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Oscillation criteria for third order delay nonlinear differential equations

Elmetwally M. Elabbasy, Taher S. Hassan*, and Bassant M. Elmatary

Abstract. The purpose of this paper is to give oscillation criteria for the third order delay nonlinear differential equation

$$[a_2(t)\{(a_1(t)(x'(t))^{\alpha_1})'\}^{\alpha_2}]' + q(t)f(x(g(t))) = 0,$$

via comparison with some first differential equations whose oscillatory characters are known. Our results generalize and improve some known results for oscillation of third order nonlinear differential equations. Some examples are given to illustrate the main results.

1. Introduction

In this paper, we are concerned with the oscillation of third order delay nonlinear differential equation

$$[a_2(t)\{(a_1(t)\,(x'(t))^{\alpha_1})'\}^{\alpha_2}]' + q(t)f(x(g(t))) = 0, \tag{1.1}$$

where the following conditions are satisfied

(A1): $a_1(t),\ a_2(t)$ and $q\,(t)\ \in C([t_0,\infty)\,,\ (0,\infty))$;
(A2): $\alpha_1,\ \alpha_2$ are quotient of odd positive integers;
(A3): $f\ \in\ C(\mathbb{R},\mathbb{R})$ such that $xf(x)\ >\ 0,\ f'(x)\ >\ 0$ for all $x\neq 0$ and $-f(-xy)\geq f(xy)\geq f(x)f(y)$ for $xy>0$;
(A4): $g(t)\in C^1([t_0,\infty)\,,\mathbb{R}),\ g(t)\leq t$ for $t\in[t_0,\infty)$ and $\lim\limits_{t\to\infty} g(t)=\infty$.

We mean by a solution of equation (1.1) a function $x\,(t):[t_x,\infty)\to\mathbb{R},\ t_x\geq t_0$ such that $x\,(t)\,,\ a_1(t)\,(x'(t))^{\alpha_1},\ a_2(t)\{(a_1(t)(x'(t))^{\alpha_1})'\}^{\alpha_2}$ are continuous and differentiable for all $t\in[t_x,\infty)$ and satisfies (1.1) for all $t\in[t_x,\infty)$ and satisfy $\sup\{|x\,(t)|\,:\,t\geq T\}>0$ for any $T\geq t_x$. A solution of equation (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. In the sequel it will be always assumed that equation (1.1) has nontrivial solutions which exist for all $t_0\geq 0$. Equation (1.1) is called oscillatory if all solutions are oscillatory. In the last few years, the oscillation theory and asymptotic behavior of differential equations and their applications have received more and more attentions, the reader is referred to the papers [**1**]- [**18**] and the references cited therein. Our aim

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*Corresponding author.

is to investigate the oscillatory criteria for all solutions of equation (1.1) with the cases, for $k = 1, 2$

$$\int_{t_0}^{\infty} a_k^{-\frac{1}{\alpha_k}}(t)dt = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} a_k^{-\frac{1}{\alpha_k}}(t)dt < \infty. \tag{1.3}$$

Our results have different natural as they are Riccati transformation technique and depend on new comparison principles that enable us to deduce properties of the third order nonlinear differential equation from oscillation the first order nonlinear delay differential equation. Recently, [**7, 12**] establish oscillation criteria for the third order nonlinear differential equation of the form

$$(a(t)\,(x''(t))^{\alpha})' + q(t)f(x(g(t))) = 0,$$

via comparison with first order oscillatory differential equations.

The purpose of this paper is to extend the above mentioned oscillation criteria which is established by [**7, 12**] , for the more general third order delay differential equation (1.1) for both of the cases (1.2) and (1.3). Hence our results will improve and extend results in [**7, 12**], and many known results on nonlinear oscillation.

2. Main Results

Before stating our main results, we start with the following lemmas which will play an important role in the proofs of our main results. We let,

$$\delta(t, t_0) := \int_{t_0}^{t} a_1^{-\frac{1}{\alpha_1}}(v)dv,\ \delta_k(t) := \int_{t}^{\infty} a_k^{-\frac{1}{\alpha_k}}(v)dv,\ k = 1, 2.$$

Lemma 2.1. *Assume that, for all sufficiently large $T_1 \in [t_0, \infty)$, there is a $T > T_1$ such that $g(t) > T_1$ for $t \geq T$ and*
(H1) either

$$\int_{t_0}^{\infty} a_2^{-\frac{1}{\alpha_2}}(t)dt = \infty, \tag{2.1}$$

or

$$\int_{T}^{\infty} \left(a_2^{-\frac{1}{\alpha_2}}(s) \left(\int_{T}^{s} \left(q(r)f(\delta_2^{\frac{1}{\alpha_1}}(g(r)))f(\delta(g(r), T)) \right) dr \right)^{\frac{1}{\alpha_2}} \right) ds = \infty, \tag{2.2}$$

(H2) either

$$\int_{t_0}^{\infty} a_1^{-\frac{1}{\alpha_1}}(t)dt = \infty, \tag{2.3}$$

or

$$\int_{t_0}^{\infty}\left(a_1^{-\frac{1}{\alpha_1}}(s)\left(\int_{t_0}^{s}a_2^{-\frac{1}{\alpha_2}}(u)\left(\int_{t_0}^{u}\left(q(v)f\left(\delta_1(v)\right)\right)dv\right)^{\frac{1}{\alpha_2}}du\right)^{\frac{1}{\alpha_1}}\right)ds=\infty, \quad (2.4)$$

hold. Let x be an eventually positive solution of the equation (1.1). *Then, either*

(1) $x'(t)>0,\ (a_1(t)\,(x'(t))^{\alpha_1})'>0$ for all $t\geq T$;
or
(2) $x'(t)<0,\ (a_1(t)\,(x'(t))^{\alpha_1})'>0$ for all $t\geq T$.

Proof. Pick $t_1\geq t_0$ such that $x(g(t))>0$, for $t\geq t_1$. From equation (1.1), (A1) and (A3), we have, $[a_2(t)\{(a_1(t)\,(x'(t))^{\alpha_1})'\}^{\alpha_2}]'<0$, for all $t\geq t_1$. Then $a_2(t)\,(a_1(t)\,(x'(t))^{\alpha_1})'$ is strictly decreasing on $[t_1,\infty)$, and thus $x'(t)$ and $(a_1(t)\,(x'(t))^{\alpha_1})'$ are eventually of one sign. We claim that $(a_1(t)\,(x'(t))^{\alpha_1})'>0$ on $[t_1,\infty)$. If not, then, we have two cases.
Case (1) There exists $t_2\geq t_1$, sufficiently large, such that

$$x'(t)>0 \quad\text{and}\quad (a_1(t)\,(x'(t))^{\alpha_1})'<0 \quad\text{for } t\geq t_2.$$

Case (2) There exists $t_2\geq t_1$, sufficiently large, such that

$$x'(t)<0 \quad\text{and}\quad (a_1(t)\,(x'(t))^{\alpha_1})'<0 \quad\text{for } t\geq t_2.$$

For the case (1), we have, $a_1(t)\,(x'(t))^{\alpha_1}$is strictly decreasing on $[t_2,\infty)$ and there exists a negative constant M such that

$$a_2(t)\{(a_1(t)(x'(t))^{\alpha_1})'\}^{\alpha_2}<M \text{ for all } t\geq t_2.$$

Dividing by $a_2(t)$ and integrating from t_2 to t, we get

$$a_1(t)(x'(t))^{\alpha_1}\leq a_1(t_2)\,(x'(t_2))^{\alpha_1}+M^{\frac{1}{\alpha_2}}\int_{t_2}^{t}a_2^{-\frac{1}{\alpha_2}}(s)ds.$$

Letting $t\to\infty$, and using (2.1) then $a_1(t)\,(x'(t))^{\alpha_1}\to-\infty$, which contradicts that $x'(t)>0$. Hence (2.2) is satisfied, we have

$$\begin{aligned}x(t)-x(t_3) &= \int_{t_3}^{t}x'(u)\,du\\ &= \int_{t_3}^{t}a_1^{-\frac{1}{\alpha_1}}(u)\left(a_1(u)\,(x'(u))^{\alpha_1}\right)^{\frac{1}{\alpha_1}}du\\ &\geq \left(a_1(t)\,(x'(t))^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\int_{t_3}^{t}a_1^{-\frac{1}{\alpha_1}}(u)du, \quad \text{for } t\geq t_3,\end{aligned}$$

and hence

$$x(t)\geq\left(a_1(t)\,(x'(t))^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\int_{t_3}^{t}a_1^{-\frac{1}{\alpha_1}}(u)du \quad \text{for } t\geq t_3.$$

There exists a $t_4 \geq t_3$ with $g(t) \geq t_3$ for all $t \geq t_4$ such that

$$x(g(t)) \geq \left(a_1(g(t))\,(x'(g(t)))^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \delta(g(t), t_3) \quad \text{for } t \geq t_4.$$

From Eq.(1.1), (A3) and the above inequality, we get, for $t \geq t_4$,

$$0 \geq (a_2(t)(y'(t))^{\alpha_2})' + q(t)f(y^{\frac{1}{\alpha_1}}(g(t)))f\left(\delta(g(t), t_3)\right), \tag{2.5}$$

where $y(t) := a_1(t)\,(x'(t))^{\alpha_1}$. It is clear that $y(t) > 0$ and $y'(t) < 0$. It follows that

$$-a_2(t)(y'(t))^{\alpha_2} \geq -a_2(t_4)(y'(t_4)) \quad \text{for} \quad t \geq t_4,$$

thus

$$-y'(t) \geq -\frac{a_2^{\frac{1}{\alpha_2}}(t_4)y'(t_4)}{a_2^{\frac{1}{\alpha_2}}(t)} \quad \text{for} \quad t \geq t_4.$$

Integrating the above inequality from t to ∞, we get

$$y(t) \geq -a_2^{\frac{1}{\alpha_2}}(t_4)y'(t_4)\delta_2(t),$$

then,

$$y(t) \geq k_1\delta_2(t), \quad \text{for} \quad t \geq t_5,$$

where $k_1 := -a_2^{\frac{1}{\alpha_2}}(t_4)y'(t_4) > 0$. There exists a $t_5 \geq t_4$ with $g(t) \geq t_4$ for all $t \geq t_5$ such that

$$y(g(t)) \geq k_1\delta_2(g(t)) \quad \text{for all} \quad t \geq t_5.$$

By integrating (2.5) from t_5 to t and using the above inequality, we obtain

$$\int_{t_5}^{t} q(r)f(k_1^{\frac{1}{\alpha_1}}\delta_2^{\frac{1}{\alpha_1}}(g(r)))f(\delta(g(r), t_3)))dr \leq a_2(t_5)(y'(t_5))^{\alpha_2} - a_2(t)(y'(t))^{\alpha_2},$$

Using (A3), we get

$$\left(\frac{b}{a_2(t)}\int_{t_5}^{t}\left(q(r)f(\delta_2^{\frac{1}{\alpha_1}}(g(r))f(\delta(g(r), t_3))\right)dr\right)^{\frac{1}{\alpha_2}} \leq -y'(t),$$

where $b := f(k_1^{\frac{1}{\alpha_1}})$. Integrating the above inequality from t_5 to ∞, we get

$$b^{\frac{1}{\alpha_2}}\int_{t_5}^{\infty}\left(a_2^{\frac{-1}{\alpha_2}}(s)\left(\int_{t_5}^{s}\left(q(r)f(\delta_2^{\frac{1}{\alpha_1}}(g(r)))f(\delta(g(r), t_3))\right)dr\right)^{\frac{1}{\alpha_2}}\right)ds \leq y(t_5) < \infty,$$

which contradicts the condition (2.2).

For the case (2), we have

$$a_1(t)\,(x'(t))^{\alpha_1} \leq a_1(t_2)\,(x'(t_2))^{\alpha_1} = k < 0.$$

Dividing by $a_1(t)$ and integrating from t_2 to t, we get

$$x(t) \leq x(t_2) + k^{\frac{1}{\alpha_1}}\int_{t_2}^{t} a_1^{-\frac{1}{\alpha_1}}(s)ds.$$

Letting $t \to \infty$, then (2.3) yields $x(t) \to -\infty$ this contradicts the fact that $x(t) > 0$. Otherwise, if (2.4) is satisfied. One can choose $t_3 \geq t_2$ with $g(t) \geq t_2$ for all $t \geq t_3$ such that

$$\begin{aligned} x(g(t)) &> -\left(a_1(g(t))\,(x'(g(t)))^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \delta_1(g(t)) \\ &\geq k_2\delta_1(g(t)), \quad \text{for all} \quad t \geq t_3, \end{aligned}$$

where $k_2 := -\left(a_1(t_2)\,(x'(t_2))^{\alpha_1}\right)^{\frac{1}{\alpha_1}} > 0$. Thus equation (1.1) and (A3) yield

$$\begin{aligned} \left(a_2(t)\{(a_1(t)\,(x'(t))^{\alpha_1})'\}^{\alpha_2}\right)' &= -q(t)f(x(g(t))) \\ &\leq Lq(t)f\left(\delta_1(g(t))\right), \end{aligned}$$

where $L := -f(k_2)$. Integrating the above inequality from t_3 to t, we get

$$a_2(t)\{(a_1(t)\,(x'(t))^{\alpha_1})'\}^{\alpha_2} \leq L \int_{t_3}^{t} (q(s)f(\delta_1(g(s)))\,ds.$$

Hence,

$$(a_1(t)\,(x'(t))^{\alpha_1})' \leq L^{\frac{1}{\alpha_2}} a_2^{-\frac{1}{\alpha_2}}(t) \left(\int_{t_3}^{t} (q(s)f(\delta_1(g(s)))\,ds\right)^{\frac{1}{\alpha_2}}.$$

Again integrating the above inequality from t_3 to t, we get

$$a_1(t)\,(x'(t))^{\alpha_1} \leq L^{\frac{1}{\alpha_2}} \int_{t_3}^{t} a_2^{-\frac{1}{\alpha_2}}(s) \left(\int_{t_3}^{s} (q(u)f(\delta_1(g(u)))\,du\right)^{\frac{1}{\alpha_2}} ds.$$

It follows that

$$x'(t) \leq k a_1^{-\frac{1}{\alpha_1}}(t) \left(\int_{t_3}^{t} a_2^{-\frac{1}{\alpha_2}}(s) \left(\int_{t_3}^{s} (q(u)f(\delta_1(g(u))\,du\right)^{\frac{1}{\alpha_2}} ds\right)^{\frac{1}{\alpha_1}},$$

where $k := L^{\frac{1}{\alpha_1\alpha_2}}$. Finally, integrating the last inequality from t_3 to t, we have

$$x(t) \leq$$
$$k \int_{t_3}^{t} \left(a_1^{-\frac{1}{\alpha_1}}(s) \left(\int_{t_3}^{s} a_2^{-\frac{1}{\alpha_2}}(u) \left(\int_{t_3}^{u} (q(v)f(\delta_1(g(v)))\,dv\right)^{\frac{1}{\alpha_2}} du\right)^{\frac{1}{\alpha_1}}\right) ds.$$

From condition (2.4), we get $x(t) \to -\infty$ as $t \to \infty$ which contradicts that $x(t)$ is a positive solution of (1.1). Then, we have $\left(a_1(t)\,(x'(t))^{\alpha_1}\right)' > 0$ for $t \geq t_1$ and of one sign thus either $x'(t) > 0$ or $x'(t) < 0$. The proof is complete. ∎

Lemma 2.2. *Assume that (H1) and (H2) hold. Let $x(t)$ be an eventually positive solution of the equation* (1.1) *for all* $t \in [t_0, \infty)$ *and suppose that Case* (2) *of Lemma*

2.1 holds. If

$$\int_{t_0}^{\infty} a_1^{-\frac{1}{\alpha_1}}(v)\left[\int_v^{\infty} a_2^{-\frac{1}{\alpha_2}}(u)\left(\int_u^{\infty} q(s)ds\right)^{\frac{1}{\alpha_2}} du\right]^{\frac{1}{\alpha_1}} dv = \infty, \tag{2.6}$$

then $x(t) \to 0$ *as* $t \to \infty$.

PROOF. Pick $t_1 \geq t_0$ such that $x(g(t)) > 0$, for $t \geq t_1$. Since $x(t)$ is positive decreasing solution of the equation (1.1) then, we get, $\lim_{t\to\infty} x(t) = l_1 \geq 0$. Assume $l_1 > 0$, then, $x(g(t)) \geq l_1$ for $t \geq t_2 \geq t_1$. Integrating equation (1.1) from t to ∞, we find

$$a_2(t)\{(a_1(t)\,(x'(t))^{\alpha_1})'\}^{\alpha_2} \geq \int_t^{\infty} q(s)f(x(g(s)))ds.$$

It follows from (A3) and (A4) that

$$(a_1(t)\,(x'(t))^{\alpha_1})' \geq \left(\frac{f(l_1)}{a_2(t)}\right)^{\frac{1}{\alpha_2}} \left(\int_t^{\infty} q(s)ds\right)^{\frac{1}{\alpha_2}}.$$

Integrating the above inequality from t to ∞, we get

$$-x'(t) \geq \frac{f^{\frac{1}{\alpha_1\alpha_2}}(l_1)}{a_1^{\frac{1}{\alpha_1}}(t)}\left[\int_t^{\infty} a_2^{-\frac{1}{\alpha_2}}(u)\left(\int_u^{\infty} q(s)ds\right)^{\frac{1}{\alpha_2}} du\right]^{\frac{1}{\alpha_1}}.$$

By integrating the last inequality from t_2 to ∞, we find that

$$x(t_2) \geq f^{\frac{1}{\alpha_1\alpha_2}}(l_1)\int_{t_2}^{\infty} a_1^{-\frac{1}{\alpha_1}}(v)\left[\int_v^{\infty} a_2^{-\frac{1}{\alpha_2}}(u)\left(\int_u^{\infty} q(s)ds\right)^{\frac{1}{\alpha_2}} du\right]^{\frac{1}{\alpha_1}} dv.$$

This contradicts to the condition (2.6), then $\lim_{t\to\infty} x(t) = 0$. ∎

THEOREM 2.1. *Let (H1), (H2) and* $g'(t) > 0$ *on* $[t_0, \infty)$ *hold and there exists a function* $\xi(t)$ *such that*

$$\xi'(t) \geq 0,\ \xi(t) > t \text{ and } g(\xi(\xi(t))) < t. \tag{2.7}$$

If both first order delay equations

$$y'(t) + q(t)f(y^{\frac{1}{\alpha_1\alpha_2}}(g(t)))f\left(\int_{t_0}^{g(t)} a_1^{-\frac{1}{\alpha_1}}(s)\left[\int_{t_0}^{s} a_2^{-\frac{1}{\alpha_2}}(u)du\right]^{\frac{1}{\alpha_1}} ds\right) = 0, \tag{2.8}$$

and

$$x'(t) + a_1^{-\frac{1}{\alpha_1}}(t)f^{\frac{1}{\alpha_1\alpha_2}}(x((\eta(t)))\left[\int_t^{\xi(t)} a_2^{-\frac{1}{\alpha_2}}(s)\left(\int_s^{\xi(s)} q(u)du\right)^{\frac{1}{\alpha_2}} ds\right]^{\frac{1}{\alpha_1}} = 0, \tag{2.9}$$

where $\eta(t) := g(\xi(\xi(t)))$, *are oscillatory, then equation* (1.1) *is oscillatory.*

PROOF. Assume (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$, sufficiently large such that $x(t) > 0$ and $x(g(t)) > 0$ on $[t_1, \infty)$. From equation (1.1), (A1) and (A3), we have $[a_2(t)\{(a_1(t)(x'(t))^{\alpha_1})'\}^{\alpha_2}]' < 0$ for all $t \geq t_1$. That is $a_2(t)\ (a_1(t)(x'(t))^{\alpha_1})'$ is strictly decreasing on $[t_1, \infty)$ and thus $(a_1(t)(x'(t))^{\alpha_1})'$ and $x'(t)$ are eventually of one sign. Then, from Lemma 2.1, we have the following cases, for $t_2 \geq t_1$, is sufficiently large
(1) $x'(t) > 0,\ (a_1(t)(x'(t))^{\alpha_1})' > 0$;
(2) $x'(t) < 0,\ (a_1(t)(x'(t))^{\alpha_1})' > 0$.
For the case (1), we have

$$\begin{aligned} a_1(t)(x'(t))^{\alpha_1} &= a_1(t_2)(x'(t_2))^{\alpha_1} + \int_{t_2}^{t} a_2^{-\frac{1}{\alpha_2}}(s) y^{\frac{1}{\alpha_2}}(s) ds \\ &\geq y^{\frac{1}{\alpha_2}}(t) \int_{t_2}^{t} a_2^{-\frac{1}{\alpha_2}}(s) ds, \end{aligned}$$

where $y(t) := a_2(t)\{(a_1(t)(x'(t))^{\alpha_1})'\}^{\alpha_2}$. It follows that

$$x'(t) \geq a_1^{-\frac{1}{\alpha_1}}(t) y^{\frac{1}{\alpha_1\alpha_2}}(t) \left[\int_{t_2}^{t} a_2^{-\frac{1}{\alpha_2}}(s) ds\right]^{\frac{1}{\alpha_1}}.$$

Integrating the above inequality from t_2 to t, we get

$$\begin{aligned} x(t) &\geq \int_{t_2}^{t} a_1^{-\frac{1}{\alpha_1}}(s) y^{\frac{1}{\alpha_1\alpha_2}}(s) \left[\int_{t_2}^{s} a_2^{-\frac{1}{\alpha_2}}(u) du\right]^{\frac{1}{\alpha_1}} ds \\ &\geq y^{\frac{1}{\alpha_1\alpha_2}}(t) \int_{t_2}^{t} a_1^{-\frac{1}{\alpha_1}}(s) \left[\int_{t_2}^{s} a_2^{-\frac{1}{\alpha_2}}(u) du\right]^{\frac{1}{\alpha_1}} ds. \end{aligned}$$

There exists $t_3 \geq t_2$ such that $g(t) \geq t_2$ for all $t \geq t_3$. Then

$$x(g(t)) \geq y^{\frac{1}{\alpha_1\alpha_2}}(g(t)) \int_{t_2}^{g(t)} a_1^{-\frac{1}{\alpha_1}}(s) \left[\int_{t_2}^{s} a_2^{-\frac{1}{\alpha_2}}(u) du\right]^{\frac{1}{\alpha_1}} ds, \text{ for all } t \geq t_3.$$

Thus equation (1.1) and (A3) yield, for all $t \geq t_3$.

$$\begin{aligned} -y'(t) &= q(t) f(x(g(t))) \\ &\geq q(t) f(y^{\frac{1}{\alpha_1\alpha_2}}(g(t))) f\left(\int_{t_2}^{g(t)} a_1^{-\frac{1}{\alpha_1}}(s) \left[\int_{t_2}^{s} a_2^{-\frac{1}{\alpha_2}}(u) du\right]^{\frac{1}{\alpha_1}} ds\right). \end{aligned}$$

Integrating the above inequality from t to ∞, we get

$$y(t) \geq \int_{t}^{\infty} q(s) f(y^{\frac{1}{\alpha_1\alpha_2}}(g(s))) f\left(\int_{t_2}^{g(s)} a_1^{-\frac{1}{\alpha_1}}(v) \left[\int_{t_2}^{v} a_2^{-\frac{1}{\alpha_2}}(u) du\right]^{\frac{1}{\alpha_1}} dv\right) ds.$$

The function $y(t)$ is obviously strictly decreasing. Hence, by Theorem 1 in [**18**] there exists a positive solution of equation (2.8) which tends to zero this contradicts that (2.8) is oscillatory.
For the case (2). Integrating equation (1.1) from t to $\xi(t)$, we obtain

$$a_2(t)\{(a_1(t)(x'(t))^{\alpha_1})'\}^{\alpha_2} \geq \int_t^{\xi(t)} q(s)f(x(g(s)))ds.$$

Using (2.7)and (A3), we get

$$(a_1(t)(x'(t))^{\alpha_1})' \geq a_2^{-\frac{1}{\alpha_2}}(t)f^{\frac{1}{\alpha_2}}(x(g(\xi(t)))) \left(\int_t^{\xi(t)} q(s)ds\right)^{\frac{1}{\alpha_2}}.$$

Integrating again the last inequality from t to $\xi(t)$, we get

$$-a_1(t)(x'(t))^{\alpha_1} \geq \int_t^{\xi(t)} a_2^{-\frac{1}{\alpha_2}}(u)f^{\frac{1}{\alpha_2}}(x(g(\xi(u)))) \left(\int_u^{\xi(u)} q(s)ds\right)^{\frac{1}{\alpha_2}} du.$$

It follows that

$$-x'(t) \geq f^{\frac{1}{\alpha_2\alpha_1}}(x(\eta(t)))a_1^{-\frac{1}{\alpha_1}}(t) \left[\int_t^{\xi(t)} a_2^{-\frac{1}{\alpha_2}}(u) \left(\int_u^{\xi(u)} q(s)ds\right)^{\frac{1}{\alpha_2}} du\right]^{\frac{1}{\alpha_1}}.$$

By integrate the above inequality from t to ∞, we have

$$x(t) \geq f^{\frac{1}{\alpha_2\alpha_1}}(x(\eta(t))) \int_t^{\infty} a_1^{-\frac{1}{\alpha_1}}(v) \left[\int_v^{\xi(v)} a_2^{-\frac{1}{\alpha_2}}(u) \left(\int_u^{\xi(u)} q(s)ds\right)^{\frac{1}{\alpha_2}} du\right]^{\frac{1}{\alpha_1}} dv$$

In view of Theorem 1 in [**18**] there exists a positive solution of equation (2.9) which tends to zero which contradicts that (2.9) is oscillatory then equation (1.1) is oscillatory. The proof is complete. ∎

The following result is obtained by combining case (1) in the proof of Theorem 2.1 with Lemma 2.2.

Theorem 2.2. *Assume that the first order delay equation* (2.8) *is oscillatory,* (2.6), *(H1) and (H2) hold. Then every solution* $x(t)$ *of equation* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Remark 2.1. Let $a_1(t) = 1$ and $\alpha_1 = 1$ Theorem 2.1 and Theorem 2.2 are reduced to [**7**, Theorem 3 and Theorem 2].

In the following examples are given to illustrate the main results.

Example 2.1. Consider the third order delay differential equation

$$\left(\left[t\left(\frac{1}{t^2}\left(y'(t)\right)^{\frac{1}{3}}\right)'\right]^3\right)' + \frac{1}{t}y\left(t^{\frac{1}{5}}\right) = 0, \quad t \geq 1. \tag{2.10}$$

We note that

$$f(y) = y, \quad g(t) = t^{\frac{1}{5}} < t, \quad g'(t) > 0, \quad \lim_{t\to\infty} g(t) = \lim_{t\to\infty} t^{\frac{1}{5}} = \infty,$$

and

$$a_1(t) = \frac{1}{t^2}, \quad a_2(t) = t, \quad \alpha_1 = \frac{1}{3}, \quad \alpha_2 = 3,$$

and

$$\int_1^\infty a_1^{-\frac{1}{\alpha_1}}(u)du = \infty, \quad \int_1^\infty a_2^{-\frac{1}{\alpha_2}}(u)du = \infty.$$

It easy to see that condition (2.6) holds and Eq.(2.8), reduces to

$$y'(t) + \frac{1}{t}\left(b_1 t^{\frac{9}{5}} + b_2 t^{\frac{5}{3}} + b_3 t^{\frac{23}{15}} - b_4 t^{\frac{7}{5}}\right) y\left(t^{\frac{1}{5}}\right) = 0. \tag{2.11}$$

where $b_1,\ b_2,\ b_3,\ b_4$ are constants. On the other hand, Theorem 2.1.1 in [**17**] guarantees oscillation of (2.11) provided that

$$\lim_{t\to\infty} \int_{t^{\frac{1}{5}}}^{t} \frac{1}{s}\left(b_1 s^{\frac{9}{5}} + b_2 s^{\frac{5}{3}} + b_3 s^{\frac{23}{15}} - b_4 s^{\frac{7}{5}}\right) ds > \frac{1}{e},$$

and according to Theorem 2.2. every nonoscillatory solution of Eq.(2.10) tends to zero as $t \to \infty$.

Example 2.2. Consider the third order delay differential equation

$$\left(t^3\left(t^6\left(y'(t)\right)\right)'\right)' + t^{11}y\left(\frac{t}{2}\right) = 0, \quad t \geq 1. \tag{2.12}$$

We note that

$$f(y) = y,\ g(t) = t^{\frac{1}{5}} < t,\ g'(t) > 0,\ \lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{t}{2} = \infty,$$

and

$$a_1(t) = t^4, \quad a_2(t) = t^3, \quad \alpha_1 = \alpha_2 = 1,$$

and

$$\int_1^\infty a_1^{-\frac{1}{\alpha_1}}(u)du = \frac{1}{5} < \infty, \quad \int_1^\infty a_2^{-\frac{1}{\alpha 2}}(u)du = \frac{1}{2} < \infty.$$

It easy to see that conditions(2.6), (2.2) and (2.4) hold. Eq.(2.8), reduces to

$$y'(t) + t^{11}\frac{(t^7 - 112t^2 + 320)}{35t^7} y\left(\frac{t}{2}\right) = 0. \tag{2.13}$$

on the other hand, Theorem 2.1.1 in [**17**] guarantees oscillation of (2.13) provided that

$$\lim_{t\to\infty}\int_{t/2}^{t} t^{11}\frac{(t^7-112t^2+320)}{35t^7} > \frac{1}{e},$$

and according to Theorem 2.2. every nonoscillatory solution of Eq.(2.12) tends to zero as $t\to\infty$.

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Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt
E-mail address: emelabbasy@mans.edu.eg (E. M. Elabbasy)

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt
E-mail address: tshassan@mans.edu.eg (T. S. Hassan)

Department of Mathematics, Faculty of Science, Mansoura University, New Damietta 34517, Egypt
E-mail address: bassantmarof@yahoo.com (B. M. Elmatary)