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Necessary and Sufficient Conditions for Nonoscillatory Solutions of Impulsive Delay Differential Equations*†

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Abstract

Monotonicity of solutions is an important property in the investigation of oscillatory behaviors of differential equations. A number of papers provide some existence criteria for eventually positive increasing solutions. However, relatively little attention is paid to eventually positive solutions that are also eventually decreasing solutions. For this reason, we establish several new and sharp oscillatory criteria for impulsive functional differential equations from this viewpoint.

1 Introduction

Let $\Upsilon = \{t_0, t_1, ...\}$ where $0 = t_0 < t_1 < t_2 < \cdots$ and let $\mathbf{R}$ and $\mathbf{N}$ be the sets of real numbers and positive integers respectively. In this paper, we intend to establish necessary and sufficient conditions of existence of nonoscillatory solutions of impulsive differential equation

$$\begin{aligned} (r(t)x'(t))' + p(t)f(x(g(t))) &= 0,\ t \in [0,\infty)\backslash\Upsilon, & (1)\\ x(t_k^+) &= a_k x(t_k),\ k \in \mathbf{N}, & (2)\\ x'(t_k^+) &= b_k x'(t_k),\ k \in \mathbf{N} & (3)\end{aligned}$$

under some or all of the following conditions

(A1) $\lim_{k\to\infty} t_k = +\infty$;

(A2) p is a function on $[0,\infty)$, and r is a positive and differentiable function on $[0,\infty)$;

(A3) g is a continuous function on $[0,\infty)$ such that $g(t) \le t$ for $t \ge 0$ and $\lim_{t\to\infty} g(t) = \infty$;

(A4) f is a continuous function on $\mathbf{R}$ with $uf(u) > 0$ for $u \neq 0$ and $\inf_{|u|\ge T}\{|f(u)|\} > 0$ for any $T > 0$;

(A5) $a_k > 0$ and $b_k > 0$ for $k \in \mathbf{N}$;

(A6) there exists $m > 0$ such that $A(0,t) \ge m$ for $t \ge 0$; and

(A7) there exists $M > 0$ such that $A(0,t) \le M$ for $t \ge 0$,

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where

$$A(s,t)=\begin{cases}\prod\limits_{s\le t_k<t} a_k & \text{if } [s,t)\cap\Upsilon\neq\emptyset\\ 1 & \text{if } [s,t)\cap\Upsilon=\emptyset\end{cases}\quad\text{for } t\ge s\ge 0.$$

"Monotonicity" of solutions is an important property for investigating oscillatory behaviors of functional differential equations. There are many papers (e.g. [2, 5, 7, 10–12, 13, 15]) which provide sufficient conditions to guarantee the solutions are increasing. For instance, if

$$\int_0^\infty \frac{B(0,t)}{A(0,t)r(t)}dt=\infty \tag{4}$$

where

$$B(s,t)=\begin{cases}\prod\limits_{s\le t_k<t} b_k & \text{if } [s,t)\cap\Upsilon\neq\emptyset\\ 1 & \text{if } [s,t)\cap\Upsilon=\emptyset\end{cases}\quad\text{for } t\ge s\ge 0,$$

by Lemma 2 in [12], then an eventually positive solution x of system (1)-(3) will satisfy $x'(t)\ge 0$ eventually. However, the case where a solution x satisfies the condition $x(t)x'(t)<0$ eventually has rarely been touched upon. To fill this gap, we will establish new and sharp oscillatory criteria from this viewpoint. Our technique is based on comparing our systems with their linearized systems (cf. [4–9]). However, the important point is that we are able to establish necessary and sufficient conditions.

Let Λ_1 and Λ_2 be intervals of $\mathbf{R}$. We set

$$\begin{aligned}PC(\Lambda_1,\Lambda_2) &= \{\varphi:\Lambda_1\to\Lambda_2|\varphi \text{ is continuous in each interval } \Lambda_1\cap(t_k,t_{k+1}], k\in\mathbf{N}\cup\{0\}\\ &\quad \text{with jump discontinuities only}\}\end{aligned}$$

and

$$PC'(\Lambda_1,\Lambda_2)=\{\varphi\in PC(\Lambda_1,\Lambda_2)|\varphi \text{ is continuously differentiable a.e. in } \Lambda_1\}.$$

For any $\varphi_1,\varphi_2\in PC(\Lambda_1,\Lambda_2)$, we say that $\varphi_1\le\varphi_2$ if and only if $\varphi_1(t)\le\varphi_2(t)$ a.e. on Λ_1.

In the subsequent discussions, we let

$$g_\eta=\min_{t\ge\eta} g(t) \text{ for any } \eta\ge 0.$$

Note that if (A3) is assumed, then g_η exists.

Definition 1.1 *Let Λ be an interval in $[0,\infty)$ and $\sigma=\inf\Lambda$. For any $\phi\in PC([g_\sigma,\sigma],\mathbf{R})$, a function x defined on $[g_\sigma,\sigma]\cup\Lambda$ is said to be a solution of system (1)-(3) on Λ satisfying the initial value condition $x(t)=\phi(t)$ for $t\in[g_\sigma,\sigma]$ if*

(i) $x,x'\in PC'(\Lambda,\mathbf{R})$;

(ii) $x(t)$ satisfies (1) a.e. on Λ; and

(iii) $x(t)$ satisfies (2) and (3) on Λ.

Definition 1.2 *Let a function $\varphi = \varphi(t)$ be defined for all sufficiently large t. We say that φ is eventually positive (or negative) if there exists a number T such that $\varphi(t) > 0$ (respectively $\varphi(t) < 0$) for every $t \geq T$. We say that $\varphi(t)$ is nonoscillatory if $\varphi(t)$ is eventually positive or eventually negative. Otherwise, $\varphi(t)$ is called oscillatory.*

A partially ordered set is called a complete lattice if all subsets admit a supremum and an infimum. A complete lattice is recalled here because we will employ the well known Knaster-Tarski fixed point theorem.

Theorem 1.1 (Knaster-Tarski fixed point theorem) *Let X be a set and f a function on X such that $f(X) \subseteq X$. Assume that $(X, \leq)$ is a complete lattice and $f(x_1) \leq f(x_2)$ for $x_1, x_2 \in X$ with $x_1 \leq x_2$. Then f has a fixed point in X.*

Let $a_k^* > 0$ and $b_k^* > 0$ for $k \in \mathbf{N}$. We define the functions

$$A^*(s,t) = \begin{cases} \prod\limits_{s \leq t_k < t} a_k^* & \text{if } [s,t) \cap \Upsilon \neq \emptyset \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset \end{cases} \quad \text{and } B^*(s,t) = \begin{cases} \prod\limits_{s \leq t_k < t} b_k^* & \text{if } [s,t) \cap \Upsilon \neq \emptyset \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset \end{cases}$$

for $t \geq s \geq 0$.

2 Main results

We first provide a criterion to illustrate that the nonoscillatory solution x with $x(t)x'(t) < 0$ eventually indeed exists.

Lemma 2.1 *Assume that (A1)–(A6) hold, $p \in PC([0,\infty), [0,\infty))$ and*

$$\int_\tau^\infty \frac{p(t)}{B(0,t)} dt = \infty \text{ for some } \tau \geq 0. \tag{5}$$

If the system (1)-(3) has a nonoscillatory solution x, then $x(t)x'(t) < 0$ eventually.

Proof. For the sake of convenience, we may assume that $\tau = 0$. We first assume that the solution x is eventually positive, say that $x(t) > 0$ for $t \geq g_0$. Let $\overline{c} = \inf_{t \geq g_0} \{x(t)\}$ and $m_f = \inf_{u \geq \overline{c}} \{f(u)\}$. We note that

$$\frac{r(t_k^+)x'(t_k^+)}{B(0,t_k^+)} = \frac{b_k r(t_k)x'(t_k)}{b_k B(0,t_k)} = \frac{r(t_k)x'(t_k)}{B(0,t_k)}$$

for $k \in \mathbf{N}$. It follows that $r(t)x'(t)/B(0,t)$ is a continuous function on $[0,\infty)$. By equation (1), we may see that $r(t)x'(t)$ is decreasing on each interval $(t_{k-1}, t_k]$ for $k \in \mathbf{N}$. For $k \in \mathbf{N}$, we may further see that $B(0,s) = B(0,t)$ and

$$\frac{r(s)x'(s)}{B(0,s)} \geq \frac{r(t)x'(t)}{B(0,t)}$$

for $t_{k-1} < s \leq t \leq t_k$. By continuity of $r(t)x'(t)/B(0,t)$, then

$$\frac{r(s)x'(s)}{B(0,s)} \geq \frac{r(t_{k_1})x'(t_{k_1})}{B(0,t_{k_1})} \geq \cdots \geq \frac{r(t_{k_n})x'(t_{k_n})}{B(0,t_{k_n})} \geq \frac{r(t)x'(t)}{B(0,t)} \tag{6}$$

for $t \geq t_{k_n} \geq \cdots \geq t_{k_1} \geq s > 0$ where $t_{k_1}, \ldots, t_{k_n} \in \Upsilon$.

There are now two cases. First assume that there exists $T > 0$ such that $x'(T) < 0$. In view of (6),

$$r(t)x'(t) \leq \frac{B(0,t)r(T)x'(T)}{B(0,T)} < 0 \text{ for } t \geq T.$$

Since $r(t) > 0$ for $t \geq 0$, we may further see that $x'(t) < 0$ for $t \geq T$. Next, if $x'(t) \geq 0$ for all $t \geq g_0$, then $x(t)$ is increasing on each interval $(t_{k-1}, t_k]$ for $k \in \mathbf{N}$. Similarly, we can verify that $x(t)/A(0,t)$ is continuous and increasing for $t \geq 0$. It follows from (A6) that

$$x(t) \geq A(0,t)x(0) \geq mx(0) > 0 \text{ for } t \geq 0.$$

So $\overline{c} \geq \min\left\{mx(0), \min_{t\in[g_0,0]} x(t)\right\} > 0$ and $x(g(t)) \geq \overline{c}$ for $t \geq 0$. In view of (A4), we see that $f(x(g(t))) \geq m_f > 0$ for $t \geq 0$. We now divide (1) by $B(0,t)$, and then integrate from 0 to t. By continuity of $r(t)x'(t)/B(0,t)$, we have

$$\int_0^t \left(\frac{r(s)x'(s)}{B(0,s)}\right)' ds = \frac{r(t)x'(t)}{B(0,t)} - \frac{r(0)x'(0)}{B(0,0)}$$

and

$$\begin{aligned} \frac{r(t)x'(t)}{B(0,t)} &= r(0)x'(0) - \int_0^t \frac{p(s)}{B(0,s)} f(x(g(s)))ds \\ &\leq r(0)x'(0) - m_f \int_0^t \frac{p(s)}{B(0,s)} ds \end{aligned} \tag{7}$$

for $t \geq 0$. By (5) and (A2), we may see that $x'(t) < 0$ eventually. This is a contradiction. Therefore, in both cases, $x'(t) < 0$ eventually.

Second, we assume that the solution x is eventually negative. Let $y(t) = -x(t)$ for sufficiently large t. Then y is an eventually positive solution of

$$(r(t)y'(t))' + p(t)F(y(g(t))) = 0, \ t \in [0,\infty)\backslash\Upsilon, \tag{8}$$

$$y(t_k^+) = a_k y(t_k), \ k \in \mathbf{N}, \tag{9}$$

$$y'(t_k^+) = b_k y'(t_k), \ k \in \mathbf{N} \tag{10}$$

where $F(u) = -f(-u)$ for $u \in \mathbf{R}$. We may observe that F is a continuous function on $\mathbf{R}$ with $uF(u) > 0$ for $u \neq 0$, and $\inf_{|u|\geq T} \{|F(u)|\} > 0$ for any $T > 0$. In view of the above discussions, $y'(t) < 0$ eventually, which implies $x'(t) > 0$ eventually. The proof is complete. ■

Corollary 2.1 *Assume that (A1)–(A6), (4) and (5) hold and* $p \in PC([0,\infty),[0,\infty))$. *Then the system (1)-(3) is oscillatory.*

Proof. Assume that the system (1)-(3) has a nonoscillatory solution x. By Lemma 2.1, we may see that $x'(t) < 0$ eventually. By (4) and Lemma 2 in [12], we may further see that $x'(t) \geq 0$ eventually. This is a contradiction. The proof is complete. ■

The following two comparison theorems hold under the condition that nonoscillatory solution is "decreasing".

Theorem 2.1 *Let $\sigma \geq 0$, $\phi \in PC([g_\sigma, \sigma], (0, \infty))$ and $\varepsilon_n \in (0, 1)$ for $n \in \mathbf{N}$. Assume that (A1)–(A5) hold, $p \in PC([0, \infty), [0, \infty))$, $g(t) < t$ for $t \geq 0$, $\phi'(\sigma)$ exists, $\lim_{n\to\infty} \varepsilon_n = 1$ and*

$$\{t \geq \sigma : p(t) = 0\} \text{ has measure zero.} \tag{11}$$

If for any $n \in \mathbf{N}$, the system

$$\begin{aligned} (r(t)x'(t))' + \varepsilon_n p(t)x(g(t)) &= 0,\ t \in [0, \infty)\backslash\Upsilon, &(12)\\ x(t_k^+) &= a_k x(t_k),\ k \in \mathbf{N}, &(13)\\ x'(t_k^+) &= b_k x'(t_k),\ k \in \mathbf{N} &(14)\end{aligned}$$

has a positive solution x_{ε_n} satisfying the initial condition $x_{\varepsilon_n}(t) = \phi(t)$ on $[g_\sigma, \sigma]$ and $x'_{\varepsilon_n}(t) < 0$ on $[\sigma, \infty)$, then the system

$$\begin{aligned} (r(t)x'(t))' + p(t)x(g(t)) &= 0,\ t \in [0, \infty)\backslash\Upsilon, &(15)\\ x(t_k^+) &= a_k x(t_k),\ k \in \mathbf{N}, &(16)\\ x'(t_k^+) &= b_k x'(t_k),\ k \in \mathbf{N} &(17)\end{aligned}$$

has a positive solution $\widetilde{x}$ on $[\sigma, \infty)$ satisfying the initial condition $\widetilde{x}(t) = \phi(t)$ on $[g_\sigma, \sigma]$.

Proof. For the sake of convenience, we assume that $\sigma = 0$. For $n \in \mathbf{N}$, we let

$$y_n(t) = \begin{cases} \frac{x_{\varepsilon_n}(t)}{A(0,t)} & \text{if } t > 0 \\ \phi(t) & \text{if } 0 \geq t \geq g_0 \end{cases} \quad \text{for } t \geq 0.$$

By assumption, we see that $y_n(t)$ are positive, strictly decreasing and continuous for $t > 0$ and $n \in \mathbf{N}$. So for $n \in \mathbf{N}$,

$$y_n(t) < y_n(0) = \frac{x_{\varepsilon_n}(0)}{A(0, 0)} = x_{\varepsilon_n}(0) = \phi(0) \text{ for } t > 0.$$

It follows that $\{y_n(t) : n \in \mathbf{N}\}$ is uniformly bounded. For any $n \in \mathbf{N}$, we divide (12) by $B(0, t)$, and then integrate from 0 to t. We have

$$\frac{r(t)x'_{\varepsilon_n}(t)}{B(0, t)} = r(0)x'_{\varepsilon_n}(0) - \varepsilon_n \int_0^t \frac{p(s)x_{\varepsilon_n}(g(s))}{B(0, s)} ds$$

for $t \geq 0$. We further divide the above equation by $A(0, t)$, and then integrate from 0 to t. Then

$$\frac{x_{\varepsilon_n}(t)}{A(0, t)} = \phi(0) + \int_0^t \frac{r(0)\phi'(0)B(0, s) - \varepsilon_n \int_0^s B(v, s)p(v)x_{\varepsilon_n}(g(v))dv}{r(s)A(0, s)} ds$$

for $t \geq 0$. It follows that

$$y_n(t) = \phi(0) + \int_0^t \left(\frac{r(0)\phi'(0)B(0, s)}{r(s)A(0, s)} - \varepsilon_n \int_0^s \frac{B(v, s)p(v)y_n(g(v))}{A(H(v), s)r(s)} dv \right) ds \tag{18}$$

for $t \geq 0$ where $H(t) = \max\{0, g(t)\}$. Given $d > 0$, there exists $M_1 > 0$ such that

$$\left|\frac{r(0)\phi'(0)B(0,t)}{r(t)A(0,t)}\right| + \int_0^t \left|\frac{B(s,t)p(s)y_n(g(s))}{A(H(s),t)r(t)}\right| ds \leq M_1 \text{ for } 0 \leq t \leq d. \tag{19}$$

For $0 \leq \eta_1 \leq \eta_2 \leq d$, by (19),

$$\begin{aligned}
|y_n(\eta_2) - y_n(\eta_1)| &\leq \int_{\eta_1}^{\eta_2} \left|\frac{r(0)\phi'(0)B(0,s)}{r(s)A(0,s)} - \varepsilon_n \int_0^s \frac{B(v,s)p(v)y_n(g(v))}{A(H(v),s)r(s)} dv\right| ds \\
&\leq \int_{\eta_1}^{\eta_2} \left(\left|\frac{r(0)\phi'(0)B(0,s)}{r(s)A(0,s)}\right| + \int_0^s \left|\frac{B(v,s)p(v)y_n(g(v))}{A(H(v),s)r(s)}\right| dv\right) ds \\
&\leq M_1(\eta_2 - \eta_1).
\end{aligned}$$

So $\{y_n(t) : n \in \mathbf{N}\}$ is equi-continuous on $[0, d]$. By the Arzela-Ascoli Theorem, there exists a nonnegative, decreasing and continuous function $\widetilde{y}_d$ defined on $[0, d]$ such that $\{y_n\}$ converges uniformly to $\widetilde{y}_d$. Then

$$\widetilde{y}_d(t) = \lim_{n\to\infty} y_n(t) = \phi(0) + \int_0^t \left(\frac{r(0)\phi'(0)B(0,s)}{r(s)A(0,s)} - \int_0^s \frac{B(v,s)p(v)\widetilde{y}_d(g(v))}{A(H(v),s)r(s)} dv\right) ds$$

for $0 \leq t \leq d$. Since d is arbitrary, there exists a nonnegative, decreasing and continuous function $\widetilde{y}(t)$ defined on $[0, \infty)$ such that

$$\widetilde{y}(t) = \phi(0) + \int_0^t \left(\frac{r(0)\widetilde{y}'(0)B(0,s)}{r(s)A(0,s)} - \int_0^s \frac{B(v,s)p(v)\widetilde{y}(g(v))}{A(H(v),s)r(s)} dv\right) ds$$

for $t \geq 0$. Let

$$\widetilde{x}(t) = \begin{cases} \phi(t) & \text{if } 0 \geq t \geq g_0 \\ A(0,t)\widetilde{y}(t) & \text{if } t > 0 \end{cases} \tag{20}$$

for $t \geq 0$. Clearly, $\widetilde{x}(t) \geq 0$ for $t > 0$. Assume that there exists $T > 0$ such that $\widetilde{x}(T) = 0$ and $\widetilde{x}(t) > 0$ for $0 < t < T$. Since $\widetilde{y}$ is decreasing and $A(0, t)$ is positive, by (20), we may see that $\widetilde{x}(t) = 0$ for $t \geq T$, which implies $\widetilde{x}'(t) = 0$ for $t > T$. We note that

$$\begin{aligned}
\widetilde{x}'(t) &= A(0,t)\widetilde{y}'(t) \qquad (21)\\
&= \frac{r(0)\widetilde{y}'(0)B(0,t)}{r(t)} - \frac{B(0,t)}{r(t)} \int_0^t \frac{p(s)A(0,g(s))\widetilde{y}(g(s))}{B(0,s)} ds
\end{aligned}$$

and

$$(r(t)\widetilde{x}'(t))' = -p(t)A(0,g(t))\widetilde{y}(g(t)) = -p(t)\widetilde{x}(g(t)) \tag{22}$$

for $t \geq 0$. Since g is continuous and $g(T) < T$, there exists $T' > T$ such that $T > g(t)$ for $T < t < T'$. So $\widetilde{x}(g(t)) > 0$ for $T < t < T'$. By (22),

$$0 = (r(t)\widetilde{x}'(t))' = -p(t)\widetilde{x}(g(t))$$

for $T < t < T'$, which implies $p(t) = 0$ for $T < t < T'$. It is a contradiction in view of (11). So $\widetilde{x}(t) > 0$ for $t > 0$. In addition, by (20) and (21), we may see that $\widetilde{x}(t_k^+) = a_k\widetilde{x}(t_k)$ and $\widetilde{x}'(t_k^+) = b_k\widetilde{x}'(t_k)$ for $k \in \mathbf{N}$. So $\widetilde{x}$ is a positive solution of system (15)-(17) on $[0, \infty)$ satisfying the initial condition $\widetilde{x}(t) = \phi(t)$ on $[g_0, 0]$. The proof is complete. ■

Corollary 2.2 *Assume that the hypotheses of Theorem 2.1 hold. If the system (12)-(14) has a negative solution x_{ε_n} satisfying the initial condition $x_{\varepsilon_n}(t) = \phi(t)$ on $[g_\sigma, \sigma]$ and $x'_{\varepsilon_n}(t) > 0$ on $[\sigma, \infty)$, then the system (15)-(17) has a negative solution $\widetilde{x}$ on $[\sigma, \infty)$ satisfying the initial condition $\widetilde{x}(t) = \phi(t)$ on $[g_\sigma, \sigma]$.*

Theorem 2.2 *Let $\sigma \geq 0$, $\phi \in PC'([g_\sigma, \sigma], (0, \infty))$, and F be a continuous function on $[0, \infty)$ with $f(u) \leq F(u)$ for $u \geq 0$. Assume that (A1)–(A6) hold, $p \in PC([0, \infty), [0, \infty))$, $a_k \geq a_k^*$, $b_k \leq b_k^*$ for $k \in \mathbf{N}$, $r'(t) \leq 0$ for $t \geq 0$ and*

$$\frac{f(v)}{v} \leq \frac{f(u)}{u} \text{ for } 0 < u \leq v. \tag{23}$$

If the system

$$\begin{aligned} (r(t)y'(t))' + p(t)F(y(g(t))) &\leq 0, \ t \in [0, \infty)\backslash\Upsilon, \\ y(t_k^+) &= a_k^* y(t_k), \ k \in \mathbf{N}, \\ y'(t_k^+) &= b_k^* y'(t_k), \ k \in \mathbf{N} \end{aligned} \tag{24}$$

has a positive solution y on $[\sigma, \infty)$ satisfying the initial condition $y(t) = \phi(t)$ on $[g_\sigma, \sigma]$ such that $y'(t) < 0$ for $t \geq \sigma$, then the system (1)-(3) has a positive solution x on $[\sigma, \infty)$ satisfying the initial condition $x(t) = \phi(t)$ on $[g_\sigma, \sigma]$.

Proof. For the sake of convenience, we assume that $\sigma = 0$. Let $\beta(t) = y'(t)/y(t)$ for $t > 0$. Clearly, $\beta \in PC'([0, \infty), (-\infty, 0))$. We note that $\beta(t) = y'(t)/y(t)$ is continuous on the intervals $(t_{k-1}, t_k]$ where $k \in \mathbf{N}$. **Given $t > 0$. There exists $k^* \in N$ such that $t_{k^*-1} < t \leq t_{k^*}$. We may observe that**

$$\begin{aligned} \int_0^{t_1} \beta(s)ds &= \int_0^{t_1} \frac{y'(s)}{y(s)} ds = \ln\left(\frac{y(t_1)}{y(0)}\right), \\ \int_{t_1^+}^{t_2} \beta(s)ds &= \ln\left(\frac{y(t_2)}{y(t_1^+)}\right) = \ln\left(\frac{y(t_2)}{a_1^* y(t_1)}\right), \\ &\vdots \\ \int_{t_{k^*-1}^+}^{t} \beta(s)ds &= \ln\left(\frac{y(t)}{y(t_{k^*-1}^+)}\right) = \ln\left(\frac{y(t)}{a_{k^*-1}^* y(t_{k^*-1})}\right). \end{aligned}$$

Then

$$\begin{aligned} \int_0^t \beta(s)ds &= \ln\left(\frac{y(t_1)}{y(0)}\right) + \ln\left(\frac{y(t_2)}{a_1^* y(t_1)}\right) + \cdots + \ln(\frac{y(t)}{a_{k^*-1}^* y(t_{k^*-1})}) \\ &= \ln\left(\frac{y(t_1)}{y(0)} \frac{y(t_2)}{a_1^* y(t_1)} \cdots \frac{y(t)}{a_{k^*-1}^* y(t_{k^*-1})}\right) \\ &= \ln\left(\frac{y(t)}{A^*(0, t) y(0)}\right). \end{aligned}$$

Since $y(0) = \phi(0)$, we may then see that

$$y(t) = A^*(0, t)\phi(0) \exp\left(\int_0^t \beta(s)ds\right) \text{ for } t \geq 0. \tag{25}$$

Since $A^*(0,t)$ is a step function on $[0,\infty)$, we can further see that by (25)

$$\frac{y''(t)}{y(t)} = \beta'(t) + \beta^2(t) \text{ for a.e. } t \ge 0. \tag{26}$$

Let $h(t) = \min\{0, g(t)\}$ and $H(t) = \max\{0, g(t)\}$ for $t \ge 0$. We assert that

$$y(g(t)) = A^*(0, H(t))\phi(h(t)) \exp\left(\int_0^{H(t)} \beta(s)ds\right) \tag{27}$$

for $t \ge 0$. Indeed, let $\widetilde{t} \ge 0$. If $g(\widetilde{t}) > 0$, then $h(\widetilde{t}) = 0$ and $H(\widetilde{t}) = g(\widetilde{t})$. It follows that

$$A^*(0, H(\widetilde{t}))\phi(h(\widetilde{t})) \exp\left(\int_0^{H(\widetilde{t})} \beta(s)ds\right) = A^*(0, g(\widetilde{t}))\phi(0) \exp\left(\int_0^{g(\widetilde{t})} \beta(s)ds\right) = y(g(\widetilde{t})).$$

If $0 \ge g(\widetilde{t}) \ge g_0$, then $h(\widetilde{t}) = g(\widetilde{t})$ and $H(\widetilde{t}) = 0$. It follows that

$$A^*(0, H(\widetilde{t}))\phi(h(\widetilde{t})) \exp\left(\int_0^{H(\widetilde{t})} \beta(s)ds\right) = \phi(g(\widetilde{t})) = y(g(\widetilde{t})).$$

Our assertion is now proven.

We note that $(r(t)y'(t))' = r'(t)y'(t) + r(t)y''(t)$ for a.e. $t \ge 0$. We divide (24) by $y(t)$. By (25), (26) and (27), it is easy to see that

$$\beta'(t) \le -\beta^2(t) - \frac{r'(t)}{r(t)}\beta(t) - \frac{p(t)F\left(A^*(0, H(t))\phi(h(t)) \exp\left(\int_0^{H(t)} \beta(s)ds\right)\right)}{r(t)A^*(0,t)\phi(0) \exp\left(\int_0^t \beta(s)ds\right)} \tag{28}$$

for a.e. $t \ge 0$. We divide (28) by $B^*(0,t)/A^*(0,t)$, and then integrate both sides. Since $A^*(0,t)\beta(t)/B^*(0,t)$ is continuous for $t \ge 0$, we have

$$\begin{aligned} \beta(t) \quad &\le \quad \frac{B^*(0,t)}{A^*(0,t)}\beta(0) - \int_0^t \frac{B^*(s,t)}{A^*(s,t)} \left(\beta^2(s) + \frac{r'(s)\beta(s)}{r(s)}\right) ds \\ &\quad - \int_0^t \frac{B^*(s,t)}{A^*(s,t)} \frac{p(s)F\left(A^*(0, H(s))\phi(h(s)) \exp\left(\int_0^{H(s)} \beta(v)dv\right)\right)}{r(s)A^*(0,s)\phi(0) \exp\left(\int_0^s \beta(v)dv\right)} ds \end{aligned} \tag{29}$$

for $t \ge 0$. Let

$$X = \{\delta \in PC([0,\infty), [0,\infty)) : \beta(t) \le \delta(t) \le 0\}.$$

Clearly, X is a complete lattice. For any $\delta \in X$, we define an operator

$$\begin{aligned} T(\delta)(t) \quad &= \quad \frac{B(0,t)}{A(0,t)}\delta(0) - \int_0^t \frac{B(s,t)}{A(s,t)} \left(\delta^2(s) + \frac{r'(s)\delta(s)}{r(s)}\right) ds \\ &\quad - \int_0^t \frac{B(s,t)}{A(s,t)} \frac{p(s)f\left(A(0, H(s))\phi(h(s)) \exp\left(\int_0^{H(s)} \delta(v)dv\right)\right)}{r(s)A(0,s)\phi(0) \exp\left(\int_0^s \delta(v)dv\right)} ds \end{aligned}$$

for $t \geq 0$. Let $\delta_1, \delta_2 \in X$ with $\delta_1 \leq \delta_2$. Since $a_k^* \leq a_k$ and $b_k^* \geq b_k$ for $k \in \mathbf{N}$, we may see that $A^*(s,t) \leq A(s,t)$ and $B^*(s,t) \geq B(s,t)$ for $t \geq s \geq 0$. So we observe that

$$\begin{aligned} \frac{\phi(h(t))}{A(H(t),t)\phi(0)\exp\left(\int_{H(t)}^t \delta_2(s)ds\right)} &\leq \frac{\phi(h(t))}{A(H(t),t)\phi(0)\exp\left(\int_{H(t)}^t \delta_1(s)ds\right)} \\ &\leq \frac{\phi(h(t))}{A^*(H(t),t)\phi(0)\exp\left(\int_{H(t)}^t \beta(s)ds\right)} \qquad (30) \end{aligned}$$

and

$$\begin{aligned} A^*(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \beta(s)ds\right) &\leq A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \delta_1(s)ds\right) \\ &\leq A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \delta_2(s)ds\right) \qquad (31) \end{aligned}$$

for $t \geq 0$. In view of (23) and (31),

$$\begin{aligned} \frac{f\left(A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \delta_2(s)ds\right)\right)}{A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \delta_2(s)ds\right)} &\leq \frac{f\left(A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \delta_1(s)ds\right)\right)}{A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \delta_1(s)ds\right)} \\ &\leq \frac{f\left(A^*(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \beta(s)ds\right)\right)}{A^*(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \beta(s)ds\right)} \\ &\leq \frac{F\left(A^*(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \beta(s)ds\right)\right)}{A^*(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \beta(s)ds\right)} \qquad (32) \end{aligned}$$

for $t \geq 0$. By (30) and (32),

$$\begin{aligned} \frac{f\left(A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \delta_2(s)ds\right)\right)}{A(0,t)\phi(0)\exp\left(\int_0^t \delta_2(s)ds\right)} &\leq \frac{f\left(A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \delta_1(s)ds\right)\right)}{A(0,t)\phi(0)\exp\left(\int_0^t \delta_1(s)ds\right)} \\ &\leq \frac{F\left(A^*(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \beta(s)ds\right)\right)}{A^*(0,t)\phi(0)\exp\left(\int_0^t \beta(s)ds\right)} \end{aligned}$$

for $t \geq 0$. By $r'(t) \leq 0$ for $t \geq 0$, we may see that

$$\delta_2^2(t) + \frac{r'(t)\delta_2(t)}{r(t)} \leq \delta_1^2(t) + \frac{r'(t)\delta_1(t)}{r(t)} \leq \beta^2(t) + \frac{r'(t)\beta(t)}{r(t)} \text{ for } t \geq 0,$$

from which and from (29) and (32) we see that $\beta(t) \leq T(\delta_1)(t) \leq T(\delta_2)(t) \leq 0$ for $t \geq 0$. So $T(X) \subseteq X$ and T is increasing on X. By the Knaster-Tarski fixed point Theorem, there exists $\alpha \in X$ such that $T(\alpha) = \alpha$. Let

$$x(t) = \begin{cases} A(0,t)\phi(0)\exp\left(\int_0^t \alpha(s)ds\right) & \text{if } t > 0 \\ \phi(t) & \text{if } 0 \geq t \geq g_0 \end{cases}$$

for $t \geq g_0$. Clearly, $x(t) > 0$ for $t > 0$. **We assert that $x'(t) = \alpha(t)x(t)$ and $x''(t) = \left(\alpha'(t) + \alpha^2(t)\right)x(t)$ for $t \in (t_{k-1}, t_k]$ where $k \in \mathbf{N}$. Indeed, we note that $A'(0,t) = B'(0,t) = 0$ on the intervals $(t_{k-1}, t_k]$ where $k \in \mathbf{N}$. Then $x'(t) = \alpha(t)x(t)$ for $t \in (t_{k-1}, t_k]$ where $k \in \mathbf{N}$. Because $T(\alpha) = \alpha$, we see that $\alpha'(t)$ exists for $t \in (t_{k-1}, t_k]$ where $k \in \mathbf{N}$. Then $x''(t) = \left(\alpha'(t)x(t) + \alpha^2(t)\right)x(t)$ for $t \in (t_{k-1}, t_k]$ where $k \in \mathbf{N}$. We have thus verified our assertion.** Similarly, we can see that

$$A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)} \alpha(s)ds\right) = x(g(t)) \tag{33}$$

for $t > 0$. We note that

$$\alpha'(t) = T(\alpha)'(t) = -\alpha^2(t) - \frac{r'(t)\alpha(t)}{r(t)} - \frac{p(t)f\left(A(0,H(t))\phi(h(t))\exp\left(\int_0^{H(t)}\alpha(s)ds\right)\right)}{r(t)A(0,t)\phi(0)\exp\left(\int_0^t \alpha(s)ds\right)} \tag{34}$$

for $t \in [0,\infty)\backslash\Upsilon$. By (33) and (34), we see that

$$\frac{x''(t)}{x(t)} = \alpha'(t) + \alpha^2(t) = -\frac{r'(t)}{r(t)}\frac{x'(t)}{x(t)} - \frac{p(t)f\left(x(g(t))\right)}{x(t)}$$

for $t \in [0,\infty)\backslash\Upsilon$. So

$$\left(r(t)x'(t)\right)' + p(t)f\left(x(g(t))\right) = 0$$

for $t \in [0,\infty)\backslash\Upsilon$. We further note that $x(t_k^+) = a_k x(t_k)$ and

$$x'(t_k^+) = \alpha(t_k^+)x(t_k^+) = \frac{b_k}{a_k}\alpha(t_k)a_k x(t_k) = b_k x'(t_k)$$

for $k \in \mathbf{N}$. So $x(t)$ is a positive solution of the system (1)-(3) on $[0,\infty)$. The proof is complete. ■

We can give two examples to illustrate the condition (23). In the first example, the function f is concave on $[0,\infty)$. Indeed, we note that

$$f(u_1) \geq \left(1 - \frac{u_1}{u_2}\right)f(0) + \frac{u_1}{u_2}f(u_2) = \frac{u_1}{u_2}f(u_2)$$

for $0 < u_1 < u_2$. So f satisfies the condition (23). In particular, $f(u) = u$. In the second example, f is concave on $[0,d]$ and decreasing on $[d,\infty)$. Similarly, we may verify that f satisfies the condition (23).

Corollary 2.3 *Let $\sigma \geq 0$, $\phi \in PC'([g_\sigma,\sigma],(0,\infty))$ and F be a continuous function on $[0,\infty)$ with $f(u) \geq F(u)$ for $u \leq 0$. Assume that (A1)–(A6) hold, $p \in PC([0,\infty),[0,\infty))$, $r'(t) \leq 0$ for $t \geq 0$ and*

$$\frac{f(v)}{v} \geq \frac{f(u)}{u} \text{ for } u \leq v < 0.$$

If the system

$$\begin{aligned} \left(r(t)y'(t)\right)' + p(t)F(y(g(t))) &\geq 0,\ t \in [0,\infty)\backslash\Upsilon, \\ y(t_k^+) &= a_k^* y(t_k),\ k \in \mathbf{N}, \\ y'(t_k^+) &= b_k^* y'(t_k),\ k \in \mathbf{N} \end{aligned}$$

has a negative solution y on $[\sigma,\infty)$ satisfying the initial condition $y(t)=\phi(t)$ on $[g_\sigma,\sigma]$ such that $y'(t)>0$ for $t\ge\sigma$, then the system (1)-(3) has a negative solution x on $[\sigma,\infty)$ satisfying the initial condition $x(t)=\phi(t)$ on $[g_\sigma,\sigma]$.

Theorem 2.3 *Let $d_1>0$ and $d_2>0$. Assume that (A1)–(A7) hold, $p\in PC([0,\infty),[0,\infty))$, $d_2>Md_1/m$ and f is increasing on $(-d_2,-d_1)\cup(d_1,d_2)$. Then*

$$\int_\tau^\infty \frac{1}{r(t)}\int_\tau^t B(s,t)p(s)dsdt<\infty \tag{35}$$

for some $\tau\ge 0$ if, and only if, the system (1)-(3) has a nonoscillatory solution x such that $|x(t)|\ge d_1$ and $x(t)x'(t)\le 0$ eventually. Furthermore, if $\left\{t\ge \overline{d}: p(t)=0\right\}$ has measure zero for any $\overline{d}\ge 0$, then $x'(t)<0$ eventually.

Proof. In view of (A6) and (A7), we may see that

$$\frac{m}{M}\le A(s,t)\le \frac{M}{m} \text{ for } t\ge s\ge 0.$$

Let $\delta=\max\{f(u):d_1\le u\le d_2\}$. Clearly, $\delta>0$. Assume that (35) holds. There exists $T\in\Upsilon$ such that $T>\tau$ and

$$\int_T^\infty \frac{1}{r(t)}\int_T^t B(s,t)p(s)dsdt\le \frac{m}{M\delta}\left(d_2-\frac{Md_1}{m}\right). \tag{36}$$

Let

$$X=\{y\in PC([T,\infty),[0,\infty)): d_1\le y(t)\le d_2 \text{ for } t\ge T\}.$$

Clearly, X is a complete lattice and X is nonempty because of the fact that $d_1\in X$. We define an operator S in X by

$$S(y)(t)=\frac{A(0,t)d_1}{m}+\int_t^\infty \frac{1}{A(t,s)r(s)}\int_T^s B(v,s)p(v)f(w(y)(g(v)))dvds$$

for $t\ge T$ and $y\in X$, where

$$w(y)(t)=\begin{cases} y(t) & \text{if } t>T\\ d_1 & \text{if } T\ge t\ge g_T\end{cases}.$$

Given $y_1,y_2\in X$ with $y_1\le y_2$. Then

$$d_1\le w(y_1)(g(t))\le w(y_2)(g(t))\le d_2$$

for $t\ge T$. By the monotonicity of f, we may see that

$$f\left(w(y_1)(g(t))\right)\le f\left(w(y_2)(g(t))\right)\le\delta \text{ for } t\ge T.$$

It follows that $S(y_1)(t)\le S(y_2)(t)$ for $t\ge T$. In view of (36),

$$d_1\le S(y)(t)\le \frac{Md_1}{m}+\frac{M}{m}\delta\int_t^\infty \frac{1}{r(s)}\int_T^s B(v,s)p(v)dvds\le d_2$$

for $y \in X$. So $S(X) \subseteq X$ and S is increasing in X. By the Knaster-Tarski fixed point Theorem, there exists $x \in X$ such that $S(x) = x$. Clearly, $x(t) \geq d_1 > 0$ for $t \geq T$. Let $T_1 > T$ such that $g_{T_1} > T$. We note that $x(g(t)) = w(x)(g(t))$,

$$x'(t) = -\frac{1}{r(t)} \int_T^t B(s,t)p(s)f(w(x)(g(s)))ds \leq 0 \tag{37}$$

and

$$(r(t)x'(t))' = p(t)f(x(g(t)))$$

for $t \geq T_1$. Furthermore, $x(t_k^+) = a_k x(t_k)$ and $x'(t_k^+) = b_k x'(t_k)$ for $t_k \geq T_1$. So x is an eventually positive solution of system (1)-(3) with $x'(t) \leq 0$ eventually. If $\{t \geq \overline{d} : p(t) = 0\}$ has measure zero for any $\overline{d} \geq 0$, then by (37), $x'(t) < 0$ eventually.

To see the converse, we first assume that system (1)-(3) has an eventually positive solution x with $x(t) \geq d_1$ and $x'(t) \leq 0$ eventually. Without loss of generality, we assume that $x(t) \geq d_1$ and $x'(t) \leq 0$ for $t \geq g_0$. We divide (1) by $B(0,t)$, and then integrate from 0 to t. We have

$$\frac{r(t)x'(t)}{B(0,t)} = r(0)x'(0) - \int_0^t \frac{p(s)f(x(g(s)))}{B(0,s)}ds \text{ for } t \geq 0.$$

Then

$$x'(t) \leq -\frac{1}{r(t)} \int_0^t B(s,t)p(s)f(x(g(s)))ds \text{ for } t \geq 0. \tag{38}$$

We further divide (38) by $A(0,t)$, and then integrate from 0 to t. We have

$$\frac{x(t)}{A(0,t)} \leq x(0) - \int_0^t \frac{1}{A(0,s)r(s)} \int_0^s B(v,s)p(v)f(x(g(v)))dvds \text{ for } t \geq 0. \tag{39}$$

In view of (38), $x(t)$ is decreasing on each interval $(t_{k-1}, t_k]$ for $k \in \mathbf{N}$. By continuity of $x(t)/A(0,t)$, we note that $x(t) \leq x(0)A(0,t) \leq Mx(0)$ for $t \geq 0$. By (A4) and continuity of f, there exists $\widetilde{\delta} > 0$ such that $f(u) \geq \widetilde{\delta}$ for $Mx(0) \geq u \geq d_1$. By (39), it follows that

$$\frac{x(t)}{A(0,t)} \leq x(0) - \frac{\widetilde{\delta}}{M} \int_0^t \frac{1}{r(s)} \int_0^s B(v,s)p(v)dvds$$

for $t \geq 0$. Since $x(t) > 0$ for $t \geq 0$, we may further see that (35) holds. Second, we assume that system (1)-(3) has an eventually negative solution $x(t)$ with $x(t) \leq -d_1$ and $x'(t) \geq 0$ eventually. Then the system (8)-(10) has an eventually positive solution $y(t)$ with $y(t) \geq d_1$ and $y'(t) \leq 0$ eventually. By the above discussion, we may verify that (35) holds. The proof is complete. ∎

Lemma 2.2 *Assume that (A1)–(A7) hold, and that* $p \in PC([0,\infty),[0,\infty))$ *and*

$$\int_\tau^\infty \frac{1}{r(t)} \int_\tau^t B(s,t)p(s)dsdt = \infty \tag{40}$$

for some $\tau \geq 0$. *If the system (1)-(3) has a nonoscillatory solution* $x(t)$ *with* $x(t)x'(t) < 0$ *eventually, then* $x(t)$ *converges to* 0 *as* $t \to \infty$.

Proof. Without loss of generality, we may assume that $x(t) > 0$ and $x'(t) < 0$ for $t \geq g_0$. Since

$$\frac{x(t_k^+)}{A(0,t_k^+)} = \frac{a_k x(t_k)}{a_k A(0,t_k)} = \frac{x(t_k)}{A(0,t_k)} \text{ and } \left(\frac{x(t)}{A(0,t)}\right)' = \frac{x'(t)}{A(0,t)} < 0$$

for $t \geq 0$ and $k \in \mathbf{N}$, we may see that $x(t)/A(0,t)$ is positive, strictly decreasing and continuous for $t \geq 0$. There exist $\widetilde{M} > 0$ and $\widetilde{m} \geq 0$ such that $\widetilde{m} = \lim_{t\to\infty} x(t)/A(0,t)$ and $x(t)/A(0,t) \leq \widetilde{M}$ for $t \geq 0$. Let $\widetilde{m}_f = \inf_{u \geq m\widetilde{m}} \{f(u)\}$. Assume that $\widetilde{m} > 0$. In view of (A4), we may see that $\widetilde{m}_f > 0$. Let $T'' > \tau$ such that $g_{T''} > 0$. We can observe that $x(t) \geq A(0,t)\widetilde{m} \geq m\widetilde{m}$ for $t \geq g_{T''}$, which implies that $f(x(g(t))) \geq \widetilde{m}_f$ for $t \geq T''$. We now divide (1) by $B(0,t)$, and then integrate from T'' to t. We have

$$\frac{r(t)x'(t)}{B(0,t)} \leq \frac{r(T'')x'(T'')}{B(0,T'')} - \widetilde{m}_f \int_{T''}^{t} \frac{p(s)}{B(0,s)} ds \tag{41}$$

for $t \geq T''$. We divide (41) by $x(t)$. Then

$$\frac{r(t)x'(t)}{B(0,t)x(t)} \leq \frac{-\widetilde{m}_f}{x(t)} \int_{T''}^{t} \frac{p(s)}{B(0,s)} ds \leq \frac{-\widetilde{m}_f}{\widetilde{M}A(0,t)} \int_{T''}^{t} \frac{p(s)}{B(0,s)} ds$$

for $t \geq T''$, form which it follows that

$$\frac{x'(t)}{x(t)} \leq \frac{-\widetilde{m}_f}{\widetilde{M}A(0,t)r(t)} \int_{T''}^{t} B(s,t)p(s)ds \text{ for } t \geq T''. \tag{42}$$

We integrate (42) from T'' to t. We have

$$\ln \frac{A(0,T'')x(t)}{x(T'')A(0,t)} \leq -\frac{\widetilde{m}_f}{\widetilde{M}M} \int_{T''}^{t} \frac{1}{r(s)} \int_{T''}^{s} B(v,s)p(v)dvds \tag{43}$$

for $t \geq T''$. Since (40) holds, we may see from (43) that

$$\lim_{t\to\infty} \left(\ln \frac{A(0,T'')x(t)}{x(T'')A(0,t)}\right) = -\infty,$$

from which it follows that $\widetilde{m} = \lim_{t\to\infty} x(t)/A(0,t) = 0$. It is a contradiction. Then $\lim_{t\to\infty} x(t) = 0$ because $A(0,t)$ has an upper bound. The proof is complete. ■

Corollary 2.4 *Let $d > 0$. Assume that (A1)–(A7), (5) and (40) hold, $p \in PC([0,\infty),[0,\infty))$ and $r'(t) \leq 0$ for $t \geq 0$. Assume that f is concave on interval $[0,d)$ and $f'(0)$ exists. Then the system*

$$\begin{aligned}
(r(t)x'(t))' + p(t)f'(0)x(g(t)) &= 0,\ t \in [0,\infty)\backslash\Upsilon, &(44)\\
x(t_k^+) &= a_k x(t_k),\ k \in \mathbf{N}, &(45)\\
x'(t_k^+) &= b_k x'(t_k),\ k \in \mathbf{N} &(46)
\end{aligned}$$

has an eventually positive solution if, and only if, the system (1)-(3) has an eventually positive solution.

Proof. Assume that the system (44)-(46) has an eventually positive solution. Let

$$F(u) = \begin{cases} f(u) & \text{if } 0 \leq u < d \\ f(d) & \text{if } u \geq d \end{cases}.$$

Clearly, $F(u) \leq f'(0)u$ for $u \geq 0$. See Figure 1. We note that F satisfies (23) (see the two examples described before Corollary 2.3). By Theorem 2.2, the system

$$\begin{aligned}
(r(t)x'(t))' + p(t)F(x(g(t))) &= 0, \ t \in [0,\infty)\backslash\Upsilon, \\
x(t_k^+) &= a_k x(t_k), \ k \in \mathbf{N}, \\
x'(t_k^+) &= b_k x'(t_k), \ k \in \mathbf{N}
\end{aligned}$$

has an eventually positive solution x. By Lemmas 2.1 and 2.2, $x(t) < 0$ eventually and $\lim_{t\to\infty} x(t) = 0$. It follows that $0 < x(t) < d$ eventually. Then $F(x(g(t))) = f(x(g(t)))$ eventually. So x is an eventually positive solution of system (1)-(3). Conversely, assume that the system (1)-(3) has an eventually positive solution x. By Lemmas 2.1 and 2.2, $x(t) < 0$ eventually and $\lim_{t\to\infty} x(t) = 0$. For sufficiently small $\varepsilon > 0$, there exists $0 < d_\varepsilon < d$ such that $f(u) > (f'(0) - \varepsilon)\, u$ for $0 < u \leq d_\varepsilon$. See Figure 2. Since $0 < x(t) \leq d_\varepsilon$ eventually, we can see that x is an eventually positive solution of

$$\begin{aligned}
(r(t)x'(t))' + p(t)\,(f'(0) - \varepsilon)\, x(g(t)) &\leq 0, \ t \in [0,\infty)\backslash\Upsilon, \\
x(t_k^+) &= a_k x(t_k), \ k \in \mathbf{N}, \\
x'(t_k^+) &= b_k x'(t_k), \ k \in \mathbf{N}
\end{aligned}$$

for sufficiently small $\varepsilon > 0$. By Theorem 2.2, the system

$$\begin{aligned}
(r(t)x'(t))' + p(t)\,(f'(0) - \varepsilon)\, x(g(t)) &= 0, \ t \in [0,\infty)\backslash\Upsilon, \\
x(t_k^+) &= a_k x(t_k), \ k \in \mathbf{N}, \\
x'(t_k^+) &= b_k x'(t_k), \ k \in \mathbf{N}
\end{aligned}$$

has an eventually positive solution for sufficiently small $\varepsilon > 0$. By Theorem 2.1, the system (44)-(46) has an eventually positive solution. The proof is complete. ■

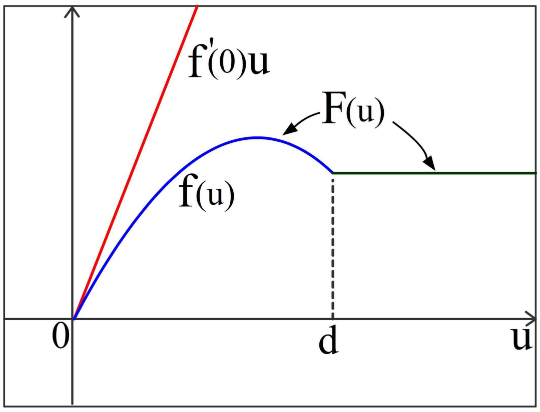

Figure 1

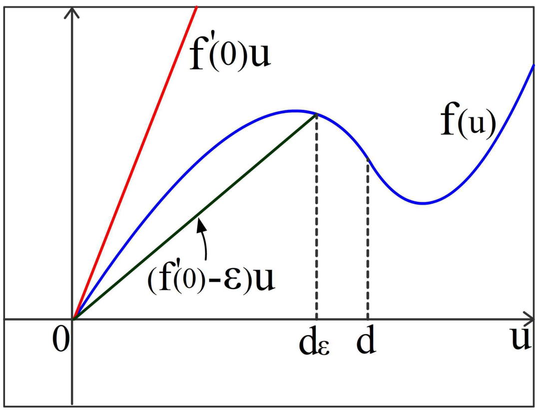

Figure 2

Corollary 2.5 *Let $d > 0$. Assume that (A1)–(A7), (5) and (40) hold, $p \in PC([0,\infty),[0,\infty))$ and $r'(t) \leq 0$ for $t \geq 0$. Assume that f is convex on the interval $(-d, 0]$ and $f'(0)$ exists. Then the system (44)-(46) has an eventually negative solution if, and only if, the system (1)-(3) has an eventually negative solution.*

Remark 2.1 *By Corollaries 2.4 and 2.5, we may obtain oscillatory criteria from those for the corresponding linear systems. In particular, since in [3], the oscillation of linear impulsive delay differential equation with constant coefficients may be determined by its characteristic equation, we may then give the following corollary.*

Corollary 2.6 *Let $\theta > 0$, $p > 0$, $\tau > 0$, $1 \geq a_k > 0$ and $b_k \geq 1$ for $k \in \mathbf{N}$. Assume that (A4) and (A6) hold and*

$$\sum_{k\in\mathbf{N}} \frac{t_{k+1} - t_k}{\prod\limits_{0<i\leq k} b_i} = \infty,$$

and that f'' exists and is continuous in interval $(-\theta, \theta)$, $f'(0) > 0$ and $f''(0) \neq 0$. Then all solutions of system

$$\begin{aligned} x''(t) + pf(x(t-\tau)) &= 0,\ t \in [0,\infty)\backslash\Upsilon, && (47)\\ x(t_k^+) &= a_k x(t_k),\ k \in \mathbf{N}, && (48)\\ x'(t_k^+) &= b_k x'(t_k),\ k \in \mathbf{N} && (49)\end{aligned}$$

are oscillatory.

Proof. Assume that the system (47)-(49) has a nonoscillatory solution x. We may assume that x is eventually positive. The case that x is eventually negative is similar so we ignore it. We note that

$$\int_0^\infty \frac{p}{B(0,t)} dt = p\left(t_1 + \sum_{k\in\mathbf{N}} \frac{t_{k+1} - t_k}{\prod_{0<i\leq k} b_i}\right) = \infty$$

and

$$\int_0^\infty \int_0^t B(s,t) p ds dt \geq \int_0^\infty \int_0^t p ds dt = \infty.$$

By Lemmas 2.1 and 2.2, we may assume that $x(t) > 0$ and $x'(t) < 0$ for $t \geq g_0$. Furthermore, $\lim_{t\to\infty} x(t) = 0$. In view of $f''(0) \neq 0$, we can see that there exists $\theta > \delta > 0$ such that $f''(u) \geq 0$ for $\delta \geq u \geq 0$, or $f''(u) \leq 0$ for $\delta \geq u \geq 0$. In the former case, f is convex on $[0, \delta]$. So $f(u) \geq f'(0)u$ for $\delta \geq u \geq 0$. Since $0 \leq x(t) \leq \delta$ eventually, we may see that $x(t)$ is an eventually positive solution of

$$\begin{aligned} x''(t) + pf'(0)x(t-\tau) &\leq 0,\ t \in [0,\infty)\backslash\Upsilon,\\ x(t_k^+) &= a_k x(t_k),\ k \in \mathbf{N},\\ x'(t_k^+) &= b_k x'(t_k),\ k \in \mathbf{N}.\end{aligned}$$

By Theorem 2.2, we may further see that the equation

$$x''(t) + pf'(0)x(t-\tau)) = 0 \qquad (50)$$

has an eventually positive solution, which implies that its characteristic equation

$$\lambda^2 + pf'(0)e^{-\tau\lambda} = 0$$

has a real root. But this is impossible because $\lambda^2 + pf'(0)e^{-\tau\lambda} > 0$ for $\lambda \in \mathbf{R}$. In the later case, f is concave on $[0, \delta]$. By Corollary 2.4, we can see that

$$\begin{aligned} x''(t) + pf'(0)x(t-\tau) &= 0,\ t \in [0,\infty)\backslash\Upsilon, \\ x(t_k^+) &= a_k x(t_k),\ k \in \mathbf{N}, \\ x'(t_k^+) &= b_k x'(t_k),\ k \in \mathbf{N} \end{aligned}$$

has an eventually positive solution. By Theorem 2.2, we can further see that the equation (50) has an eventually positive solution. By the above discussion, this is also impossible. The proof is complete. ■

3 Examples

We illustrate our results by two examples.

Example 1. Let r and p be continuous functions on $[0, \infty)$ with $r(t) > 0$ and $p(t) \geq 0$ for $t \geq 0$, and

$$f(u) = sgn(u)\left(1 + e^{-|u|} - 2e^{-2|u|}\right) \text{ for } u \in \mathbf{R}.$$

Consider the impulsive delay differential equation

$$\begin{aligned} (r(t)x'(t))' + p(t)f(x(g(t))) &= 0,\ t \in [0,\infty)\backslash\Upsilon, && (51) \\ x(t_k^+) &= a_k x(t_k),\ k \in \mathbf{N}, && (52) \\ x'(t_k^+) &= b_k x'(t_k),\ k \in \mathbf{N}, && (53) \end{aligned}$$

where $a_k = 2$ for k even, and $a_k = 0.5$ for k odd. Clearly, $0.5 \leq A(s,t) \leq 2$ for $t \geq s \geq 0$. By elementary analysis, we may see that $f(u)$ is concave on $[0, \ln 8)$, is convex on $(-\ln 8, 0]$, is strictly decreasing on $(-\infty, -\ln 4] \cup [\ln 4, \infty)$, $f'(0) = 3$ and $\inf_{|u|\geq T}\{|f(u)|\} > 0$ for any $T > 0$. See Figure 3.

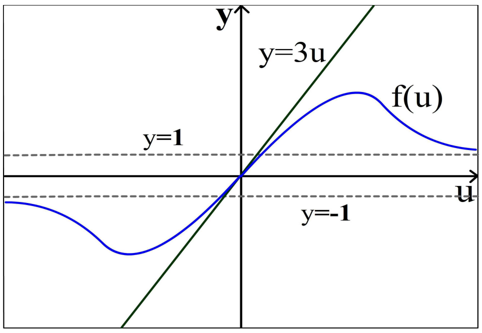

Figure 3

We have the following conclusions:

(i) Assume that $a_k = b_k$ for $k \in \mathbf{N}$, and that $r(t) = \exp(-t)$ and $p(t) = t$ for $t \geq 0$. It is easy to check that (4) and (5) hold. By Corollary 2.1, the system (51)-(53) is oscillatory.

(ii) Assume that $\Upsilon = \mathbf{N}$, $b_k = 1/e$, $r(t) = e^{-0.5t}$ and $p(t) = e^{2t}$ for $t \geq 0$ and $k \in \mathbf{N}$. We note that

$$e^{s-t-1} \leq B(s,t) \leq e^{s-t+1} \text{ for } t \geq 1 \text{ and } t \geq s \geq 0. \quad (54)$$

Then

$$\int_0^\infty \frac{B(0,t)}{A(0,t)r(t)}dt \le 2\left(\int_0^1 e^{0.5t}dt + e\int_1^\infty e^{-0.5t}dt\right) < \infty,$$

$$\int_0^\infty \frac{p(t)}{B(0,t)}dt \ge \int_1^\infty \frac{e^{2t}}{e^{-t+1}}dt = \infty,$$

and

$$\int_0^\infty \frac{1}{r(t)}\int_0^t B(s,t)p(s)dsdt \ge \int_0^\infty \frac{1}{e^{-0.5t}}\int_0^t e^{s-t-1}e^{2s}dsdt = \infty.$$

By Corollaries 2.4 and 2.5, the system (51)-(53) is oscillatory if, and only if,

$$\begin{aligned}\left(e^{-0.5t}x'(t)\right)' + 3e^{2t}x(g(t)) &= 0,\ t \in [0,\infty)\backslash\mathbf{N},\\ x(t_k^+) &= a_k x(t_k),\ k \in \mathbf{N},\\ x'(t_k^+) &= b_k x'(t_k),\ k \in \mathbf{N}\end{aligned}$$

is oscillatory.

(iii). Assume that $r(t) = e^{-0.5t}$, $p(t) = e^{-2t}$ and $b_k = 1/e$, for $t \ge 0$ and $k \in \mathbf{N}$. By (54), we see that

$$\int_0^\infty \frac{1}{r(t)}\int_0^t B(s,t)p(s)dsdt \le \int_0^\infty \frac{1}{e^{-t}}\int_0^t e^{s-t+1}e^{-2s}dsdt < \infty.$$

By Theorem 2.3, the system (51)-(53) has a nonoscillatory solution.

Example 2. Let $p, \tau > 0$, $t_k = 2^k$, $a_k = 1 - 1/(2k)^2$ and $b_k = 2$ for $k \in \mathbf{N}$. Consider the impulsive delay differential equation

$$\begin{aligned}x''(t) + p + p\left(x(t-\tau) - 1\right)^3 &= 0,\ t \in [0,\infty)\backslash\Upsilon, &(55)\\ x(t_k^+) &= a_k x(t_k),\ \ k \in \mathbf{N}, &(56)\\ x'(t_k^+) &= b_k x'(t_k),\ \ k \in \mathbf{N}. &(57)\end{aligned}$$

We note that $a_k < 1$ for $k \in \mathbf{N}$. Then $A(0,t) \ge \lim_{s\to\infty} A(0,s)$ for $t \ge 0$. Since

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots,$$

we can see that

$$\frac{2}{\pi} = \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^2 2^2}\right)\left(1 - \frac{1}{3^2 2^2}\right)\cdots = \prod_{k\in\mathbf{N}} a_i.$$

It follows that $A(0,t) \ge 2/\pi$ for $t \ge 0$. Let $f(u) = 1 + (u-1)^3$ for $u \in \mathbf{R}$. It is easy to check that condition (A4) is satisfied, f'' is continuously differentiable on $\mathbf{R}$, $f'(0) = 3 > 0$, $f''(0) = 6 \ne 0$ and

$$\sum_{k\in\mathbf{N}} \frac{t_{k+1} - t_k}{\prod\limits_{0<i\le k} b_i} = \sum_{k\in\mathbf{N}} 1 = \infty.$$

By Corollary 2.6, all solutions of system (55)-(57) are oscillatory.

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