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Existence of infinitely many solutions for a Steklov problem involving the $p(x)$-Laplace operator

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Abstract. In this article, we study the nonlinear Steklov boundary-value problem

$$\begin{aligned} \Delta_{p(x)}u &= |u|^{p(x)-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= f(x,u) \qquad \text{on } \partial\Omega. \end{aligned}$$

We prove the existence of infinitely many non-negative solutions of the problem by applying a general variational principle due to B. Ricceri and the theory of the variable exponent Sobolev spaces.

Keywords: $p(x)$-Laplace operator, infinitely many solutions, Ricceri's variational principle.

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1 Introduction

Motivated by the developments in elastic mechanics, electrorheological fluids and image restoration [4, 16, 17, 20, 21], the interest in variational problems and differential equations with variable exponent has grown in recent decades; see for example [6, 12, 13, 15]. We refer the reader to [3, 7, 8, 18, 19] for developments in $p(x)$-Laplacian equations.

The purpose of this article is to study the existence and multiplicity of solutions for the Steklov problem involving the $p(x)$-Laplacian,

$$\begin{aligned} \Delta_{p(x)}u &= |u|^{p(x)-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= f(x,u) \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, p is a continuous function on $\overline{\Omega}$ with $N < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < +\infty$ and $f\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a

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generalization of the classical p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant. Many authors have studied the inhomogeneous Steklov problems involving the p-Laplacian [14]. The authors have studied this class of inhomogeneous Steklov problems in the cases of $p(x) \equiv p = 2$ and of $p(x) \equiv p > 1$, respectively. From now, we put $X = W^{1,p(x)}(\Omega)$ and $w := \frac{2\pi^{N/2}}{N\Gamma(\frac{N}{2})}$ the measure of the N-dimensional unit ball.

The main results of this paper are as follows.

Theorem 1.1. *We assume that $f(x,t) = 0$ for all $t \le 0$, a.e $x \in \partial\Omega$, and $\inf_{\eta\ge 0} F(x,\eta) \ge 0$ for a.e. $x \in \partial\Omega$. Moreover, suppose that there exist two sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ in $(0,+\infty)$ with $a_k < b_k$, $\lim_{k\to+\infty} b_k = +\infty$ such that*

(1) $\lim_{k\to+\infty} \frac{b_k^{p^-}}{a_k^{p^++\beta}} = +\infty$ for some non-negative constant β;

(2) $\max_{\overline{\Omega}\times[a_k,b_k]} f \le 0$ for all $k \in \mathbb{N}$;

(3) there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{k\to+\infty} \xi_k = +\infty$ and a constant $h_0 > \frac{|\Omega|}{p^-|\partial\Omega|}$, such that $F(x,\xi_k) \ge h_0\xi_k^{p^+}$ for a.e. $x \in \partial\Omega$;

(4) $\limsup_{k\to+\infty} \frac{\max_{\partial\Omega\times[0,a_k]} F(x,\eta)}{b_k^{p^-}} < \frac{1}{C_0^{p^-} p^+|\partial\Omega|}$, where $C_0 = \sup_{u\in X\setminus\{0\}} \frac{|u|_\infty}{\|u\|}$.

Then problem (1.1) admits an unbounded sequence of non-negative weak solutions in X.

Theorem 1.2. *We assume that $f(x,t) = 0$ for all $t \le 0$, a.e. $x \in \partial\Omega$, and $\inf_{\eta\ge 0} F(x,\eta) \ge 0$ for a.e. $x \in \partial\Omega$. Moreover, suppose that there exist two sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ in $(0,+\infty)$ with $a_k < b_k$, $\lim_{k\to+\infty} b_k = 0$ such that*

(1) $\lim_{k\to+\infty} \frac{b_k^{p^-}}{a_k^{p^--\alpha}} = +\infty$ for some non-negative constant $\alpha < p^-$;

(2) $\max_{\overline{\Omega}\times[a_k,b_k]} f \le 0$ for all $k \in \mathbb{N}$;

(3) there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{k\to+\infty} \xi_k = 0^+$ and a constant $h_0 > \frac{|\Omega|}{p^-|\partial\Omega|}$, such that $F(x,\xi_k) \ge h_0\xi_k^{p^-}$ for a.e. $x \in \partial\Omega$;

(4) $\limsup_{k\to+\infty} \frac{\max_{\partial\Omega\times[0,a_k]} F(x,\eta)}{b_k^{p^-}} < \frac{1}{C_0^{p^-} p^+|\partial\Omega|}$, where $C_0 = \sup_{u\in X\setminus\{0\}} \frac{|u|_\infty}{\|u\|}$.

Then problem (1.1) admits a sequence of non-zero non-negative weak solutions, which strongly converges to 0 in X.

Example 1.3. An example of functions satisfying the assumptions of Theorem 1.1

$$F(x,t) = \begin{cases} e^{2z(x)\ln 2} h(x) \left(\frac{t-b_k}{a_{k+1}-b_k}\right)^{z(x)} \left(\frac{a_{k+1}-t}{a_{k+1}-b_k}\right)^{z(x)} t^{q(x)}, & \text{if } t \in (b_k, a_{k+1}); \\ 0, & \text{otherwise,} \end{cases}$$

where $h \in C(\overline{\Omega})$ with $\min_{x\in\overline{\Omega}} h(x) \ge h_0$, $z \in C(\overline{\Omega})$ with $\min_{x\in\overline{\Omega}} z(x) > 1$ and $q \in C(\overline{\Omega})$ with $p^+ \le q(x) \le p^+ + \beta$ for all $x \in \overline{\Omega}$. Note that in this occasion we can choose $\xi_k = \frac{a_{k+1}+b_k}{2}$.

Example 1.4. An example of functions satisfying the assumptions of Theorem 1.2

$$F(x,t) = \begin{cases} e^{2z(x)\ln 2} h(x) \left(\frac{t-b_{k+1}}{a_k-b_{k+1}}\right)^{z(x)} \left(\frac{a_k-t}{a_k-b_{k+1}}\right)^{z(x)} t^{q(x)}, & \text{if } t \in (b_{k+1}, a_k); \\ 0, & \text{otherwise}, \end{cases}$$

where $h \in C(\overline{\Omega})$ with $\min_{x\in\overline{\Omega}} h(x) \geq h_0$, $z \in C(\overline{\Omega})$ with $\min_{x\in\overline{\Omega}} z(x) > 1$ and $q \in C(\overline{\Omega})$ with $p^- - \alpha \leq q(x) \leq p^-$ for all $x \in \overline{\Omega}$. Note that in this occasion we can choose $\xi_k = \frac{b_{k+1}+a_k}{2}$.

Existence of infinitely many solutions for boundary value problems have received a great deal of interest in recent years, see, for instance, the paper [2, 5] and references therein. In [1] we have considered the existence and multiplicity of solutions for the Steklov problem involving the $p(x)$-Laplacian of the type

$$(\mathcal{S}) \begin{cases} \Delta_{p(x)} u = |u|^{p(x)-2} u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = \lambda f(x,u) & \text{on } \partial\Omega. \end{cases}$$

Under the following assumptions of the function f,

(f_1) $| f(x,s) | \leq a(x) + b \mid s \mid^{\alpha(x)-1}, \qquad \forall (x,s) \in \partial\Omega \times \mathbb{R}$,
where $a(x) \in L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)$ and $b \geq 0$ is a constant, $\alpha(x) \in C_+(\partial\Omega)$.

(f_2) $f(x,t) < 0$, when $|t| \in (0,1)$,
$f(x,t) \geq m > 0$, when $t \in (t_0, \infty)$, $t_0 > 1$,

we have established the existence of at least three solutions of this problem.

This article is organized as follows. First, we will introduce some basic preliminary results and lemmas in Section 2. In Section 3, we will give the proofs of our main results.

2 Preliminaries

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For more details, see [9, 10]. Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$ where

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) \quad \text{and} \quad \inf_{x\in\overline{\Omega}} p(x) > 1 \right\}.$$

Denote by $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\},$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \tau > 0; \int_\Omega \left| \frac{u}{\tau} \right|^{p(x)} dx \leq 1 \right\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm

$$\|u\| = \inf\left\{\tau > 0; \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \left|\frac{u}{\tau}\right|^{p(x)}\right) dx \le 1\right\},$$
$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}.$$

We refer the reader to [9, 10] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

Lemma 2.1 ([10]). *Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable and uniformly convex Banach spaces.*

Lemma 2.2 ([10]). *Hölder inequality holds, namely*

$$\int_\Omega |uv|\, dx \le 2|u|_{p(x)}|v|_{q(x)} \quad \forall u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Lemma 2.3 ([10]). *Let $I(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx$, for $u \in W^{1,p(x)}(\Omega)$ we have*

- $\|u\| < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1, > 1)$;
- $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le I(u) \le \|u\|^{p^-}$;
- $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le I(u) \le \|u\|^{p^+}$.

Lemma 2.4 ([9]). *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ and $1 \le q(x) < p^*(x)$ for $x \in \overline{\Omega}$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N; \\ +\infty, & \text{if } p(x) \ge N. \end{cases}$$

Lemma 2.5 ([9]). *The embedding $W^{1,p(x)}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact whenever $N < p^-$.*

Lemma 2.6 ([11]). *Let X be a separable and reflexive real Banach space, $\phi, \psi \colon X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that ϕ is (strongly) continuous and satisfies $\lim_{\|u\|\to+\infty} \phi(u) = +\infty$. For each $\rho > \inf_X \phi$, put*

$$\varphi(\rho) = \inf_{x \in \phi^{-1}((-\infty,\rho))} \frac{\psi(x) - \inf_{\overline{\phi^{-1}((-\infty,\rho))_w}} \psi}{\rho - \phi(x)},$$

where $\overline{\phi^{-1}((-\infty,\rho))_w}$ is the closure of $\phi^{-1}((-\infty,\rho))_w$ in the weak topology.

1. *If there exist a sequence $\{r_k\} \subset (\inf_X \phi, +\infty)$ with $r_k \to +\infty$ and a sequence $\{u_k\} \subset X$ such that for each $k \in \mathbb{N}$,*
$$\phi(u_k) < r_k \tag{2.1}$$
and
$$\psi(u_k) - \inf_{\overline{\phi^{-1}((-\infty,\rho))_w}} \psi < r_k - \phi(u_k), \tag{2.2}$$
and in addition,
$$\liminf_{\|u\|\to+\infty} (\phi(u) + \psi(u)) = -\infty, \tag{2.3}$$
then there exists a sequence $\{v_k\}$ of local minima of $\phi + \psi$ such that $\phi(v_k) \to +\infty$ as $k \to +\infty$.

2. *If there exist a sequence $\{r_k\} \subset (\inf_X \phi, +\infty)$ with $r_k \to (\inf_X \phi)^+$ and a sequence $\{u_k\} \subset X$ such that for each k the conditions (2.1) and (2.2) are satisfied, and in addition, every global minimizer of ϕ is not a local minimizer of $\phi + \psi$, then there exists a sequence $\{v_k\}$ of pairwise distinct local minimizers of $\phi + \psi$ such that $\lim_{k\to+\infty} \phi(v_k) = \inf_X \phi$, and $\{v_k\}$ weakly converges to a global minimizer of ϕ.*

Definition 2.7. We say that $u \in X$ is a weak solution of (1.1) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx + \int_\Omega |u|^{p(x)-2}uv\, dx = \int_{\partial\Omega} f(x,u)v\, d\sigma \quad \text{for all } v \in X.$$

For each $u \in X$, we define

$$\begin{aligned}
\phi(u) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx,\\
\psi(u) &= -\int_{\partial\Omega} F(x,u)\, d\sigma,\\
J(u) &= \phi(u) + \psi(u),
\end{aligned}$$

where $F(x,t) = \int_0^t f(x,s)\, ds$. Then we have

$$\begin{aligned}
\langle \phi'(u), v\rangle &= \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv\right) dx,\\
\langle \psi'(u), v\rangle &= -\int_{\partial\Omega} f(x,u)v\, d\sigma,
\end{aligned}$$

for all $v \in X$.

Then it is easy to see that $\phi, \psi \in C^1(X, \mathbb{R})$ and $u \in X$ is a weak solution of (1.1) if and only if u is a critical point of the functional J.

Notice that ϕ is convex and continuous functional so it is a weakly lower semi-continuous. Since the embedding $X \hookrightarrow C(\overline{\Omega})$ is compact, we can see that $\psi\colon X \to \mathbb{R}$ is sequentially weakly lower semi-continuous.

3 Proof of main results

For the proof of Theorems 1.1 and 1.2, we will use Lemma 2.6. We start with the following lemmas.

Lemma 3.1. *ϕ is coercive.*

Proof. When $\|u\| \geq 1$, we have

$$\phi(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx \geq \frac{1}{p^+}\|u\|^{p^-}, \tag{3.1}$$

then ϕ is coercive. The proof is completed. □

Since $\phi\colon X \to \mathbb{R}$ is coercive we can define $K(r)$ as

$$K(r) = \inf\left\{\tau > 0 : \phi^{-1}((-\infty, r)) \subset \overline{B_X(0,\tau)}\right\}, \tag{3.2}$$

for $r > \inf_X \phi$, where

$$B_X(0,\tau) = \{u \in X : \|u\| < \tau\},$$

$\overline{B_X(0,\tau)}$ denotes the closure of $B_X(0,\tau)$ in X. Since ϕ is coercive, we know $0 < K(r) < +\infty$ for each $r > \inf_X \phi$. From the definition of $K(r)$, we have $\phi^{-1}((-\infty,r)) \subset \overline{B_X(0,K(r))}$ and consequently $\overline{\phi^{-1}((-\infty,r))_w} \subset \overline{B_X(0,K(r))}$. Since the embedding $X \hookrightarrow C(\overline{\Omega})$ is compact, so there is a constant $C_0 > 0$ such that

$$C_0 = \sup_{u\in X\setminus\{0\}} \frac{|u|_\infty}{\|u\|}.$$

Therefore we have

$$\overline{B_X(0,K(r))} \subset \{u \in C(\overline{\Omega}) : |u|_\infty \leq C_0K(r)\}.$$

So we have

$$\inf_{v\in\overline{\phi^{-1}((-\infty,r))_w}} \psi(v) \geq \inf_{\|v\|\leq K(r)} \psi(v) \geq \inf_{|v|_\infty\leq C_0K(r)} \psi(v). \tag{3.3}$$

Lemma 3.2. *For $r \geq \frac{1}{p^+}$, we have*

$$K(r) \leq \left(rp^+\right)^{\frac{1}{p^-}}. \tag{3.4}$$

Proof. Let $r \geq \frac{1}{p^+}$ and $u \in X$ be such that $\phi(u) < r$. When $\|u\| \geq 1$, by (3.1), we obtain

$$r > \phi(u) \geq \frac{1}{p^+}\|u\|^{p^-},$$

thus we have $\|u\| < \left(rp^+\right)^{\frac{1}{p^-}}$. When $\|u\| < 1$, it is clear that

$$\|u\| < 1 \leq rp^+,$$

which implies that $\|u\| < \left(rp^+\right)^{\frac{1}{p^-}}$. By the definition of $K(r)$, (3.4) holds. □

Proof of Theorem 1.1. We use Lemma 2.6(1) to prove Theorem 1.1.
Now put $r_k = \frac{1}{p^+}\left(\frac{b_k}{C_0}\right)^{p^-}$, then $\lim_{k\to+\infty} r_k = +\infty$. Using Lemma 3.2, we have $C_0K(r_k) \leq b_k$. Fix $x_0 \in \Omega$ and pick $\gamma > 0$ such that $B(x_0,\gamma) \subseteq \Omega$.

By condition (2), we have $\max_{\partial\Omega\times[0,a_k]} F = \max_{\partial\Omega\times[0,b_k]} F$. Now we consider the function $u_k \in X$ defined by

$$u_k = \begin{cases} 0, & \text{if } x \in \Omega\setminus B(x_0,\gamma);\\ \eta_k, & \text{if } x \in B(x_0,\frac{\gamma}{2});\\ \frac{2\eta_k}{\gamma}\left(\gamma - |x-x_0|\right), & \text{if } x \in B(x_0,\gamma)\setminus B(x_0,\frac{\gamma}{2})\ , \end{cases} \tag{3.5}$$

with $\eta_k \in (0,a_k]$ and $x_k \in \partial\Omega$ such that $F(x_k,\eta_k) = \max_{\partial\Omega\times[0,a_k]} F$. Without loss of generality, we may assume that $\eta_k \geq \max(\gamma,1)$. In view of condition (1), we choose $k_1 \in \mathbb{N}$ such that

$$\frac{b_k^{p^-}}{a_k^{p^+}} > \frac{p^+C_0^{p^-}w\gamma^{N-p^+}}{p^-2^N}\left[2^{p^+}(2^N-1) + 2^N\gamma^{p^+}\right], \tag{3.6}$$

for all $k > k_1$. For each $k > k_1$, using (3.6), we have

$$\begin{aligned}
\phi(u_k) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_k|^{p(x)} + |u_k|^{p(x)}\right) dx \\
&\le \frac{1}{p^-}\left[\int_\Omega |\nabla u_k|^{p(x)}\,dx + \int_\Omega |u_k|^{p(x)}\,dx\right] \\
&\le \frac{1}{p^-}\left[\left(\frac{2\eta_k}{\gamma}\right)^{p^+}\left[\left|B\left(x_0,\gamma\right)\right| - \left|B\left(x_0,\frac{\gamma}{2}\right)\right|\right] + \int_{B(x_0,\gamma)} |u_k|^{p(x)}\,dx\right] \\
&\le \frac{2^{p^+}\eta_k^{p^+} w\gamma^N}{p^-\gamma^{p^+}}\left(1-\frac{1}{2^N}\right) + \frac{\eta_k^{p^+} w\gamma^N}{p^-} \\
&= \frac{\eta_k^{p^+} w\gamma^N}{p^-\gamma^{p^+}2^N}\left[2^{p^+}\left(2^N-1\right) + 2^N\gamma^{p^+}\right] \\
&\le \frac{a_k^{p^+} w\gamma^N}{p^-\gamma^{p^+}2^N}\left[2^{p^+}\left(2^N-1\right) + 2^N\gamma^{p^+}\right] \\
&< r_k.
\end{aligned}$$

Then for each $k > k_1$, we have

$$\phi(u_k) < r_k. \tag{3.7}$$

For each $v \in \phi^{-1}((-\infty, r_k))$, we can easily see that for each $x \in \partial\Omega$

$$\begin{aligned}
F(x, v(x)) &\le \max_{\partial\Omega\times[0,C_0K(r_k)]} F(x,v) \\
&\le \max_{\partial\Omega\times[0,b_k]} F(x,v) \\
&= \max_{\partial\Omega\times[0,a_k]} F(x,v) \\
&= F(x_k,\eta_k).
\end{aligned}$$

By condition (4), there exists a $k_2 \in \mathbb{N}$ such that

$$\frac{\max_{\partial\Omega\times[0,a_k]} F(x,\eta)}{b_k^{p^-}} = \frac{F(x_k,\eta_k)}{b_k^{p^-}} < \frac{1}{C_0^{p^-} p^+|\partial\Omega|}, \tag{3.8}$$

for every $k > k_2$. Since $\lim_{k\to+\infty} \frac{\eta_k^{p^+}}{b_k^{p^-}} = 0$ and (3.8), for every $k > k_2$, we have

$$\frac{F(x_k,\eta_k) + \frac{\eta_k^{p^+} w\gamma^N}{|\partial\Omega| p^-\gamma^{p^+}2^N}\left[2^{p^+}\left(2^N-1\right)+2^N\gamma^{p^+}\right]}{b_k^{p^-}} < \frac{1}{C_0^{p^-} p^+|\partial\Omega|}. \tag{3.9}$$

Therefore, using (3.9), we obtain

$$\begin{aligned}
&\sup_{v\in C(\overline{\Omega}),|v|_\infty\le C_0K(r_k)} \int_{\partial\Omega} F(x,v(x))\,d\sigma - \int_{\partial\Omega} F(x,u_k(x))\,d\sigma \\
&\qquad\le F(x_k,\eta_k)|\partial\Omega| \\
&\qquad< \frac{b_k^{p^-}}{C_0^{p^-} p^+} - \frac{\eta_k^{p^+} w\gamma^N}{p^-\gamma^{p^+}2^N}\left[2^{p^+}\left(2^N-1\right)+2^N\gamma^{p^+}\right] \\
&\qquad\le r_k - \phi(u_k),
\end{aligned}$$

for each $k > k_2$. Then for every $k > k_2$, we have

$$\sup_{v\in C(\overline{\Omega}),|v|_\infty \le C_0 K(r_k)} \int_{\partial\Omega} F(x,v(x))\,d\sigma - \int_{\partial\Omega} F(x,u_k(x))\,d\sigma < r_k - \phi(u_k). \tag{3.10}$$

In view of (3.7) and (3.10), for each $k > \max\{k_1,k_2\}$, we have proved (2.1) and (2.2). We need to verify that the functional $\phi+\psi$ has no global minimum, i.e. (2.3). By condition (3), we can find a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{k\to+\infty}\xi_k = +\infty$ and $F(x,\xi_k) \ge h_0\xi_k^{p^+}$ for a.e. $x\in\partial\Omega$. Now we consider a function $w_k\in X$ defined by $w_k(x)=\xi_k$. Without loss of generality, we may assume that $\xi_k \ge 1$. So we have

$$\begin{aligned}\phi(w_k)+\psi(w_k) &\le \frac{\xi_k^{p^+}|\Omega|}{p^-} - h_0\xi_k^{p^+}|\partial\Omega| \\ &\le \xi_k^{p^+}|\partial\Omega|\left(\frac{|\Omega|}{p^-|\partial\Omega|} - h_0\right).\end{aligned}$$

Since $h_0 > \frac{|\Omega|}{p^-|\partial\Omega|}$. It forces

$$\lim_{k\to+\infty} \xi_k^{p^+}|\partial\Omega|\left(\frac{|\Omega|}{p^-|\partial\Omega|} - h_0\right) = -\infty.$$

Therefore, Lemma 2.6(1) assures that there is a sequence $\{v_k\}_{k\in\mathbb{N}}$ of local minima of $\phi+\psi$ such that $\phi(v_k)\to+\infty$ as $k\to+\infty$.
It remains to show that the weak solutions obtained are non-negative.
Define

$$f_+(x,t) = \begin{cases} f(x,t), & \text{if } t\ge 0; \\ 0, & \text{otherwise,}\end{cases}$$

and consider the following problem

$$(\mathcal{S}_+)\begin{cases}\Delta_{p(x)}u = |u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = f_+(x,u) & \text{on } \partial\Omega,\end{cases}$$

if u is weak solution of the problem $(\mathcal{S}_+)$, then one has

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx - \int_{\partial\Omega} f_+(x,u)v\,d\sigma = 0, \tag{3.11}$$

for all $v\in X$. Taking $v=u^-$ in (3.11) shows that $\|u^-\| = 0$, so $u^-=0$. Obviously, u is a non-negative solution of (1.1) in X. This completes the proof.

Proof of Theorem 1.2. We use Lemma 2.6(2) to prove Theorem 1.2.
We put $r_k = \frac{1}{p^+}\left(\frac{b_k}{C_0}\right)^{p^-}$, and consider the function $u_k\in X$ defined by

$$u_k = \begin{cases} 0, & \text{if } x\in\Omega\backslash B(x_0,\gamma); \\ \eta_k, & \text{if } x\in B(x_0,\frac{\gamma}{2}); \\ \frac{2\eta_k}{\gamma}(\gamma-|x-x_0|), & \text{if } x\in B(x_0,\gamma)\backslash B(x_0,\frac{\gamma}{2})\,. \end{cases} \tag{3.12}$$

We can easily get (2.1) and (2.2) using the same method as in the proof of Theorem 1.1. In view of condition (3), we can find a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{k\to+\infty}\xi_k = 0^+$ and

$F(x, \xi_k) \geq h_0 \xi_k^{p^-}$ for a.e. $x \in \partial\Omega$. If we take $w_k = \xi_k$, of course the sequence $\{w_k\}$ strongly converges to 0 in X and $\phi(w_k) + \psi(w_k) < 0$ for all $k \in \mathbb{N}$. Since $\phi(0) + \psi(0) = 0$, this means that 0 is not a local minimum of $\phi + \psi$.

So, since 0 is the only global minimum of ϕ. Lemma 2.6(2) ensures that there exists a sequence $\{v_k\}$ of pairwise distinct local minimizers of $\phi + \psi$ such that $\lim_{k\to+\infty} \phi(v_k) = 0$. Using the same method as in the proof of Theorem 1.1, we can get that each weak solution of problem (1.1) is non-negative. This completes the proof.

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References

[1] M. Allaoui, A. El Amrouss, A. Ourraoui, Existence and multiplicity of solutions for a Steklov problem involving the $p(x)$-Laplace operator, *Electron. J. Diff. Equ.* **2012**, No. 132, 1–12. MR2967197

[2] P. Candito, L. Li, R. Livrea, Infinitely many solutions for a perturbed nonlinear Navier boundary value problem involving the p-biharmonic, *Nonlinear Anal.* **75**(2012), 6360–6369. MR2959811; url

[3] J. Chabrowski, Y. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, *J. Math. Anal. Appl.* **306**(2005), 604–618. MR2136336; url

[4] Y. M. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* **66**(2006), 1383–1406. MR2246061; url

[5] G. Dai, J. Wei, Infinitely many non-negative solutions for a $p(x)$-Kirchhoff-type problem with Dirichlet boundary condition, *Nonlinear Anal.* **73**(2010), 3420–3430. MR2680035; url

[6] S. G. Deng, Eigenvalues of the $p(x)$-Laplacian Steklov problem, *J. Math. Anal. Appl.* **339**(2008), 925–937. MR2375248; url

[7] S. G. Deng, A local mountain pass theorem and applications to a double perturbed $p(x)$-Laplacian equations, *Appl. Math. Comput.* **211**(2009) 234–241. MR2517682; url

[8] X. Ding, X. Shi, Existence and multiplicity of solutions for a general $p(x)$-Laplacian Neumann problem, *Nonlinear Anal.* **70**(2009), 3713–3720. MR2504458; url

[9] X. L. Fan, J. S. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}$, *J. Math. Anal. Appl.* **262**(2001), 749–760. MR1859337; url

[10] X. L. Fan, D. Zhao, On the spaces $L^{p(x)}$ and $W^{m,p(x)}$, *J. Math. Anal. Appl.* **263**(2001), 424–446. MR1866056; url

[11] X.L. Fan, C. Ji, Existence of infinitely many solutions for a Neumann problem involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.* **334**(2007), 248–260. MR2332553; url

[12] X. L. Fan, S. G. Deng, Remarks on Ricceri's variational principle and applications to the $p(x)$-Laplacian equations, *Nonlinear Anal.* **67**(2007), 3064–3075. MR2347599; url

[13] P. Harjulehto, P. Hästö, V. Lê, M. Nuortio, Overview of differential equations with non-standard growth, *Nonlinear Anal.* **72**(2010), 4551–4574. MR2639204; url

[14] N. Mavinga, Mubenga N. Nkashama, Steklov spectrum and nonresonance for elliptic equations with nonlinear boundary conditions, *Electron. J. Diff. Equ. Conf.* **2010**, No. 19, 197–205. MR2754944

[15] M. Mihăilescu, V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.* **135**(2007), 2929–2937. MR2317971

[16] T. G. Myers, Thin films with high surface tension, *SIAM Review* **40**(1998), 441–462. MR1642807 ; url

[17] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Springer-Verlag, Berlin, 2000. MR1810360

[18] L. L. Wang, Y. H. Fan, W. G. Ge, Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator, *Nonlinear Anal.* **71**(2009), 4259–4270. MR2536331; url

[19] Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian equations with singular coefficients in R^N, *J. Math. Anal. Appl.* **348**(2008), 38–50. MR2449325; url

[20] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.* **29**(1986), 33–66. MR0864171

[21] V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of differential operators and integral functionals*, translated from Russian by G. A. Yosifian, Springer-Verlag, Berlin, 1994. MR1329546