\|^q, \; \left|\int_{\mathbb{R}^N} g(x)|v|^q\right| \leq C_g\|v\|^q.$$

By the Young inequality, Hardy inequality and $1 < q < 2$, it follows that

$$\begin{array}{rcl} J_{\lambda,\mu}(z) & = & \frac{1}{N}\|z\|_t^2 - (\frac{1}{q} - \frac{1}{2^*}) \int_{\mathbb{R}^N} \lambda f(x)|u|^q + \mu g(x)|v|^q \\ & \geq & \frac{1}{N}\|z\|_t^2 - (\frac{1}{q} - \frac{1}{2^*})(\lambda C_f \|u\|^q + \mu C_g \|v\|^q) \\ & \geq & \frac{1}{N}\|z\|_t^2 - (\frac{1}{q} - \frac{1}{2^*})(\lambda C_f + \mu C_g)(\frac{\overline{t} - t}{\overline{t}})^{-\frac{q}{2}} \|z\|_t^q \\ & \geq & -C_0(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}). \end{array}$$

□

3 Nehari manifold

For any $\lambda, \mu > 0$, we consider the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \{z \in E \setminus \{0\} | < J'_{\lambda,\mu}(z), z >= 0\}.$$

We recall that any nonzero solution of (1.1) belongs to $\mathcal{N}_{\lambda,\mu}$. Moreover, $z = (u, v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\|z\|_t \neq 0, \quad \|z\|_t^2 = K_{\lambda,\mu}(z) + 2 \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta,$$

where $K_{\lambda,\mu}(z) = \int_{\mathbb{R}^N} (\lambda f(x)|u|^q + \mu g(x)|v|^q)$.

Denote

$$\theta_{\lambda,\mu} = \inf_{z \in \mathcal{N}_{\lambda,\mu}} J_{\lambda,\mu}(z).$$

We will see that $\theta_{\lambda,\mu} > -\infty$. In fact, let $z \in \mathcal{N}_{\lambda,\mu}$, then from the proof the Lemma 2.3, we have

$$J_{\lambda,\mu}(z) \geq \frac{1}{N}\|z\|_t^2 - (\frac{1}{q} - \frac{1}{2^*})(\lambda C_f + \mu C_g)(\frac{\overline{t} - t}{\overline{t}})^{-\frac{q}{2}} \|z\|_t^q.$$

It follows from $1 < q < 2$ that $J_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$. So $\theta_{\lambda,\mu} > -\infty$.

Define $\Phi_{\lambda,\mu} : E \to \mathbb{R}$, by $\Phi_{\lambda,\mu}(z) =< J'_{\lambda,\mu}(z), z >$, then

$$\begin{array}{rcl} < \Phi'_{\lambda,\mu}(z), z > & = & (2 - q)\|z\|_t^2 - 2(2^* - q) \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \\ & = & (2^* - q)K_{\lambda,\mu}(z) - (2^* - 2)\|z\|_t^2. \end{array} \tag{3.1}$$

As in Tarantello [12], we divide $\mathcal{N}_{\lambda,\mu}$ in three parts

$$\begin{array}{l} \mathcal{N}^+_{\lambda,\mu} = \{z \in \mathcal{N}_{\lambda,\mu} :< \Phi'_{\lambda,\mu}(z), z >> 0\}, \\ \mathcal{N}^0_{\lambda,\mu} = \{z \in \mathcal{N}_{\lambda,\mu} :< \Phi'_{\lambda,\mu}(z), z >= 0\}, \\ \mathcal{N}^-_{\lambda,\mu} = \{z \in \mathcal{N}_{\lambda,\mu} :< \Phi'_{\lambda,\mu}(z), z >< 0\}, \end{array}$$

and consider

$$\theta^+_{\lambda,\mu} = \inf_{z\in\mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(z),\ \theta^0_{\lambda,\mu} = \inf_{z\in\mathcal{N}^0_{\lambda,\mu}} J_{\lambda,\mu}(z),\ \theta^-_{\lambda,\mu} = \inf_{z\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(z).$$

Let

$$\Lambda_1 = \left(\frac{2-q}{2(2^*-q)}\right)^{\frac{2}{2^*-2}} \left(\frac{2^*-2}{2^*-q}\right)^{\frac{2}{2-q}} \left(\frac{\bar{t}-t}{\bar{t}}\right)^{\frac{q}{2-q}} (S^t_{\alpha,\beta})^{\frac{N}{2}} (C_f^{\frac{2}{q}} + C_g^{\frac{2}{q}})^{\frac{q}{q-2}},$$

where C_f, C_g are from Lemma 2.3.
Lemma 3.1 $\mathcal{N}^0_{\lambda,\mu} = \emptyset$ if $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_1$.

Proof. Suppose by absurd that $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$ for any small $\lambda, \mu > 0$. Let $z = (u, v) \in \mathcal{N}^0_{\lambda,\mu}$, by (3.1) we get

$$\|z\|_t^2 = \frac{2(2^*-q)}{2-q} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta, \tag{3.2}$$

$$\|z\|_t^2 = \frac{2^*-q}{2^*-2} K_{\lambda,\mu}(z). \tag{3.3}$$

By the definition of $S^t_{\alpha,\beta}$ and (3.2), we have

$$\|z\|_t \geq \left(\frac{2-q}{2(2^*-q)}\right)^{\frac{1}{2^*-2}} (S^t_{\alpha,\beta})^{\frac{N}{4}}.$$

By the assumptions $(H1)$-$(H2)$, (3.3) and Hölder inequality, we have

$$\begin{aligned} \|z\|_t &\leq \left(\frac{2^*-q}{2^*-2}\right)^{\frac{1}{2-q}} (\lambda C_f + \mu C_g)^{\frac{1}{2-q}} \left(\frac{\bar{t}-t}{\bar{t}}\right)^{\frac{q}{2(q-2)}} \\ &\leq \left(\frac{2^*-q}{2^*-2}\right)^{\frac{1}{2-q}} (\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}})^{\frac{1}{2}} (C_f^{\frac{2}{q}} + C_g^{\frac{2}{q}})^{\frac{q}{2(2-q)}} \left(\frac{\bar{t}-t}{\bar{t}}\right)^{\frac{q}{2(q-2)}}. \end{aligned}$$

So $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \geq \Lambda_1$, which is a contradiction with $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_1$. □

By Lemma 3.1, we have $\theta_{\lambda,\mu} = \min\{\theta^+_{\lambda,\mu}, \theta^-_{\lambda,\mu}\}$.
Lemma 3.2 If $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_2$, then

$$\theta_{\lambda,\mu} = \theta^+_{\lambda,\mu} < 0 < \theta^-_{\lambda,\mu},$$

where $\Lambda_2 = \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_1$.

Proof. The proof is similar to Theorem 3.1 in [10]. □

Lemma 3.3

$$\lim_{(\lambda,\mu)\to(0,0)} \theta_{\lambda,\mu} = 0.$$

Proof. By Lemma 2.3, we have

$$\theta_{\lambda,\mu} \geq -C_0(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}).$$

Combining with Lemma 3.2, it is easy to verify that $\lim_{(\lambda,\mu)\to(0,0)} \theta_{\lambda,\mu} = 0$. □

Similar to Lemma 2.6 in [15], we have the following result.

Lemma 3.4 If $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_1$, then for every $z = (u, v) \in E$ with $\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta > 0$, there exist unique $s^+ = s^+(z)$ and $s^- = s^-(z) > 0$ such that $s^+ z \in \mathcal{N}^+_{\lambda,\mu}$, $s^- z \in \mathcal{N}^-_{\lambda,\mu}$. Moveover, we have

$$\begin{aligned} s^- &> \left[\frac{(2-q)\|z\|_t^2}{2(2^*-q)\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta}\right]^{\frac{1}{2^*-2}} = s_{max} > s^+, \\ J_{\lambda,\mu}(s^+ z) &= \min_{0\leq s\leq s_{max}} J_{\lambda,\mu}(sz) \end{aligned}$$

and

$$J_{\lambda,\mu}(s^- z) = \max_{s\geq 0} J_{\lambda,\mu}(sz).$$

Lemma 3.5 Assume that z is a local minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and $z \notin \mathcal{N}^0_{\lambda,\mu}$, then $J'_{\lambda,\mu}(z) = 0$ in E^{-1}.

Proof. The proof is almost the same as that of Theorem 2.3 in [14] and is omitted here. □

The following lemma provides a precise description of the $(PS)_c$-sequence for $J_{\lambda,\mu}$.

Lemma 3.6 If $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_2$, then each sequence $\{z_n\} \subset E$ satisfying

$$J_{\lambda,\mu}(z_n) = c + o(1), \quad J'_{\lambda,\mu}(z_n) = o(1) \quad \text{in} \quad E^{-1}$$

with $c \neq 0$ and

$$c < \theta_{\lambda,\mu} + \frac{2}{N}\left(\frac{S^t_{\alpha,\beta}}{2}\right)^{\frac{N}{2}},$$

has a convergent subsequence.

Proof. By Lemma 2.2, we have $\{z_n\} \subset E$ is bounded and there exists $z = (u, v) \in E$. We can assume, by passing to a subsequence if necessary, that $z_n \rightharpoonup z$ in E and $z_n \to z$ *a.e.* in $\mathbb{R}^N$. Now we will show that $z \in \mathcal{N}_{\lambda,\mu}$.

First, we prove that $z \neq 0$. On the contrary, suppose that $z = 0$.

Then by Lemma 2.1, we have

$$\int_{\mathbb{R}^N} f(x)|u_n|^q = o(1), \quad \int_{\mathbb{R}^N} g(x)|v_n|^q = o(1), \quad \text{as} \quad n \to \infty.$$

By $< J'_{\lambda,\mu}(z_n), z_n >= 0$, we get that

$$\|z_n\|_t^2 = 2\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta + o(1). \tag{3.4}$$

Moreover, because $\{z_n\}$ is a $(PS)_c$-sequence, we have

$$c = o(1) + J_{\lambda,\mu}(z_n) = \frac{1}{N}\|z_n\|_t^2 + o(1).$$

It is obvious that $c > 0$. Thus $\|z_n\|_t^2 \geq c$ for large n. Then by (3.4) and the definition of $S^t_{\alpha,\beta}$, we obtain that

$$\|z_n\|_t^2 \geq 2(\frac{S^t_{\alpha,\beta}}{2})^{\frac{N}{2}} + o(1)$$

for large n. So $c \geq \frac{2}{N}(\frac{S^t_{\alpha,\beta}}{2})^{\frac{N}{2}}$, which contradicts $\theta_{\lambda,\mu} < 0$. Therefore $z \neq 0$.

It is easy to verify that $z = (u, v)$ is a weak solution of problem (1.1) and $z \in \mathcal{N}_{\lambda,\mu}$. Let $\tilde{z_n} = z_n - z$, $\tilde{u_n} = u_n - u$ and $\tilde{v_n} = v_n - v$.

Then

$$\begin{aligned} & \int_{\mathbb{R}^N}(|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2 - t\frac{\tilde{u}^2}{|x|^2} - t\frac{\tilde{v}^2}{|x|^2}) \\ = & \int_{\mathbb{R}^N}(|\nabla u_n|^2 + |\nabla v_n|^2 - t\frac{u_n^2}{|x|^2} - t\frac{v_n^2}{|x|^2}) \\ & - \int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2 - t\frac{u^2}{|x|^2} - t\frac{v^2}{|x|^2}) + o(1). \end{aligned}$$

By Lemma 2.1 in [13]

$$\int_{\mathbb{R}^N}|\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta = \int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta - \int_{\mathbb{R}^N}|u|^\alpha|v|^\beta + o(1),$$

and by Lemma 2.1

$$\int_{\mathbb{R}^N} f(x)|\tilde{u_n}|^q = \int_{\mathbb{R}^N} f(x)|u_n|^q - \int_{\mathbb{R}^N} f(x)|u|^q + o(1) = o(1),$$

$$\int_{\mathbb{R}^N} g(x)|\tilde{v_n}|^q = \int_{\mathbb{R}^N} g(x)|v_n|^q - \int_{\mathbb{R}^N} g(x)|v|^q + o(1) = o(1),$$

we have

$$< J'_{\lambda,\mu}(\tilde{z_n}), \tilde{z_n} >=< J'_{\lambda,\mu}(z_n), z_n > - < J'_{\lambda,\mu}(z), z > +o(1) = o(1).$$

So we can get that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla\tilde{u_n}|^2 + |\nabla\tilde{v_n}|^2 - t\frac{\tilde{u_n}^2}{|x|^2} - t\frac{\tilde{v_n}^2}{|x|^2}) = \lim_{n\to\infty} 2\int_{\mathbb{R}^N}|\tilde{u_n}|^\alpha|\tilde{v_n}|^\beta = a, \qquad (3.5)$$

where a is a nonnegative constant.

If $a = 0$, the proof is completed. Assume that $a > 0$, it follows from (3.5), that

$$\begin{aligned} S^t_{\alpha,\beta}(\frac{a}{2})^{\frac{2}{2^*}} &= S^t_{\alpha,\beta} \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |\tilde{u_n}|^\alpha |\tilde{v_n}|^\beta \right)^{\frac{2}{2^*}} \\ &\leq \lim_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla \tilde{u_n}|^2 + |\nabla \tilde{v_n}|^2 - t\frac{\tilde{u_n}^2}{|x|^2} - t\frac{\tilde{v_n}^2}{|x|^2}) \\ &= a, \end{aligned}$$

which implies that $a \geq 2(\frac{S^t_{\alpha,\beta}}{2})^{\frac{N}{2}}$.

Thus

$$\begin{aligned} c &= J_{\lambda,\mu}(z_n) + o(1) \\ &= J_{\lambda,\mu}(\tilde{z_n}) + J_{\lambda,\mu}(z) + o(1) \\ &\geq J_{\lambda,\mu}(\tilde{z_n}) + \theta_{\lambda,\mu} + o(1) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla \tilde{u_n}|^2 + |\nabla \tilde{v_n}|^2 - t\frac{\tilde{u_n}^2}{|x|^2} - t\frac{\tilde{v_n}^2}{|x|^2}) - \frac{2}{2^*}\int_{\mathbb{R}^N} |\tilde{u_n}|^\alpha |\tilde{v_n}|^\beta + \theta_{\lambda,\mu} + o(1) \\ &= \theta_{\lambda,\mu} + \frac{1}{N}a \\ &\geq \theta_{\lambda,\mu} + \frac{2}{N}(\frac{S^t_{\alpha,\beta}}{2})^{\frac{N}{2}}, \end{aligned}$$

which is a contradiction. So the proof is completed. □

4 Proof of Theorem 1

First, we shall use the idea of Tarantello [12] to get the following results.

Similar to Proposition 9 in [16], we can prove the following result.

Proposition 4.1 (i) If $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_1$, then there exists a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$;

(ii) If $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_2$, then there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$.

Now, we establish the existence of a positive solution in $\mathcal{N}^+_{\lambda,\mu}$.

Theorem 4.1 If $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_1$, then $J_{\lambda,\mu}$ has a minimizer z^1 in $\mathcal{N}^+_{\lambda,\mu}$ which satisfies

(i) $J_{\lambda,\mu}(z^1) = \theta_{\lambda,\mu} = \theta^+_{\lambda,\mu} < 0$;

(ii) z^1 is a positive solution of problem (1.1).

Proof. By Proposition 4.1(i), there exists a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$. It follows from $\theta_{\lambda,\mu} < 0$ and Lemma 3.5, that there exists $z^1 = (u^1, v^1) \in \mathcal{N}_{\lambda,\mu}$ such that $z_n \to z^1$ strongly in E. So z^1 is a nontrivial solution of problem (1.1).

Similar to the proof of Theorem 4.1 in [10], we can prove that $|z^1| = (|u^1|, |v|^1) \in \mathcal{N}^+_{\lambda,\mu}$ is a positive solution of problem (1.1). □

Next, we establish the existence of a positive solution of the system (1.1) on $\mathcal{N}^-_{\lambda,\mu}$.
First, we consider

$$u_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}} U(\frac{x}{\varepsilon}),\ \varepsilon > 0,\ x \in \mathbb{R}^N,$$

which is an extremal function for S_t, where U is defined in (2.1).

Since $f^+,\ g^+$ are continuous functions in $\mathbb{R}^N$ and $\Sigma = \Sigma_f \cap \Sigma_g \neq \phi$. Following the method of [17], without loss of generality, we may assume the Σ is a domain of positive measure.

We consider the test function

$$\omega_{\varepsilon,y}(x) = \eta_y(x) u_{\varepsilon,y}(x), \quad x \in \mathbb{R}^N,$$

where $y \in \Sigma,\ u_{\varepsilon,y}(x) = u_\varepsilon(x-y)$ and $\eta_y \in C_0^\infty(\Sigma)$ with $\eta_y \geq 0$ and $\eta_y = 1$ near y.

Let Λ_2 as in Theorem 4.1, then for $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_2$, we have the following result.

Lemma 4.1 Let $z^1 = (u^1, v^1)$ be the local minimizer in Theorem 4.1. Then for every $l > 0$ and *a.e.* $y \in \Sigma$, there exists $\varepsilon_0 = \varepsilon_0(l, y) > 0$ such that

$$J_{\lambda,\mu}(u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}, v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}) < \theta_{\lambda,\mu} + \frac{2}{N}(\frac{S^t_{\alpha,\beta}}{2})^{\frac{N}{2}}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. One has

$$
\begin{aligned}
& J_{\lambda,\mu}(u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}, v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}) \\
= \ & \frac{1}{2}\|(u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}, v^1 + l\sqrt{\beta}\omega_{\varepsilon,y})\|_t^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} f(x)|u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}|^q \\
& -\frac{\mu}{q}\int_{\mathbb{R}^N} g(x)|v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}|^q - \frac{2}{2^*}\int_{\mathbb{R}^N} |u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}|^\alpha |v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}|^\beta \\
= \ & \frac{1}{2}\|(u^1, v^1)\|_t^2 + \frac{l^2}{2}(\alpha+\beta)\|\omega_{\varepsilon,y}\|_t^2 + l\left[< u^1, \sqrt{\alpha}\omega_{\varepsilon,y} >_t + < v^1, \sqrt{\beta}\omega_{\varepsilon,y} >_t\right] \\
& -\frac{1}{q}\int_{\mathbb{R}^N} \lambda f(x)|u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}|^q + \mu g(x)|v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}|^q \\
& -\frac{2}{2^*}\int_{\mathbb{R}^N} |u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}|^\alpha |v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}|^\beta \\
= \ & J_{\lambda,\mu}(u^1, v^1) + \frac{l^2}{2}(\alpha+\beta)\|\omega_{\varepsilon,y}\|_t^2 + \frac{2}{2^*}\int_{\mathbb{R}^N} |u^1|^\alpha |v^1|^\beta + \frac{1}{q}\int_{\mathbb{R}^N} \lambda f(x)|u^1|^q + \mu g(x)|v^1|^q \\
& -\frac{1}{q}\int_{\mathbb{R}^N} \lambda f(x)|u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}|^q + \mu g(x)|v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}|^q \\
& +l\int_{\mathbb{R}^N} \lambda f(x)|u^1|^{q-1}\sqrt{\alpha}\omega_{\varepsilon,y} + \mu g(x)|v^1|^{q-1}\sqrt{\beta}\omega_{\varepsilon,y} \\
& -\frac{2}{2^*}\int_{\mathbb{R}^N} |u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}|^\alpha |v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}|^\beta \\
& +\frac{2l}{2^*}\int_{\mathbb{R}^N} |u^1|^{\alpha-1}|v^1|^\beta \alpha^{\frac{3}{2}}\omega_{\varepsilon,y} + |u^1|^\alpha |v^1|^{\beta-1}\beta^{\frac{3}{2}}\omega_{\varepsilon,y}.
\end{aligned}
$$

Since

$$
\begin{aligned}
& \frac{1}{q}\int_{\mathbb{R}^N} f(x)|u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}|^q - \frac{1}{q}\int_{\mathbb{R}^N} f(x)|u^1|^q - l\int_{\mathbb{R}^N} f(x)|u^1|^{q-1}\sqrt{\alpha}\omega_{\varepsilon,y} \\
= \ & \int_{\mathbb{R}^N} f(x)\left\{\int_0^{l\sqrt{\alpha}\omega_{\varepsilon,y}} \left[(u^1+s)^{q-1} - (u^1)^{q-1}\right] ds\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{q}\int_{\mathbb{R}^N} g(x)|v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}|^q - \frac{1}{q}\int_{\mathbb{R}^N} g(x)|v^1|^q - l\int_{\mathbb{R}^N} g(x)|v^1|^{q-1}\sqrt{\beta}\omega_{\varepsilon,y} \\
= \ & \int_{\mathbb{R}^N} g(x)\left\{\int_0^{l\sqrt{\beta}\omega_{\varepsilon,y}} \left[(v^1+s)^{q-1} - (v^1)^{q-1}\right] ds\right\},
\end{aligned}
$$

it follows from $f > 0, g > 0$ in Σ and $\omega_{\varepsilon,y} \equiv 0$ in Σ^c, that

$$\begin{aligned}
& J_{\lambda,\mu}(u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}, v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}) \\
\leq\ & J_{\lambda,\mu}(u^1, v^1) + \frac{l^2}{2}(\alpha+\beta)\|\omega_{\varepsilon,y}\|_t^2 + \frac{2}{2^*}\int_{\mathbb{R}^N} |u^1|^\alpha |v^1|^\beta \\
& + \frac{2l}{2^*}\int_{\mathbb{R}^N} |u^1|^{\alpha-1}|v^1|^\beta \alpha^{\frac{3}{2}}\omega_{\varepsilon,y} + |u^1|^\alpha |v^1|^{\beta-1}\beta^{\frac{3}{2}}\omega_{\varepsilon,y} \\
& - \frac{2}{2^*}\int_{\mathbb{R}^N} |u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}|^\alpha |v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}|^\beta. \qquad (4.1)
\end{aligned}$$

Similar to the estimate in [16] and [17], we can get

$$\begin{aligned}
& \int_{\mathbb{R}^N} |u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}|^\alpha |v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}|^\beta \\
=\ & \int_{\mathbb{R}^N} |u^1|^\alpha |v^1|^\beta + l\int_{\mathbb{R}^N} |u^1|^{\alpha-1}|v^1|^\beta \alpha^{\frac{3}{2}}\omega_{\varepsilon,y} + |u^1|^\alpha |v^1|^{\beta-1}\beta^{\frac{3}{2}}\omega_{\varepsilon,y} \\
& + l^{2^*}\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}\int_{\mathbb{R}^N} |\omega_{\varepsilon,y}|^{2^*} + l^{2^*-1}\int_{\mathbb{R}^N} (\alpha^{\frac{\alpha+1}{2}}\beta^{\frac{\beta}{2}}u^1 + \alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta+1}{2}}v^1)|\omega_{\varepsilon,y}|^{2^*-1} + o(\varepsilon^{\frac{N-1}{2}})
\end{aligned}$$

and

$$\int_{\mathbb{R}^N} |\omega_{\varepsilon,y}|^{2^*} = A + O(\varepsilon^N),\ \ \|\omega_{\varepsilon,y}\|_t^2 = B + O(\varepsilon^{N-1}),$$

where $A = \|U\|_{2^*}^{2^*}, B = \|U\|_t^2$ and $S_t = \frac{B}{A^{\frac{2}{2^*}}}$.

Substituting in (4.1), we obtain

$$\begin{aligned}
& J_{\lambda,\mu}(u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}, v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}) \\
\leq\ & \theta_{\lambda,\mu} + \frac{l^2}{2}(\alpha+\beta)B - \frac{2l^{2^*}}{2^*}\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}A \\
& - l^{2^*-1}\int_{\mathbb{R}^N} (\alpha^{\frac{\alpha+1}{2}}\beta^{\frac{\beta}{2}}u^1 + \alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta+1}{2}}v^1)|\omega_{\varepsilon,y}|^{2^*-1} + o(\varepsilon^{\frac{N-1}{2}}).
\end{aligned}$$

Similar to the argument of Lemma 3.1 in [12], we can conclude that for every $l > 0$ and $a.e.\ y \in \Sigma$, there exists $\varepsilon_0 = \varepsilon_0(l, y) > 0$ such that

$$J_{\lambda,\mu}(u^1 + l\sqrt{\alpha}\omega_{\varepsilon,y}, v^1 + l\sqrt{\beta}\omega_{\varepsilon,y}) < \theta_{\lambda,\mu} + \frac{2}{N}(\frac{S_{\alpha,\beta}^t}{2})^{\frac{N}{2}}$$

for all $\varepsilon \in (0, \varepsilon_0)$. □

Theorem 4.2 There exists $\Lambda > 0$ with $\Lambda \leq \Lambda_2$, for all $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda$, then $J_{\lambda,\mu}$ has a minimizer z^2 in $\mathcal{N}_{\lambda,\mu}^-$ which satisfies

(i) $J_{\lambda,\mu}(z^2) = \theta_{\lambda,\mu}^- < \theta_{\lambda,\mu} + \frac{2}{N}(\frac{S_{\alpha,\beta}^t}{2})^{\frac{N}{2}}$;
(ii) z^2 is a positive solution of problem (1.1).

Proof. First, we will show that

$$\theta^-_{\lambda,\mu} < \theta_{\lambda,\mu} + \frac{2}{N}(\frac{S^t_{\alpha,\beta}}{2})^{\frac{N}{2}}.$$

Let

$$\begin{aligned} U_1 &= \left\{ z = (u,v) \in E : \frac{1}{\|z\|_t} s^-(\frac{z}{\|z\|_t}) > 1,\ \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta > 0 \right\} \\ &\quad \cup \left\{ z = (u,v) \in E : \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta = 0 \right\} \end{aligned}$$

and

$$U_2 = \left\{ z = (u,v) \in E : \frac{1}{\|z\|_t} s^-(\frac{z}{\|z\|_t}) < 1,\ \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta > 0 \right\}.$$

Then $\mathcal{N}^-_{\lambda,\mu}$ disconnects E in two connected components U_1 and U_2. For each $z \in \mathcal{N}^+_{\lambda,\mu}$, one has $1 < s_{max} < s^-(z)$. Since $s^-(z) = \frac{1}{\|z\|_t} s^-(\frac{z}{\|z\|_t})$, we have $\mathcal{N}^+_{\lambda,\mu} \subset U_1$. So $z^1 \in U_1$.

In the following, we will prove that there exits $l_0 > 0$ such that $z^1 + l_0(\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y}) \in U_2$. First, we show that there exists $c > 0$ such that

$$0 < s^- \left(\frac{z^1 + l(\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y})}{\|z^1 + l(\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y})\|_t} \right) < c$$

for any $l > 0$.

On the contrary, assume that there is a sequence $\{l_n\}$ with $l_n \to \infty$ such that

$$s^- \left(\frac{z^1 + l_n(\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y})}{\|z^1 + l_n(\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y})\|_t} \right) \to \infty$$

as $n \to \infty$.

Let

$$w_n = (w^1_n, w^2_n) = \frac{z^1 + l_n(\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y})}{\|z^1 + l_n(\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y})\|_t}.$$

In connection with $s^-(w_n)w_n \in \mathcal{N}^-_{\lambda,\mu}$ and the Lebesgue dominated convergence theorem,

$$\begin{aligned} \int_{\mathbb{R}^N} |w^1_n|^\alpha |w^2_n|^\beta &= \frac{\int_{\mathbb{R}^N} |u^1 + l_n\sqrt{\alpha}\omega_{\varepsilon,y}|^\alpha |v^1 + l_n\sqrt{\beta}\omega_{\varepsilon,y}|^\beta}{\|z^1 + l_n(\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y})\|^{2^*}_t} \\ &= \frac{\int_{\mathbb{R}^N} |\frac{u^1}{l_n} + \sqrt{\alpha}\omega_{\varepsilon,y}|^\alpha |\frac{v^1}{l_n} + \sqrt{\beta}\omega_{\varepsilon,y}|^\beta}{\|\frac{z^1}{l_n} + (\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y})\|^{2^*}_t} \\ &= \frac{\int_{\mathbb{R}^N} |\sqrt{\alpha}\omega_{\varepsilon,y}|^\alpha |\sqrt{\beta}\omega_{\varepsilon,y}|^\beta}{\|(\sqrt{\alpha}\omega_{\varepsilon,y}, \sqrt{\beta}\omega_{\varepsilon,y})\|^{2^*}_t} + o(1), \text{ as } n \to \infty. \end{aligned}$$

Thus

$$\begin{aligned}J_{\lambda,\mu}(s^-(w_n)w_n) &= \frac{|s^-(w_n)|^2}{2} - \frac{|s^-(w_n)|^{2^*}}{2^*}\int_{\mathbb{R}^N}|w_n^1|^\alpha|w_n^2|^\beta \\ &\quad -\frac{|s^-(w_n)|^q}{q}\int_{\mathbb{R}^N}\lambda f(x)(w_n^1)^q+\mu g(x)(w_n^2)^q \\ &\longrightarrow -\infty,\ \text{as } n\to\infty,\end{aligned}$$

which contradicts that $J_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$.

Set

$$l_0=\frac{|c^2-\|z^1\|_t^2|^{\frac{1}{2}}}{\|(\sqrt{\alpha}\omega_{\varepsilon,y},\sqrt{\beta}\omega_{\varepsilon,y})\|_t}+1,$$

then

$$\begin{aligned}&\|z^1+l_0(\sqrt{\alpha}\omega_{\varepsilon,y},\sqrt{\beta}\omega_{\varepsilon,y})\|_t^2\\ =\ &\|z^1\|_t^2+l_0^2(\alpha+\beta)\|\omega_{\varepsilon,y}\|_t^2+2l_0<u^1,\sqrt{\alpha}\omega_{\varepsilon,y}>_t+2l_0<v^1,\sqrt{\beta}\omega_{\varepsilon,y}>_t\\ =\ &\|z^1\|_t^2+l_0^2(\alpha+\beta)\|\omega_{\varepsilon,y}\|_t^2+2l_0\left(\int_{\mathbb{R}^N}\lambda f(x)|u^1|^{q-1}\sqrt{\alpha}\omega_{\varepsilon,y}+\mu g(x)|v^1|^{q-1}\sqrt{\beta}\omega_{\varepsilon,y}\right)\\ &+\frac{4l_0\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}\sqrt{\alpha}|u^1|^{\alpha-1}|v^1|^\beta\omega_{\varepsilon,y}+\frac{4l_0\beta}{\alpha+\beta}\int_{\mathbb{R}^N}\sqrt{\beta}|u^1|^\alpha|v^1|^{\beta-1}\omega_{\varepsilon,y}.\end{aligned}$$

Since $f>0, g>0$ in Σ, $\omega_{\varepsilon,y}\equiv 0$ in Σ^c and the choice of l_0, we have

$$\begin{aligned}&\|z^1+l_0(\sqrt{\alpha}\omega_{\varepsilon,y},\sqrt{\beta}\omega_{\varepsilon,y})\|_t^2\\ \geq\ &\|z^1\|_t^2+l_0^2(\alpha+\beta)\|\omega_{\varepsilon,y}\|_t^2\\ >\ &\|z^1\|_t^2+\left|c^2-\|z^1\|_t^2\right|\geq c^2\\ >\ &\left[s^-\left(\frac{z^1+l(\sqrt{\alpha}\omega_{\varepsilon,y},\sqrt{\beta}\omega_{\varepsilon,y})}{\|z^1+l(\sqrt{\alpha}\omega_{\varepsilon,y},\sqrt{\beta}\omega_{\varepsilon,y})\|_t}\right)\right]^2.\end{aligned}$$

So $z^1+l_0(\sqrt{\alpha}\omega_{\varepsilon,y},\sqrt{\beta}\omega_{\varepsilon,y})\in U_2$.

Denote

$$\theta=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}J_{\lambda,\mu}(\gamma(s)),$$

where $\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=z^1,\gamma(1)=z^1+l_0(\sqrt{\alpha}\omega_{\varepsilon,y},\sqrt{\beta}\omega_{\varepsilon,y})\}$.

Obviously, the path $\gamma_0(s)=z^1+sl_0(\sqrt{\alpha}\omega_{\varepsilon,y},\sqrt{\beta}\omega_{\varepsilon,y})$ belongs to Γ. Thus, it follows from $\gamma(0)\in U_1$ and $\gamma(1)\in U_2$, that there exists $s_0\in(0,1)$ such that $\gamma(s_0)\in\mathcal{N}^-_{\lambda,\mu}$.

By Lemma 4.1, we get

$$\theta^-_{\lambda,\mu}\leq\theta<\theta_{\lambda,\mu}+\frac{2}{N}\left(\frac{S^t_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}.$$

By Proposition 4.1(ii), there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{z_n\}\subset\mathcal{N}_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$. By Lemma 3.5, there exists $z^2=(u^2,v^2)\in\mathcal{N}_{\lambda,\mu}$ such that $z_n\to z^2$ strongly in E. So z^2 is a nontrivial solution of problem (1.1).

Similar to the proof of Theorem 4.1 in [10], we can prove that $|z^2| = (|u^2|, |v|^2) \in \mathcal{N}^-_{\lambda,\mu}$ is a positive solution of problem (1.1). $\square$

Finally, we will give the proof of Theorem 1.

Proof. Let Λ be defined as in Theorem 4.2. For all $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda_1$, by Theorem 4.1, the system (1.1) has a positive solution $|z^1| \in \mathcal{N}^+_{\lambda,\mu}$. By $\Lambda \leq \Lambda_2 < \Lambda_1$ and Theorem 4.2, the system (1.1) has a positive solution $|z^2| \in \mathcal{N}^-_{\lambda,\mu}$. It follows from $\mathcal{N}^-_{\lambda,\mu} \cap \mathcal{N}^+_{\lambda,\mu} = \emptyset$, that the system (1.1) has two positive solutions $|z^1|$ and $|z^2|$. $\square$

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