Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **98**, 1-16; **http://www.math.u-szeged.hu/ejqtde/**

Even Number of Positive Solutions for $3n^{th}$ Order Three-Point Boundary Value Problems on Time Scales

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Abstract

We establish the existence of at least two positive solutions for the $3n^{th}$ order three-point boundary value problem on time scales by using Avery-Henderson fixed point theorem. We also establish the existence of at least $2m$ positive solutions for an arbitrary positive integer m.

Key words: Green's function, boundary value problem, time scale, positive solution, cone.

AMS Subject Classification: 39A10, 34B05.

1 Introduction

The theory of time scales was introduced and developed by Hilger [13] to unify both continuous and discrete analysis. Time scales theory presents us with the tools necessary to understand and explain the mathematical structure underpinning the theories of discrete and continuous dynamic systems and allows us to connect them. The theory is widely applied to various situations like epidemic models, the stock market and mathematical modeling of physical and biological systems. Certain economically important phenomena contain processes that feature elements of both the continuous and discrete.

In recent years, the existence of positive solutions of the higher order boundary value problems (BVPs) on time scales have been studied extensively due to their striking applications to almost all area of science, engineering and technology. The existence of positive solutions are studied by many authors. A few papers along these lines are Henderson [11], Anderson [1, 2], Kaufmann

[15], Anderson and Avery [3], DaCunha, Davis and Singh [10], Peterson, Raffoul and Tisdell [18], Sun and Li [19], Luo and Ma [17], Cetin and Topal [8], Karaca [14] and Anderson and Karaca [4].

In this paper, we are concerned with the existence of positive solutions for the $3n^{th}$ order BVP on time scales,

$$(-1)^n y^{\Delta^{(3n)}}(t) = f(t, y(t)), \quad t \in [t_1, \sigma(t_3)] \tag{1.1}$$

satisfying the general three-point boundary conditions,

$$\left.\begin{aligned} \alpha_{3i-2,1}y^{\Delta^{(3i-3)}}(t_1) + \alpha_{3i-2,2}y^{\Delta^{(3i-2)}}(t_1) + \alpha_{3i-2,3}y^{\Delta^{(3i-1)}}(t_1) = 0,\\ \alpha_{3i-1,1}y^{\Delta^{(3i-3)}}(t_2) + \alpha_{3i-1,2}y^{\Delta^{(3i-2)}}(t_2) + \alpha_{3i-1,3}y^{\Delta^{(3i-1)}}(t_2) = 0,\\ \alpha_{3i,1}y^{\Delta^{(3i-3)}}(\sigma(t_3)) + \alpha_{3i,2}y^{\Delta^{(3i-2)}}(\sigma(t_3)) + \alpha_{3i,3}y^{\Delta^{(3i-1)}}(\sigma(t_3)) = 0, \end{aligned}\right\} \tag{1.2}$$

for $1 \le i \le n$, where $n \ge 1$, $\alpha_{3i-2,j}, \alpha_{3i-1,j}, \alpha_{3i,j}$, for $j = 1, 2, 3$, are real constants, $t_1 < t_2 < \sigma(t_3)$ and $f : [t_1, \sigma(t_3)] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. For convenience, we use the following notations. For $1 \le i \le n$, let us denote $\beta_{i_j} = \alpha_{3i-3+j,1}t_j + \alpha_{3i-3+j,2}$, $\gamma_{i_j} = \alpha_{3i-3+j,1}t_j^2 + \alpha_{3i-3+j,2}(t_j + \sigma(t_j)) + 2\alpha_{3i-3+j,3}$, where $j = 1, 2$; $\beta_{i_3} = \alpha_{3i,1}\sigma(t_3) + \alpha_{3i,2}$ and $\gamma_{i_3} = \alpha_{3i,1}(\sigma(t_3))^2 + \alpha_{3i,2}(\sigma(t_3) + \sigma^2(t_3)) + 2\alpha_{3i,3}$. Also, for $1 \le i \le n$, we define

$$m_{i_{jk}} = \frac{\alpha_{3i-3+j,1}\gamma_{i_k} - \alpha_{3i-3+k,1}\gamma_{i_j}}{2(\alpha_{3i-3+j,1}\beta_{i_k} - \alpha_{3i-3+k,1}\beta_{i_j})}, \quad M_{i_{jk}} = \frac{\beta_{i_j}\gamma_{i_k} - \beta_{i_k}\gamma_{i_j}}{\alpha_{3i-3+j,1}\beta_{i_k} - \alpha_{3i-3+k,1}\beta_{i_j}},$$

where $j, k = 1, 2, 3$ and let $p_i = \max\{m_{i_{12}}, m_{i_{13}}, m_{i_{23}}\}$,
$q_i = \min\left\{m_{i_{23}} + \sqrt{m_{i_{23}}^2 - M_{i_{23}}},\ \ m_{i_{13}} + \sqrt{m_{i_{13}}^2 - M_{i_{13}}}\right\}$,
$d_i = \alpha_{3i-2,1}(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) - \beta_{i_1}(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) + \gamma_{i_1}(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})$ and $l_{i_j} = \alpha_{3i-3+j,1}\sigma(s)\sigma^2(s) - \beta_{i_j}(\sigma(s) + \sigma^2(s)) + \gamma_{i_j}$, where $j = 1, 2, 3$. We assume the following conditions throughout this paper:

(A1) $\alpha_{3i-2,1} > 0, \alpha_{3i-1,1} > 0, \alpha_{3i,1} > 0$ and $\frac{\alpha_{3i,2}}{\alpha_{3i,1}} > \frac{\alpha_{3i-1,2}}{\alpha_{3i-1,1}} > \frac{\alpha_{3i-2,2}}{\alpha_{3i-2,1}}$, for all $1 \le i \le n$,

(A2) $p_i \le t_1 < t_2 < \sigma(t_3) \le q_i$ and $2\alpha_{3i-2,3}\alpha_{3i-2,1} > \alpha_{3i-2,2}^2$, $2\alpha_{3i-1,3}\alpha_{3i-1,1} < \alpha_{3i-1,2}^2$, $2\alpha_{3i,3}\alpha_{3i,1} > \alpha_{3i,2}^2$, for all $1 \le i \le n$,

(A3) $m_{i_{23}}^2 > M_{i_{23}}$, $m_{i_{12}}^2 < M_{i_{12}}$, $m_{i_{13}}^2 > M_{i_{13}}$ and $d_i > 0$, for all $1 \le i \le n$,

(A4) The point $t \in [t_1, \sigma(t_3)]$ is not left dense and right scattered at the same time.

This paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate bounds for the Green's function. In Section 3, we establish a criteria for the existence of at least two positive solutions for the BVP (1.1)-(1.2) by using an Avery-Henderson fixed point theorem [5]. We also establish the existence of at least $2m$ positive solutions for an arbitrary positive integer m. Finally as an application, we give an example to illustrate our result.

2 Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate bounds for the Green's function.

Let $G_i(t,s)$ be the Green's function for the homogeneous BVP,

$$-y^{\Delta^3}(t) = 0, \qquad t \in [t_1, \sigma(t_3)], \tag{2.1}$$

satisfying the general three-point boundary conditions,

$$\left.\begin{aligned} \alpha_{3i-2,1}y(t_1) + \alpha_{3i-2,2}y^{\Delta}(t_1) + \alpha_{3i-2,3}y^{\Delta^2}(t_1) = 0,\\ \alpha_{3i-1,1}y(t_2) + \alpha_{3i-1,2}y^{\Delta}(t_2) + \alpha_{3i-1,3}y^{\Delta^2}(t_2) = 0,\\ \alpha_{3i,1}y(\sigma(t_3)) + \alpha_{3i,2}y^{\Delta}(\sigma(t_3)) + \alpha_{3i,3}y^{\Delta^2}(\sigma(t_3)) = 0,\end{aligned}\right\} \tag{2.2}$$

for $1 \le i \le n$.

Lemma 2.1 *For $1 \le i \le n$, the Green's function $G_i(t,s)$ for the homogeneous BVP* (2.1)-(2.2) *is given by*

$$G_i(t,s) = \begin{cases} \underset{t\in[t_1,t_2]}{G_i(t,s)} = \begin{cases} G_{i_1}(t,s), & t_1 < \sigma(s) < t \le t_2 < \sigma(t_3)\\ G_{i_2}(t,s), & t_1 \le t < s < t_2 < \sigma(t_3)\\ G_{i_3}(t,s), & t_1 \le t < t_2 < s < \sigma(t_3)\end{cases}\\ \\ \underset{t\in[t_2,\sigma(t_3)]}{G_i(t,s)} = \begin{cases} G_{i_4}(t,s), & t_1 < t_2 < \sigma(s) < t \le \sigma(t_3)\\ G_{i_5}(t,s), & t_1 < t_2 \le t < s < \sigma(t_3)\\ G_{i_6}(t,s), & t_1 \le \sigma(s) < t_2 < t < \sigma(t_3)\end{cases}\end{cases} \tag{2.3}$$

where

$$\begin{aligned} G_{i_1}(t,s) =& \frac{1}{2d_i}[-(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) + t(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) - t^2(\alpha_{3i-1,1}\beta_{i_3} - \\ & \alpha_{3i,1}\beta_{i_2})]l_{i_1},\end{aligned}$$

$$
\begin{aligned}
G_{i_2}(t,s) =& \frac{1}{2d_i}\{[-(\beta_{i_1}\gamma_{i_3}-\beta_{i_3}\gamma_{i_1})+t(\alpha_{3i-2,1}\gamma_{i_3}-\alpha_{3i,1}\gamma_{i_1})-t^2(\alpha_{3i-2,1}\beta_{i_3}- \\
&\alpha_{3i,1}\beta_{i_1})]l_{i_2}+[(\beta_{i_1}\gamma_{i_2}-\beta_{i_2}\gamma_{i_1})-t(\alpha_{3i-2,1}\gamma_{i_2}-\alpha_{3i-1,1}\gamma_{i_1})+ \\
&t^2(\alpha_{3i-2,1}\beta_{i_2}-\alpha_{3i-1,1}\beta_{i_1})]l_{i_3}\}, \\
G_{i_3}(t,s) =& \frac{1}{2d_i}[(\beta_{i_1}\gamma_{i_2}-\beta_{i_2}\gamma_{i_1})-t(\alpha_{3i-2,1}\gamma_{i_2}-\alpha_{3i-1,1}\gamma_{i_1})+t^2(\alpha_{3i-2,1}\beta_{i_2}- \\
&\alpha_{3i-1,1}\beta_{i_1})]l_{i_3}, \\
G_{i_4}(t,s) =& \frac{1}{2d_i}\{[-(\beta_{i_2}\gamma_{i_3}-\beta_{i_3}\gamma_{i_2})+t(\alpha_{3i-1,1}\gamma_{i_3}-\alpha_{3i,1}\gamma_{i_2})-t^2(\alpha_{3i-1,1}\beta_{i_3}- \\
&\alpha_{3i,1}\beta_{i_2})]l_{i_1}+[(\beta_{i_1}\gamma_{i_3}-\beta_{i_3}\gamma_{i_1})-t(\alpha_{3i-2,1}\gamma_{i_3}-\alpha_{3i,1}\gamma_{i_1})+ \\
&t^2(\alpha_{3i-2,1}\beta_{i_3}-\alpha_{3i,1}\beta_{i_1})]l_{i_2}\}, \\
G_{i_5}(t,s) =& \frac{1}{2d_i}[(\beta_{i_1}\gamma_{i_2}-\beta_{i_2}\gamma_{i_1})-t(\alpha_{3i-2,1}\gamma_{i_2}-\alpha_{3i-1,1}\gamma_{i_1})+t^2(\alpha_{3i-2,1}\beta_{i_2}- \\
&\alpha_{3i-1,1}\beta_{i_1})]l_{i_3}, \\
G_{i_6}(t,s) =& \frac{1}{2d_i}[-(\beta_{i_2}\gamma_{i_3}-\beta_{i_3}\gamma_{i_2})+t(\alpha_{3i-1,1}\gamma_{i_3}-\alpha_{3i,1}\gamma_{i_2})-t^2(\alpha_{3i-1,1}\beta_{i_3}- \\
&\alpha_{3i,1}\beta_{i_2})]l_{i_1}.
\end{aligned}
$$

Lemma 2.2 *Assume that the conditions* $(A1)$-$(A4)$ *are satisfied. Then, for* $1 \leq i \leq n$, *the Green's function* $G_i(t,s)$ *of* (2.1)-(2.2) *is positive, for all* $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Proof: For $1 \leq i \leq n$, the Green's function $G_i(t,s)$ is given in (2.3). We prove the result for $G_{i_1}(t,s)$. Then, $G_{i_1}(t,s) = g_{i_1}(t)l_{i_1}(s)$, where

$$
\begin{aligned}
g_{i_1}(t) =& \frac{1}{2d_i}[-(\beta_{i_2}\gamma_{i_3}-\beta_{i_3}\gamma_{i_2})+t(\alpha_{3i-1,1}\gamma_{i_3}-\alpha_{3i,1}\gamma_{i_2})-t^2(\alpha_{3i-1,1}\beta_{i_3}- \\
&\alpha_{3i,1}\beta_{i_2})].
\end{aligned}
$$

Using the conditions $(A1)$ and $(A4)$, $g_{i_1}(t)$ has maximum at $t = m_{i_{23}}$, and hence $g_{i_1}(t) > 0$ on $[t_1, \sigma(t_3)]$ by conditions $(A2)$ and $(A3)$. From conditions $(A2)$ and $(A4)$, $l_{i_1}(s) > 0$ on $[t_1, t_3]$.
Therefore,

$$G_{i_1}(t,s) > 0, \text{ for all } (t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3].$$

Similarly, we can establish the positivity of the Green's function in the remaining cases. $\square$

Theorem 2.3 *Assume that the conditions* $(A1)$*-*$(A4)$ *are satisfied. Then, for* $1 \le i \le n$*, the Green's function* $G_i(t,s)$ *satisfies the following inequality,*

$$m_i G_i(\sigma(s),s) \le G_i(t,s) \le G_i(\sigma(s),s), \text{ for all } (t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3], \tag{2.4}$$

where

$$0 < m_i = \min\left\{ \frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}, \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}, \frac{G_{i_2}(t_1,s)}{G_{i_2}(\sigma(t_3),s)}, \frac{G_{i_4}(\sigma(t_3),s)}{G_{i_4}(t_1,s)} \right\} < 1.$$

Proof: For $1 \le i \le n$, the Green's function $G_i(t,s)$ is given (2.3) in six different cases. In each case, we prove the inequality as in (2.4).

Case 1. For $t_1 < \sigma(s) < t \le t_2 < \sigma(t_3)$.
$\frac{G_i(t,s)}{G_i(\sigma(s),s)} = \frac{G_{i_1}(t,s)}{G_{i_1}(\sigma(s),s)}$

$$= \frac{[-(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) + t(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) - t^2(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})]}{[-(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) + \sigma(s)(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) - (\sigma(s))^2(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})]}.$$

From $(A1)$-$(A4)$, we have $G_{i_1}(t,s) \le G_{i_1}(\sigma(s),s)$ and also

$$\frac{G_i(t,s)}{G_i(\sigma(s),s)} = \frac{G_{i_1}(t,s)}{G_{i_1}(\sigma(s),s)} \ge \frac{G_{i_1}(t,s)}{G_{i_1}(t_1,s)} \ge \frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}.$$

Therefore, $G_i(t,s) \le G_i(\sigma(s),s)$ and $G_i(t,s) \ge \frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)} G_i(\sigma(s),s)$, for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Case 2. For $t_1 \le t < t_2 < s < \sigma(t_3)$.
$\frac{G_i(t,s)}{G_i(\sigma(s),s)} = \frac{G_{i_3}(t,s)}{G_{i_3}(\sigma(s),s)}$

$$= \frac{[(\beta_{i_1}\gamma_{i_2} - \beta_{i_2}\gamma_{i_1}) - t(\alpha_{3i-2,1}\gamma_{i_2} - \alpha_{3i-1,1}\gamma_{i_1}) + t^2(\alpha_{3i-2,1}\beta_{i_2} - \alpha_{3i-1,1}\beta_{i_1})]}{[(\beta_{i_1}\gamma_{i_2} - \beta_{i_2}\gamma_{i_1}) - \sigma(s)(\alpha_{3i-2,1}\gamma_{i_2} - \alpha_{3i-1,1}\gamma_{i_1}) + (\sigma(s))^2(\alpha_{3i-2,1}\beta_{i_2} - \alpha_{3i-1,1}\beta_{i_1})]}.$$

From $(A1)$-$(A4)$, we have $G_{i_3}(t,s) \le G_{i_3}(\sigma(s),s)$ and also

$$\frac{G_i(t,s)}{G_i(\sigma(s),s)} = \frac{G_{i_3}(t,s)}{G_{i_3}(\sigma(s),s)} \ge \frac{G_{i_3}(t,s)}{G_{i_3}(\sigma(t_3),s)} \ge \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}.$$

Therefore, $G_i(t,s) \le G_i(\sigma(s),s)$ and $G_i(t,s) \ge \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)} G_i(\sigma(s),s)$, for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Case 3. For $t_1 \leq t < s < t_2 < \sigma(t_3)$.
From $(A1)$-$(A4)$ and case 2, we have $G_{i_2}(t,s) \leq G_{i_2}(\sigma(s),s)$ and also

$$\frac{G_i(t,s)}{G_i(\sigma(s),s)} \geq \min\left\{\frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}, \frac{G_{i_2}(t_1,s)}{G_{i_2}(\sigma(t_3),s)}\right\}.$$

Therefore, $G_i(t,s) \leq G_i(\sigma(s),s)$ and

$$G_i(t,s) \geq \min\left\{\frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}, \frac{G_{i_2}(t_1,s)}{G_{i_2}(\sigma(t_3),s)}\right\} G_i(\sigma(s),s),$$

for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Case 4. For $t_1 < t_2 < \sigma(s) < t \leq \sigma(t_3)$.
From $(A1)$-$(A4)$ and case 1, we have $G_{i_4}(t,s) \leq G_{i_4}(\sigma(s),s)$ and

$$\frac{G_i(t,s)}{G_i(\sigma(s),s)} \geq \min\left\{\frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}, \frac{G_{i_4}(\sigma(t_3),s)}{G_{i_4}(t_1,s)}\right\}.$$

Therefore, $G_i(t,s) \leq G_i(\sigma(s),s)$ and

$$G_i(t,s) \geq \min\left\{\frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}, \frac{G_{i_4}(\sigma(t_3),s)}{G_{i_4}(t_1,s)}\right\} G_i(\sigma(s),s),$$

for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Case 5. For $t_1 < t_2 \leq t < s < \sigma(t_3)$.
From case 2, we have $G_i(t,s) \leq G_i(\sigma(s),s)$ and $G_i(t,s) \geq \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)} G_i(\sigma(s),s)$, for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Case 6. For $t_1 \leq \sigma(s) < t_2 < t < \sigma(t_3)$.
From case 1, we have $G_i(t,s) \leq G_i(\sigma(s),s)$ and $G_i(t,s) \geq \frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)} G_i(\sigma(s),s)$, for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

From all above cases, for $1 \leq i \leq n$, we have

$$m_i G_i(\sigma(s),s) \leq G_i(t,s) \leq G_i(\sigma(s),s), \text{ for all } (t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3],$$

where

$$0 < m_i = \min\left\{\frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}, \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}, \frac{G_{i_2}(t_1,s)}{G_{i_2}(\sigma(t_3),s)}, \frac{G_{i_4}(\sigma(t_3),s)}{G_{i_4}(t_1,s)}\right\} < 1.$$

□

Lemma 2.4 *Assume that the conditions* (A1)-(A4) *are satisfied and* $G_i(t,s)$ *is defined as in* (2.3). *Take* $H_1(t,s) = G_1(t,s)$ *and recursively define*

$$H_j(t,s) = \int_{t_1}^{\sigma(t_3)} H_{j-1}(t,r)G_j(r,s)\Delta r, \ \ for\ 2 \leq j \leq n.$$

Then $H_n(t,s)$ *is the Green's function for the homogeneous BVP corresponding to* (1.1)-(1.2).

Lemma 2.5 *Assume that the conditions* (A1)-(A4) *hold. If we define*

$$K = \prod_{j=1}^{n-1} K_j \quad \text{and } L = \prod_{j=1}^{n-1} m_j L_j,$$

then the Green's function $H_n(t,s)$ *in Lemma 2.4 satisfies*

$$0 \leq H_n(t,s) \leq K \parallel G_n(\cdot,s) \parallel, \ for\ all\ (t,s) \in [t_1,\sigma(t_3)] \times [t_1,t_3]$$

and

$$H_n(t,s) \geq m_n L \parallel G_n(\cdot,s) \parallel, \ for\ all\ (t,s) \in [t_2,\sigma(t_3)] \times [t_1,t_3],$$

where m_n *is given as in Theorem 2.3,*

$$K_j = \int_{t_1}^{\sigma(t_3)} \parallel G_j(\cdot,s) \parallel \Delta s > 0, \ for\ 1 \leq j \leq n,$$

$$L_j = \int_{t_2}^{\sigma(t_3)} \parallel G_j(\cdot,s) \parallel \Delta s > 0, \ for\ 1 \leq j \leq n$$

and $\parallel \cdot \parallel$ *is defined by*

$$\parallel x \parallel = \max_{t \in [t_1,\sigma(t_3)]} |x(t)|.$$

3 Multiple Positive Solutions

In this section, we establish the existence of at least two positive solutions for the BVP (1.1)-(1.2) by using an Avery-Henderson functional fixed point theorem. And then, we establish the existence of at least $2m$ positive solutions for an arbitrary positive integer m.

Let B be a real Banach space. A nonempty closed convex set $P \subset B$ is called a cone, if it satisfies the following two conditions:

(i) $y \in P, \lambda \geq 0$ implies $\lambda y \in P$, and

(ii) $y \in P$ and $-y \in P$ implies $y = 0$.

Let ψ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for a positive real number c', we define the sets

$$P(\psi, c') = \{y \in P : \psi(y) < c'\}$$

and

$$P_a = \{y \in P : \| y \| < a\}.$$

In obtaining multiple positive solutions of the BVP (1.1)-(1.2), the following Avery-Henderson functional fixed point theorem will be the fundamental tool.

Theorem 3.1 [5] *Let P be a cone in a real Banach space B. Suppose α and γ are increasing, nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive numbers c' and k, $\gamma(y) \leq \theta(y) \leq \alpha(y)$ and $\| y \| \leq k\gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose that there exist positive numbers a' and b' with $a' < b' < c'$ such that $\theta(\lambda y) \leq \lambda\theta(y)$, for all $0 \leq \lambda \leq 1$ and $y \in \partial P(\theta, b')$. Further, let $T : \overline{P(\gamma, c')} \to P$ be a completely continuous operator such that*
(B1) $\gamma(Ty) > c'$, for all $y \in \partial P(\gamma, c')$,
(B2) $\theta(Ty) < b'$, for all $y \in \partial P(\theta, b')$,
(B3) $P(\alpha, a') \neq \emptyset$ and $\alpha(Ty) > a'$, for all $y \in \partial P(\alpha, a')$.
Then, T has at least two fixed points $y_1, y_2 \in \overline{P(\gamma, c')}$ such that $a' < \alpha(y_1)$ with $\theta(y_1) < b'$ and $b' < \theta(y_2)$ with $\gamma(y_2) < c'$.

Let

$$M = m_n \prod_{j=1}^{n-1} \frac{m_j L_j}{K_j} \tag{3.1}$$

Let $B = \{y : y \in C[t_1, \sigma(t_3)]\}$ be the Banach space equipped with the norm

$$\| y \| = \max_{t \in [t_1, \sigma(t_3)]} |y(t)|.$$

Define the cone $P \subset B$ by

$$P = \{y \in B : y(t) \geq 0 \text{ on } [t_1, \sigma(t_3)] \text{ and } \min_{t \in [t_2, \sigma(t_3)]} y(t) \geq M \| y \|\},$$

where M is given as in (3.1).

Define the nonnegative, increasing, continuous functionals γ, θ and α on the cone P by

$$\gamma(y) = \min_{t \in [t_2, \sigma(t_3)]} y(t), \ \theta(y) = \max_{t \in [t_2, \sigma(t_3)]} y(t) \text{ and } \alpha(y) = \max_{t \in [t_1, \sigma(t_3)]} y(t).$$

We observe that for any $y \in P$,

$$\gamma(y) \leq \theta(y) \leq \alpha(y) \tag{3.2}$$

and

$$\| y \| \leq \frac{1}{M} \min_{t \in [t_2, \sigma(t_3)]} y(t) = \frac{1}{M}\gamma(y) \leq \frac{1}{M}\theta(y) \leq \frac{1}{M}\alpha(y). \tag{3.3}$$

Theorem 3.2 *Suppose there exist* $0 < a' < b' < c'$ *such that* f *satisfies the following conditions.*
(D1) $f(t, y) > \frac{c'}{\Pi_{j=1}^{n} m_j L_j}$, *for* $t \in [t_2, \sigma(t_3)]$ *and* $y \in [c', \frac{c'}{M}]$,
(D2) $f(t, y) < \frac{b'}{\Pi_{j=1}^{n} K_j}$, *for* $t \in [t_1, \sigma(t_3)]$ *and* $y \in [0, \frac{b'}{M}]$,
(D3) $f(t, y) > \frac{a'}{\Pi_{j=1}^{n} m_j L_j}$, *for* $t \in [t_2, \sigma(t_3)]$ *and* $y \in [a', \frac{a'}{M}]$,
where m_n *and* M *are defined in Theorem 2.3 and* (3.1) *respectively. Then the BVP* (1.1)-(1.2) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$a' < \max_{t \in [t_1, \sigma(t_3)]} y_1(t) \ \textit{with} \ \max_{t \in [t_2, \sigma(t_3)]} y_1(t) < b',$$

$$b' < \max_{t \in [t_2, \sigma(t_3)]} y_2(t) \ \textit{with} \ \min_{t \in [t_2, \sigma(t_3)]} y_2(t) < c'.$$

Proof: Define the operator $T : P \to B$ by

$$Ty(t) = \int_{t_1}^{\sigma(t_3)} H_n(t, s) f(s, y(s)) \Delta s. \tag{3.4}$$

It is obvious that a fixed point of T is the solution of the BVP (1.1)-(1.2). We seek two fixed points $y_1, y_2 \in P$ of T. First, we show that $T : P \to P$. Let

$y \in P$. From Theorem 2.3 and Lemma 2.5, we have $Ty(t) \geq 0$ on $[t_1, \sigma(t_3)]$ and also,

$$\begin{aligned} Ty(t) &= \int_{t_1}^{\sigma(t_3)} H_n(t,s) f(s, y(s)) \Delta s \\ &\leq K \int_{t_1}^{\sigma(t_3)} \parallel G_n(\cdot, s) \parallel f(s, y(s)) \Delta s \end{aligned}$$

so that

$$\parallel Ty \parallel \leq K \int_{t_1}^{\sigma(t_3)} \parallel G_n(\cdot, s) \parallel f(s, y(s)) \Delta s.$$

Next, if $y \in P$, then we have

$$\begin{aligned} Ty(t) &= \int_{t_1}^{\sigma(t_3)} H_n(t,s) f(s, y(s)) \Delta s \\ &\geq m_n L \int_{t_1}^{\sigma(t_3)} \parallel G_n(\cdot, s) \parallel f(s, y(s)) \Delta s \\ &\geq \frac{m_n L}{K} \parallel Ty \parallel = M \parallel Ty \parallel . \end{aligned}$$

Hence $Ty \in P$ and so $T : P \to P$. Moreover, T is completely continuous. From (3.2) and (3.3), for each $y \in P$, we have $\gamma(y) \leq \theta(y) \leq \alpha(y)$ and $\parallel y \parallel \leq \frac{1}{M}\gamma(y)$. Also, for any $0 \leq \lambda \leq 1$ and $y \in P$, we have $\theta(\lambda y) = \max_{t \in [t_2, \sigma(t_3)]} (\lambda y)(t) = \lambda \max_{t \in [t_2, \sigma(t_3)]} y(t) = \lambda \theta(y)$. It is clear that $\theta(0) = 0$. We now show that the remaining conditions of Theorem 3.1 are satisfied.

Firstly, we shall verify that condition $(B1)$ of Theorem 3.1 is satisfied. Since $y \in \partial P(\gamma, c')$, from (3.3) we have that $c' = \min_{t \in [t_2, \sigma(t_3)]} y(t) \leq \parallel y \parallel \leq \frac{c'}{M}$. Then

$$\begin{aligned} \gamma(Ty) &= \min_{t \in [t_2, \sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} H_n(t,s) f(s, y(s)) \Delta s \\ &\geq \min_{t \in [t_2, \sigma(t_3)]} \int_{t_2}^{\sigma(t_3)} H_n(t,s) f(s, y(s)) \Delta s \\ &> \frac{c'}{\Pi_{j=1}^{n} m_j L_j} m_n L \int_{t_2}^{\sigma(t_3)} \parallel G_n(\cdot, s) \parallel \Delta s = c', \end{aligned}$$

using hypothesis $(D1)$.

Now we shall show that condition $(B2)$ of Theorem 3.1 is satisfied. Since $y \in \partial P(\theta, b')$, from (3.3) we have that $0 \leq y(t) \leq \parallel y \parallel \leq \frac{b'}{M}$, for $[t_1, \sigma(t_3)]$.

Thus

$$\begin{aligned}\theta(Ty) &= \max_{t\in[t_2,\sigma(t_3)]}\int_{t_1}^{\sigma(t_3)} H_n(t,s)f(s,y(s))\Delta s\\ &< \frac{b'}{\Pi_{j=1}^n K_j}K\int_{t_1}^{\sigma(t_3)} \parallel G_n(\cdot,s)\parallel \Delta s = b',\end{aligned}$$

by hypothesis $(D2)$.
Finally, using hypothesis $(D3)$, we shall show that condition $(B3)$ of Theorem 3.1 is satisfied. Since $0\in P$ and $a'>0$, $P(\alpha,a')\neq\emptyset$. Since $y\in\partial P(\alpha,a')$, $a'=\max_{t\in[t_1,\sigma(t_3)]} y(t)\leq\parallel y\parallel\leq\frac{a'}{M}$, for $t\in[t_2,\sigma(t_3)]$. Therefore,

$$\begin{aligned}\alpha(Ty) &= \max_{t\in[t_1,\sigma(t_3)]}\int_{t_1}^{\sigma(t_3)} H_n(t,s)f(s,y(s))\Delta s\\ &\geq \int_{t_1}^{\sigma(t_3)} H_n(t,s)f(s,y(s))\Delta s\\ &> \frac{a'}{\Pi_{j=1}^n m_jL_j}m_nL\int_{t_2}^{\sigma(t_3)} \parallel G_n(\cdot,s)\parallel \Delta s = a'.\end{aligned}$$

Thus, all the conditions of Theorem 3.1 are satisfied and so there exist at least two positive solutions $y_1,y_2\in\overline{P(\gamma,c')}$ for the BVP (1.1)-(1.2). This completes the proof of the theorem. □

Theorem 3.3 *Let m be an arbitrary positive integer. Assume that there exist numbers $a_r(r=1,2,\cdots,m+1)$ and $b_s(s=1,2,\cdots,m)$ with $0<a_1<b_1<a_2<b_2<\cdots<a_m<b_m<a_{m+1}$ such that*

$$f(t,y)>\frac{a_r}{\Pi_{j=1}^n m_jL_j},\ \textit{for } t\in[t_2,\sigma(t_3)] \textit{ and } y\in[a_r,\frac{a_r}{M}],\ r=1,2,\cdots,m+1, \tag{3.5}$$

$$f(t,y)<\frac{b_s}{\Pi_{j=1}^n K_j},\ \textit{for } t\in[t_1,\sigma(t_3)] \textit{ and } y\in[0,\frac{b_s}{M}],\ s=1,2,\cdots,m. \tag{3.6}$$

Then the BVP (1.1)-(1.2) *has at least $2m$ positive solutions in $\overline{P}_{a_{m+1}}$.*

Proof: We use induction on m. For $m=1$, we know from (3.5) and (3.6) that $T:\overline{P}_{a_2}\to P_{a_2}$, then, it follows from Avery-Henderson fixed point theorem that the BVP (1.1)-(1.2) has at least two positive solutions in $\overline{P}_{a_2}$. Next, we assume that this conclusion holds for $m=l$. In order to prove this conclusion holds for $m=l+1$. We suppose that there exist numbers $a_r(r=1,2,\cdots,l+2)$

and $b_s(s = 1, 2, \cdots, l+1)$ with $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{l+1} < b_{l+1} < a_{l+2}$ such that

$$f(t, y) > \frac{a_r}{\Pi_{j=1}^{n} m_j L_j}, \text{ for } t \in [t_2, \sigma(t_3)] \text{ and } y \in [a_r, \frac{a_r}{M}], \ r = 1, 2, \cdots, l+2, \tag{3.7}$$

$$f(t, y) < \frac{b_s}{\Pi_{j=1}^{n} K_j}, \text{ for } t \in [t_1, \sigma(t_3)] \text{ and } y \in [0, \frac{b_s}{M}], \ s = 1, 2, \cdots, l+1. \tag{3.8}$$

By assumption, the BVP (1.1)-(1.2) has at least $2l$ positive solutions $y_i(i = 1, 2, \cdots, 2l)$ in $\overline{P}_{a_{l+1}}$. At the same time, it follows from Theorem 3.2, (3.7) and (3.8) that the BVP (1.1)-(1.2) has at least two positive solutions y_1, y_2 in $\overline{P}_{a_{l+2}}$ such that $a_{l+1} < \alpha(y_1)$ with $\theta(y_1) < b_{l+1}$ and $b_{l+1} < \theta(y_2)$ with $\gamma(y_2) < a_{l+2}$. Obviously y_1 and y_2 are different from $y_i(i = 1, 2, \cdots, 2l)$. Therefore, the BVP (1.1)-(1.2) has at least $2l + 2$ positive solutions in $\overline{P}_{a_{l+2}}$, which shows that this conclusion holds for $m = l + 1$. □

4 Example

Let us consider an example to illustrate the usage of the Theorem 3.2. Let $n = 2$ and $\mathbb{T} = \{0\} \cup \{\frac{1}{2^{n+1}} : n \in \mathbb{N}\} \cup [\frac{1}{2}, \frac{3}{2}]$. Now, consider the following BVP,

$$y^{\Delta^6}(t) = \frac{800(y+1)^4}{73(y^2 + 999)}, \ t \in [0, \sigma(1)] \cap \mathbb{T} \tag{4.1}$$

subject to the boundary conditions,

$$\left.\begin{aligned} \frac{1}{2}y(0) - y^{\Delta}(0) + 2y^{\Delta^2}(0) = 0, \\ 2y\Big(\frac{1}{2}\Big) - 3y^{\Delta}\Big(\frac{1}{2}\Big) + 2y^{\Delta^2}\Big(\frac{1}{2}\Big) = 0, \\ y(\sigma(1)) + \frac{1}{2}y^{\Delta}(\sigma(1)) + \frac{1}{3}y^{\Delta^2}(\sigma(1)) = 0, \\ \frac{3}{4}y^{\Delta^3}(0) - 2y^{\Delta^4}(0) + 3y^{\Delta^5}(0) = 0, \\ y^{\Delta^3}\Big(\frac{1}{2}\Big) - 2y^{\Delta^4}\Big(\frac{1}{2}\Big) + y^{\Delta^5}\Big(\frac{1}{2}\Big) = 0, \\ y^{\Delta^3}(\sigma(1)) + \frac{1}{2}y^{\Delta^4}(\sigma(1)) + \frac{1}{2}y^{\Delta^5}(\sigma(1)) = 0. \end{aligned}\right\} \tag{4.2}$$

Then the conditions $(A1)$-$(A4)$ are satisfied. The Green's function $G_1(t,s)$ in Lemma 2.1 is

$$G_1(t,s) = \begin{cases} \underset{t\in[0,\frac{1}{2}]}{G_1(t,s)} = \begin{cases} G_{1_1}(t,s), & 0 < \sigma(s) < t \le \frac{1}{2} < \sigma(1) \\ G_{1_2}(t,s), & 0 \le t < s < \frac{1}{2} < \sigma(1) \\ G_{1_3}(t,s), & 0 \le t < \frac{1}{2} < s < \sigma(1) \end{cases} \\ \\ \underset{t\in[\frac{1}{2},\sigma(1)]}{G_1(t,s)} = \begin{cases} G_{1_4}(t,s), & 0 < \frac{1}{2} < \sigma(s) < t \le \sigma(1) \\ G_{1_5}(t,s), & 0 < \frac{1}{2} \le t < s < \sigma(1) \\ G_{1_6}(t,s), & 0 \le \sigma(s) < \frac{1}{2} < t < \sigma(1) \end{cases} \end{cases}$$

where

$$\begin{aligned}
G_{1_1}(t,s) =& \frac{12}{481}\Big[\frac{91}{12} + \frac{23}{6}t - 5t^2\Big]\Big[\frac{1}{2}\sigma(s)\sigma^2(s) + (\sigma(s)+\sigma^2(s)) + 4\Big], \\
G_{1_2}(t,s) =& \frac{12}{481}\Big\{\Big[\frac{26}{3} - \frac{8}{3}t - \frac{7}{4}t^2\Big]\Big[2\sigma(s)\sigma^2(s) + 2(\sigma(s)+\sigma^2(s)) + \frac{3}{2}\Big] \\
&+ \Big[\frac{13}{2} + \frac{29}{4}t + t^2\Big]\Big[\sigma(s)\sigma^2(s) - \frac{3}{2}(\sigma(s)+\sigma^2(s)) + \frac{8}{3}\Big]\Big\}, \\
G_{1_3}(t,s) =& \frac{12}{481}\Big[\frac{13}{2} + \frac{29}{4}t + t^2\Big]\Big[\sigma(s)\sigma^2(s) - \frac{3}{2}(\sigma(s)+\sigma^2(s)) + \frac{8}{3}\Big], \\
G_{1_4}(t,s) =& \frac{12}{481}\Big\{\Big[\frac{91}{12} + \frac{23}{6}t - 5t^2\Big]\Big[\frac{1}{2}\sigma(s)\sigma^2(s) + (\sigma(s)+\sigma^2(s)) + 4\Big] \\
&+ \Big[-\frac{26}{3} + \frac{8}{3}t + \frac{7}{4}t^2\Big]\Big[2\sigma(s)\sigma^2(s) + 2(\sigma(s)+\sigma^2(s)) + \frac{3}{2}\Big]\Big\}, \\
G_{1_5}(t,s) =& \frac{12}{481}\Big[\frac{13}{2} + \frac{29}{4}t + t^2\Big]\Big[\sigma(s)\sigma^2(s) - \frac{3}{2}(\sigma(s)+\sigma^2(s)) + \frac{8}{3}\Big], \\
G_{1_6}(t,s) =& \frac{12}{481}\Big[\frac{91}{12} + \frac{23}{6}t - 5t^2\Big]\Big[\frac{1}{2}\sigma(s)\sigma^2(s) + (\sigma(s)+\sigma^2(s)) + 4\Big].
\end{aligned}$$

The Green's function $G_2(t,s)$ in Lemma 2.1 is

$$G_2(t,s) = \begin{cases} \underset{t\in[0,\frac{1}{2}]}{G_2(t,s)} = \begin{cases} G_{2_1}(t,s), & 0 < \sigma(s) < t \le \frac{1}{2} < \sigma(1) \\ G_{2_2}(t,s), & 0 \le t < s < \frac{1}{2} < \sigma(1) \\ G_{2_3}(t,s), & 0 \le t < \frac{1}{2} < s < \sigma(1) \end{cases} \\ \\ \underset{t\in[\frac{1}{2},\sigma(1)]}{G_2(t,s)} = \begin{cases} G_{2_4}(t,s), & 0 < \frac{1}{2} < \sigma(s) < t \le \sigma(1) \\ G_{2_5}(t,s), & 0 < \frac{1}{2} \le t < s < \sigma(1) \\ G_{2_6}(t,s), & 0 \le \sigma(s) < \frac{1}{2} < t < \sigma(1) \end{cases} \end{cases}$$

where

$$G_{2_1}(t,s) = \frac{16}{635}\Big[\frac{39}{8} + \frac{11}{4}t - 3t^2\Big]\Big[\frac{3}{4}\sigma(s)\sigma^2(s) + 2(\sigma(s)+\sigma^2(s)) + 6\Big],$$

$$G_{2_2}(t,s) = \frac{16}{635}\Big\{\Big[15 - \frac{15}{4}t - \frac{25}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) + \frac{3}{2}(\sigma(s)+\sigma^2(s)) + \frac{1}{4}\Big] + \Big[\frac{17}{2} + \frac{93}{16}t + \frac{7}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) - \frac{3}{2}(\sigma(s)+\sigma^2(s)) + 3\Big]\Big\},$$

$$G_{2_3}(t,s) = \frac{16}{635}\Big[\frac{17}{2} + \frac{93}{16}t + \frac{7}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) - \frac{3}{2}(\sigma(s)+\sigma^2(s)) + 3\Big],$$

$$G_{2_4}(t,s) = \frac{16}{635}\Big\{\Big[\frac{39}{8} + \frac{11}{4}t - 3t^2\Big]\Big[\frac{3}{4}\sigma(s)\sigma^2(s) + 2(\sigma(s)+\sigma^2(s)) + 6\Big] + \Big[-15 + \frac{15}{4}t + \frac{25}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) + \frac{3}{2}(\sigma(s)+\sigma^2(s)) + \frac{1}{4}\Big]\Big\},$$

$$G_{2_5}(t,s) = \frac{16}{635}\Big[\frac{17}{2} + \frac{93}{16}t + \frac{7}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) - \frac{3}{2}(\sigma(s)+\sigma^2(s)) + 3\Big],$$

$$G_{2_6}(t,s) = \frac{16}{635}\Big[\frac{39}{8} + \frac{11}{4}t - 3t^2\Big]\Big[\frac{3}{4}\sigma(s)\sigma^2(s) + 2(\sigma(s)+\sigma^2(s)) + 6\Big].$$

From Theorem 2.3 and Lemma 2.5, we get

$$m_1 = 0.4406779661, \quad K_1 = 0.6552328771, \quad L_1 = 0.183991684,$$

$$m_2 = 0.5596707819, \quad K_2 = 0.7449516076, \quad L_2 = 0.2551181102.$$

Therefore, $K = 0.6552328771$, $L = 0.08108108108$ and $M = 0.06925585335$. Clearly f is continuous and increasing on $[0,\infty)$. If we choose $a' = 0.0001$, $b' = 0.04$ and $c' = 100$ then $0 < a' < b' < c'$ and f satisfies

(i) $f(t,y) > 8637.8676 = \frac{c'}{\Pi_{j=1}^2 m_j L_j}$, for $t \in [\frac{1}{2}, \sigma(1)]$ and $y \in [100, 1443.9212]$,

(ii) $f(t,y) < 0.081947 = \frac{b'}{\Pi_{j=1}^2 K_j}$, for $t \in [0, \sigma(1)]$ and $y \in [0, 0.577568]$,

(iii) $f(t,y) > 0.008637 = \frac{a'}{\Pi_{j=1}^2 m_j L_j}$, for $t \in [\frac{1}{2}, \sigma(1)]$ and $y \in [0.0001, 0.001443]$.

Then all the conditions of Theorem 3.2 are satisfied. Thus by Theorem 3.2, the BVP (4.1)-(4.2) has at least two positive solutions y_1 and y_2 satisfying

$$0.0001 < \max_{t\in[0,\sigma(1)]} y_1(t) \text{ with } \max_{t\in[\frac{1}{2},\sigma(1)]} y_1(t) < 0.04,$$

$$0.04 < \max_{t\in[\frac{1}{2},\sigma(1)]} y_2(t) \text{ with } \min_{t\in[\frac{1}{2},\sigma(1)]} y_2(t) < 100.$$

Acknowledgements: The authors thank the referees for their valuable suggestions and comments.

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(Received October 9, 2011)