Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **71**, 1–17; **http://www.math.u-szeged.hu/ejqtde/**

ANALYSIS OF A DYNAMIC THERMO-ELASTIC-VISCOPLASTIC CONTACT PROBLEM

Azeb Ahmed Abdelaziz[1]
Department of Mathematics, University of Eloued, Algeria
Boutechebak Souraya[2]
Department of Mathematics, University of Setif 1, Algeria

Abstract. We consider a dynamic frictionless contact problem for thermo-elastic-viscoplastic materials with damage and adhesion. The contact is modeled with normal compliance condition. We derive a weak formulation of the system, then we prove existence and uniqueness of the solution. The proof is based on arguments of monotonicity and fixed point.

Keywords: dynamic process; damage field; adhesion field; temperature; thermo-elastic-viscoplastic; variational inequality; fixed-point.

2010 Mathematics Subject Classification: 74M15, 74C10, 74F05.

1. Introduction

Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal forming processes are just a few examples. The constitutive laws with internal variables has been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metal and rocks polymers. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials, the absolute temperature and the damage field. See for examples [6, 26, 27, 28, 29, 35, 36] for the case of hardening, temperature and other internal state variables and the references [18, 20, 27] for the case of damage field and the adhesion field which is denoted in this paper by β. It describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [15, 16], the bonding field satisfies the restrictions $0 \le \beta \le 1$. When $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active. When $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion. When $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [31, 33, 34].

In this paper we deal with the study of a dynamic problem of frictionless adhesive contact for general thermo-elastic-viscoplastic materials. For this, we consider a rate-type constitutive equation with two internal variables of the form

$$\sigma(t) = \mathcal{A}\big(\varepsilon(\dot u(t))\big) + \mathcal{E}\big(\varepsilon(u(t))\big) + \int_0^t \mathcal{G}\big(\sigma(s) - \mathcal{A}\big(\varepsilon(\dot u(s))\big), \varepsilon\big(u(s)\big), \theta(s), \varsigma(s)\big)ds, \tag{1.1}$$

in which u, σ represent, respectively, the displacement field and the stress field where the dot above denotes the derivative with respect to the time variable, θ represents the absolute temperature, ς is the damage field, $\mathcal{A}$ and $\mathcal{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function which describes the visco-plastic behavior of the material. It follows from (1.1) that at each time moment, the stress tensor $\sigma(t)$ is split into two parts: $\sigma(t) = \sigma^V(t) + \sigma^R(t)$, where $\sigma^V(t) = \mathcal{A}(\varepsilon(\dot u(t)))$ represents the

[1]Corresponding author: *Email address: aziz-azebahmed@univ-eloued.dz*
[2]*Email address: bou_souraya@yahoo.fr*

purely viscous part of the stress, whereas $\sigma^R(t)$ satisfies a rate-type elastic-viscoplastic relation with absolute temperature and damage

$$\sigma^R(t) = \mathcal{E}\big(\varepsilon(u(t))\big) + \int_0^t \mathcal{G}\big(\sigma^R(s), \varepsilon\big(u(s)\big), \theta(s), \varsigma(s)\big)ds. \tag{1.2}$$

When $\mathcal{G} = 0$ in (1.1) reduces to the Kelvin–Voigt viscoelastic constitutive law given by

$$\sigma(t) = \mathcal{A}\big(\varepsilon(\dot{u}(t))\big) + \mathcal{E}\big(\varepsilon(u(t))\big). \tag{1.3}$$

The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [17, 18] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [19]. In all these papers the damage of the material is described with a damage function ς, restricted to have values between zero and one. When $\varsigma = 1$ there is no damage in the material, when $\varsigma = 0$ the material is completely damaged, when $0 < \varsigma < 1$ there is partial damage and the system has a reduced load carrying capacity. In this paper the inclusion used for the evolution of the damage field is

$$\rho\dot{\varsigma} - k_1\Delta\varsigma + \partial_K\varphi(\varsigma) \ni \phi\big(\sigma - \mathcal{A}\big(\varepsilon(\dot{u}(s))\big), \varepsilon(u), \theta, \varsigma\big)$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in V : 0 \le \xi(x) \le 1 \ \text{ a.e. } x \in \Omega\},$$

k_1 is a positive coefficient, $\partial_K\varphi(\varsigma)$ represents the subdifferential of the indicator function of the set K and ϕ is a given constitutive function which describes the sources of the damage in the system. Examples and mechanical interpretation of elastic-viscoplastic can be found in [12, 21]. Dynamic and quasistatic contact problems are the topic of numerous papers, e.g. [1, 2, 4, 11, 14, 32]. More recently in [5], we study an electro-elastic-visco-plastic frictionless contact problem with damage and adhesion. The mathematical problem modelled the quasi-static evolution of damage in thermo-viscoplastic materials has been studied in [27].

We model the material's behavior with an elastic-viscoplastic constitutive law with damage. We derive a variational formulation of the problem and prove the existence of a unique weak solution. The paper is organized as follows. In Section 2 we present the mechanical problem of the dynamic evolution of damage and adhesion in thermo-elastic-viscoplastic materials. We introduce some notations and preliminaries and we derive the variational formulation of the problem. We prove in Section 3 the existence and uniqueness of the solution.

2. Statement of the Problem

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with a Lipschitz boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$. We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$, respectively, by

$$u \cdot v = u_i v_i \quad \forall u, v \in \mathbb{R}^n, \quad \sigma \cdot \tau = \sigma_{ij}\tau_{ij} \quad \forall \sigma, \tau \in \mathbb{S}_n,$$
$$|u| = (u \cdot u)^{1/2} \quad \forall u \in \mathbb{R}^n, \quad |\sigma| = (\sigma \cdot \sigma)^{1/2} \quad \forall \sigma \in \mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used. We shall use the notation

$$\begin{aligned}
H &= L^2(\Omega)^n = \{u = \{u_i\} : u_i \in L^2(\Omega)\},\\
\mathcal{H} &= \{\sigma = \{\sigma_{ij}\} : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},\\
H_1 &= \{u \in H : \varepsilon(u) \in \mathcal{H}\},\\
\mathcal{H}_1 &= \{\sigma \in \mathcal{H} : \mathrm{Div}(\sigma) \in H\},\\
V &= H^1(\Omega).
\end{aligned}$$

Here $\varepsilon : H_1 \to \mathcal{H}$ and $\mathrm{Div} : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}(\sigma) = (\sigma_{ij,j}).$$

The sets H, $\mathcal{H}$, H_1, $\mathcal{H}_1$ and V are real Hilbert spaces endowed with the canonical inner products:

$$\begin{aligned}
(u, v)_H &= \int_\Omega u_i v_i dx, \quad (\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} dx,\\
(u, v)_{H_1} &= (u, v)_H + (\varepsilon(u), \varepsilon(v)_{\mathcal{H}},\\
(\sigma, \tau)_{\mathcal{H}_1} &= (\sigma, \tau)_{\mathcal{H}} + (\mathrm{Div}(\sigma), \mathrm{Div}(\tau))_H,\\
(f, g)_V &= (f, g)_{L^2(\Omega)} + (f_{x_i}, g_{x_i})_{L^2(\Omega)}\,.
\end{aligned}$$

The associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}_1}$ and $\|\cdot\|_V$. Since the boundary Γ is Lipschitz continuous, the unit outward normal vector field ν on the boundary is defined a.e. For every vector field $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary given by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.$$

Let $H_\Gamma = (H^{1/2}(\Gamma))^n$ and $\gamma : H_1 \to H_\Gamma$ be the trace map. We denote by $\mathcal{V}$ the closed subspace of H_1 defined by

$$\mathcal{V} = \{v \in H_1 : \gamma v = 0 \text{ on } \Gamma_1\}.$$

We also denote by H'_Γ the dual of H_Γ. Moreover, since $\mathrm{meas}(\Gamma_1) > 0$, Korn's inequality holds and thus, there exists a positive constant C_0 depending only on Ω, Γ_1 such that

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq C_0 \|v\|_{H_1} \quad \forall v \in \mathcal{V}.$$

On the space $\mathcal{V}$ we consider the inner product given by

$$(u, v)_{\mathcal{V}} = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$

and let $\|\cdot\|_{\mathcal{V}}$ be the associated norm, defined by

$$\|v\|_{\mathcal{V}} = \|\varepsilon(v)\|_{\mathcal{H}}. \tag{2.1}$$

It follows from Korn's inequality that $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{V}}$ are equivalent norms on $\mathcal{V}$. Therefore $(\mathcal{V}, |\cdot|_{\mathcal{V}})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a positive constant C_0 which depends only on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^n} \leq C_0 \|v\|_{\mathcal{V}} \quad \forall v \in \mathcal{V}. \tag{2.2}$$

Furthermore, if $\sigma \in \mathcal{H}_1$ there exists an element $\sigma\nu \in H'_\Gamma$ such that the following Green formula holds

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (\mathrm{Div}(\sigma), v)_H = (\sigma\nu, \gamma v)_{H'_\Gamma \times H_\Gamma} \quad \forall v \in H_1.$$

In addition, if σ is sufficiently regular (say $\mathcal{C}^1$), then

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (\mathrm{Div}(\sigma), v)_H = \int_\Gamma \sigma\nu \cdot \gamma v d\Gamma \qquad \forall v \in H_1, \tag{2.3}$$

where $d\Gamma$ denotes the surface element. Similarly, for a regular tensor field $\sigma : \Omega \to \mathbb{S}_n$ we define its normal and tangential components on the boundary by

$$\sigma_\nu = \sigma\nu \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

Moreover, we denote by $\mathcal{V}'$ and V' the dual of the spaces $\mathcal{V}$ and V, respectively. Identifying H, respectively $L^2(\Omega)$, with its own dual, we have the inclusions

$$\mathcal{V} \subset H \subset \mathcal{V}', \quad V \subset L^2(\Omega) \subset V'.$$

We use the notation $\langle \cdot, \cdot \rangle_{\mathcal{V}' \times \mathcal{V}}$, $\langle \cdot, \cdot \rangle_{V' \times V}$ to represent the duality pairing between $\mathcal{V}', \mathcal{V}$ and V', V, respectively. Let $T > 0$. For every real space X, we use the notation $C(0,T;X)$, and $C^1(0,T;X)$ for the space of continuous an continuously differentiable functions from $[0,T]$ to X respectively, $C(0,T;X)$ is a real Banach space with the norm

$$|f|_{C(0,T;X)} = \max_{t \in [0,T]} |f(t)|_X.$$

While $C^1(0,T;X)$ is a real Banach space with the norm

$$|f|_{C^1(0,T;X)} = \max_{t \in [0,T]} |f(t)|_X + \max_{t \in [0,T]} |\dot{f}(t)|_X.$$

Finally, for $k \in \mathbb{N}$ and $p \in [1, \infty]$, we use the standard notation for the Lebesgue space $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, for a real number r, we use r_+ to represent its positive part that is $r_+ = \max(0, r)$, and if X_1 and X_2 are real Hilbert spaces, than $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1 \times X_2}$.

The physical setting is the following. A body occupies the domain Ω, and is clamped on Γ_1 and so the displacement field vanishes there. Surface tractions of density f_0 act on $\Gamma_2 \times (0,T)$ and a volume force of density f is applied in $\Omega \times (0,T)$. We assume that the body is in adhesive frictionless contact with an obstacle, the so-called foundation, over the potential contact surface Γ_3. We admit a possible external heat source applied in $\Omega \times (0,T)$, given by the function q. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. We use an elastic-viscoplastic constitutive law with damage to model the material's behaviour and an ordinary differential equation to describe the evolution of the adhesion field.
The mechanical formulation of the frictionless problem with normal compliance is as follow.

Problem P. Find the displacement field $u : \Omega \times [0,T] \to \mathbb{R}^n$, the stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}_n$, the temperature $\theta : \Omega \times [0,T] \to \mathbb{R}$, the damage field $\varsigma : \Omega \times [0,T] \to \mathbb{R}$ and the adhesion field $\beta : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{gathered}\sigma(t) = \mathcal{A}\big(\varepsilon(\dot{u}(t))\big) + \mathcal{E}\big(\varepsilon(u(t))\big) + \int_0^t \mathcal{G}\big(\sigma(s) - \mathcal{A}\big(\varepsilon(\dot{u}(s))\big), \varepsilon\big(u(s)\big), \theta(s), \varsigma(s)\big) ds \\ \text{in } \Omega \text{ a.e. } t \in (0,T),\end{gathered} \tag{2.4}$$

$$\rho\ddot{u} = \mathrm{Div}(\sigma) + f \quad \text{in } \Omega \times (0,T), \tag{2.5}$$

$$\rho\dot{\theta} - k_0 \Delta\theta = \psi\big(\sigma - \mathcal{A}\big(\varepsilon(\dot{u})\big), \varepsilon(u), \theta, \varsigma\big) + q \quad \text{in } \Omega \times (0,T), \tag{2.6}$$

$$\rho\dot{\varsigma} - k_1 \Delta\varsigma + \partial_K \varphi(\varsigma) \ni \phi\big(\sigma - \mathcal{A}\big(\varepsilon(\dot{u})\big), \varepsilon(u), \theta, \varsigma\big) \quad \text{in } \Omega \times (0,T), \tag{2.7}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.8}$$

$$\sigma\nu = f_0 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.9}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$-\sigma_\tau = p_\tau(\beta) R_\tau(u_\tau) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.11}$$

$$\dot\beta = -\big(\beta\big[\gamma_\nu (R_\nu(u_\nu))^2 + \gamma_\tau |R_\tau(u_\tau)|^2\big] - \varepsilon_a\big)_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{2.12}$$

$$k_0 \frac{\partial\theta}{\partial\nu} + \alpha\theta = 0 \quad \text{on } \Gamma \times (0,T), \tag{2.13}$$

$$\frac{\partial\varsigma}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{2.14}$$

$$u(0) = u_0, \quad \dot u(0) = w_0, \quad \theta(0) = \theta_0, \quad \varsigma(0) = \varsigma_0 \quad \text{in } \Omega, \tag{2.15}$$

$$\beta(0) = \beta_0 \qquad \text{on } \Gamma_3. \tag{2.16}$$

This problem represents the dynamic evolution of damage and adhesion in thermo-elastic-viscoplastic materials. Equation (2.4) is the thermo-elastic-viscoplastic constitutive law where $\mathcal{A}$ and $\mathcal{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function which describes the viscoplastic behavior of the material. (2.5) represents the equation of motion in which the dot above denotes the derivative with respect to the time variable and ρ is the density of mass. Equation (2.6) represents the energy conservation where ψ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and q is a given volume heat source. Inclusion (2.7) describes the evolution of damage field. Equalities (2.8) and (2.9) are the displacement-traction boundary conditions, respectively. Condition (2.10) represents the normal compliance condition with adhesion where γ_ν is a given adhesion coefficient and p_ν is a given positive function which will be described below. In this condition the interpenetrability between the body and the foundation is allowed, that is u_ν can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\beta^2 R_\nu(u_\nu)$ the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but only as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } \; s < -L, \\ -s & \text{if } \; -L \le s \le 0, \\ 0 & \text{if } \; s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The contact condition (2.10) was used in various papers, see e.g. [9, 10, 34, 37]. Condition (2.11) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and R_τ is the truncation operator given by

$$R_\tau(v) = \begin{cases} v & \text{if } \; |v| \le L, \\ L\frac{v}{|v|} & \text{if } \; |v| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the adhesion field and on the tangential displacement, but only as long as it does not exceed the adhesion length L. The frictional tangential traction is assumed to be much smaller than the adhesive one, and therefore omitted. The introduction of the operator R_ν, together with the operator R_τ defined above, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows.

Next, equation (2.12) represents the ordinary differential equation which describes the evolution of the adhesion field and it was already used in [9, 34], see also [33] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ϵ_a. Notice that in this model once debonding occurs, adhesion cannot be reestablished, since, as it follows from (2.12), $\dot{\beta} \leq 0$. (2.13) and (2.14) represent, respectively a Fourier boundary condition for the temperature and a homogeneous Neumann boundary condition for the damage field on Γ. Finally the functions u_0, w_0, θ_0 and ς_0 in (2.15) and β_0 in (2.16) are the initial data. To obtain the variational formulation of the problem(2.4)–(2.16) we introduce for the adhesive field the set

$$\mathcal{Z} = \{\omega \in L^\infty\big(0,T;L^2(\Gamma_3)\big) :\ 0 \leq \omega(t) \leq 1\ \forall t \in [0,T],\ \text{a.e. on } \Gamma_3\}.$$

In the study of the mechanical problem (P), we consider the following hypotheses.
The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}_n \to \mathbb{S}_n$ satisfies the following properties:

$$\left\{\begin{array}{ll} \text{(a)} & \text{There exists a constant } L_\mathcal{A} > 0 \text{ such that} \\ & |\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)| \leq L_\mathcal{A}|\varepsilon_1 - \varepsilon_2| \text{ for all } \varepsilon_1,\varepsilon_2 \in \mathbb{S}_n,\ \text{a.e. } x \in \Omega. \\ \text{(b)} & \text{There exists a constant } m_\mathcal{A} \text{ such that} \\ & (\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m_\mathcal{A}|\varepsilon_1 - \varepsilon_2|^2 \text{ for all } \varepsilon_1,\varepsilon_2 \in \mathbb{S}_n \text{ a.e. } x \in \Omega. \\ \text{(c)} & \text{The mapping } x \mapsto \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}_n. \\ \text{(d)} & \text{The mapping } x \mapsto \mathcal{A}(x,0) \in \mathcal{H}. \end{array}\right. \tag{2.17}$$

The elasticity operator $\mathcal{E} : \Omega \times \mathbb{S}_n \to \mathbb{S}_n$ satisfies the following properties:

$$\left\{\begin{array}{ll} \text{(a)} & \text{There exists a constant } L_\mathcal{E} > 0 \text{ such that} \\ & |\mathcal{E}(x,\varepsilon_1) - \mathcal{E}(x,\varepsilon_2)| \leq L_\mathcal{E}|\varepsilon_1 - \varepsilon_2| \text{ for all } \varepsilon_1,\varepsilon_2 \in \mathbb{S}_n, \text{a.e. } x \in \Omega. \\ \text{(b)} & \text{The mapping } x \mapsto \mathcal{E}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}_n. \\ \text{(c)} & \text{The mapping } x \mapsto \mathcal{E}(x,0) \in \mathcal{H}. \end{array}\right. \tag{2.18}$$

The viscoplasticity operator $\mathcal{G} : \Omega \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}_n$ satisfies the following properties:

$$\left\{\begin{array}{ll} \text{(a)} & \text{There exists a constant } L_\mathcal{G} > 0 \text{ such that } |\mathcal{G}(x,\sigma_1,\varepsilon_1,\theta_1,\varsigma_1) - \\ & \mathcal{G}(x,\sigma_2,\varepsilon_2,\theta_2,\varsigma_2)| \leq L_\mathcal{G}(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\varsigma_1 - \varsigma_2|) \\ & \text{for all } \sigma_1,\sigma_2 \in \mathbb{S}_n,\ \text{for all } \varepsilon_1,\varepsilon_2 \in \mathbb{S}_n \text{ for all } \theta_1,\theta_2 \in \mathbb{R}, \\ & \text{for all } \varsigma_1,\varsigma_2 \in \mathbb{R},\quad \text{a.e. } x \in \Omega. \\ \text{(b)} & \text{The mapping } x \mapsto \mathcal{G}(\mathbf{x},\sigma,\varepsilon,\theta,\varsigma) \text{ is Lebesgue measurable on } \Omega \\ & \text{for all } \sigma,\varepsilon \in \mathbb{S}_n, \text{for all } \theta,\varsigma \in \mathbb{R}. \\ \text{(c)} & \text{The mapping } x \mapsto \mathcal{G}(x,0,0,0,0) \in \mathcal{H}. \end{array}\right. \tag{2.19}$$

The nonlinear constitutive function $\psi : \Omega \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following properties:

$$\left\{\begin{array}{ll} \text{(a)} & \text{There exists a constant } L_\psi > 0 \text{ such that } |\psi(x,\sigma_1,\varepsilon_1,\theta_1,\varsigma_1) - \\ & \psi(x,\sigma_2,\varepsilon_2,\theta_2,\varsigma_2)| \leq L_\psi(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\varsigma_1 - \varsigma_2|) \\ & \text{for all } \sigma_1,\sigma_2 \in \mathbb{S}_n,\ \text{for all } \varepsilon_1,\varepsilon_2 \in \mathbb{S}_n,\ \text{for all } \theta_1,\theta_2 \in \mathbb{R}, \\ & \text{for all } \varsigma_1,\varsigma_2 \in \mathbb{R} \text{ a.e. } \ x \in \Omega. \\ \text{(b)} & \text{The mapping } x \mapsto \psi(x,\sigma,\varepsilon,\theta,\varsigma) \text{ is Lebesgue measurable on } \Omega \\ & \text{for all } \sigma,\varepsilon \in \mathbb{S}_n,\ \text{for all } \theta,\varsigma \in \mathbb{R}. \\ \text{(c)} & \text{The mapping } x \mapsto \psi(x,0,0,0,0) \in L^2(\Omega). \end{array}\right. \tag{2.20}$$

The damage source function $\phi : \Omega \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following properties:

$$\begin{cases} \text{(a)} & \text{There exists a constant } L_\phi > 0 \text{ such that} \\ & |\phi(x,\sigma_1,\varepsilon_1,\theta_1,\varsigma_1) - \phi(x,\sigma_2,\varepsilon_2,\theta_2,\varsigma_2)| \le L_\phi(|\sigma_1-\sigma_2| + |\varepsilon_1-\varepsilon_2| \\ & +|\theta_1-\theta_2| + |\varsigma_1-\varsigma_2|) \ \text{ for all } \sigma_1,\sigma_2 \in \mathbb{S}_n, \text{for all } \varepsilon_1,\varepsilon_2 \in \mathbb{S}_n, \\ & \text{for all } \theta_1,\theta_2 \in \mathbb{R}, \ \text{for all } \varsigma_1,\varsigma_2 \in \mathbb{R}, \quad \text{a.e. } x \in \Omega. \\ \text{(b)} & \text{The mapping } x \mapsto \phi(x,\sigma,\varepsilon,\theta,\varsigma) \text{ is Lebesgue measurable on } \Omega \\ & \text{for all } \sigma,\varepsilon \in \mathbb{S}_n, \ \text{for all } \theta,\varsigma \in \mathbb{R}. \\ \text{(c)} & \text{The mapping } x \mapsto \phi(x,0,0,0,0) \in L^2(\Omega). \end{cases} \tag{2.21}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a)} & \text{There exists a constant } L_\nu > 0 \text{ such that} \\ & |p_\nu(x,r_1) - p_\nu(x,r_2)| \le L_\nu |r_1 - r_2| \quad \forall \ r_1,r_2 \in \mathbb{R}, \ \text{a.e. } x \in \Gamma_3. \\ \text{(b)} & \text{The mapping } x \mapsto p_\nu(x,r) \text{ is measurable on, } \Gamma_3, \quad \forall r \in \mathbb{R}. \\ \text{(c)} & \text{The mapping } x \mapsto p_\nu(x,r) = 0 \text{ for any } r \le 0, \quad \text{a.e. } x \in \Gamma_3. \end{cases} \tag{2.22}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a)} & \text{There exists a constant } L_\tau > 0 \text{ such that} \\ & \|p_\tau(x,d_1) - p_\tau(x,d_2)| \le L_\tau \ |d_1 - d_2| \ \forall \ d_1,d_2 \in \mathbb{R}, \ \text{a.e. } x \in \Gamma_3, \\ \text{(b)} & \text{There exists a constant } M_\tau > 0 \text{ such that} \\ & |p_\tau(x,d)| \le M_\tau \ \forall \ d \in \mathbb{R}, \ \text{a.e. } x \in \Gamma_3. \\ \text{(c)} & \text{The mapping } x \mapsto p_\tau(x,d) \text{ is measurable on } \Gamma_3, \ \forall \ d \in \mathbb{R}. \\ \text{(d)} & \text{The mapping } x \mapsto p_\tau(x,0) \in L^2(\Gamma_3). \end{cases} \tag{2.23}$$

The mass density satisfies:

$$\rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho \ge \rho^* \text{ a.e. } x \in \Omega. \tag{2.24}$$

The adhesion coefficient and the limit bound satisfy:

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \epsilon_a \ge 0. \tag{2.25}$$

The body forces, surface tractions and the volume heat source have the regularity

$$f \in L^2(0,T;H), \quad f_0 \in L^2(0,T;L^2(\Gamma_2)^n), \ q \in L^2(0,T;L^2(\Omega)), \tag{2.26}$$

$$u_0 \in \mathcal{V}, \quad w_0 \in H, \quad \theta_0 \in V, \quad \varsigma_0 \in K, \tag{2.27}$$

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \le \beta_0 \le 1, \quad \text{a.e on } \Gamma_3, \tag{2.28}$$

$$k_i > 0, \quad i = 0,1. \tag{2.29}$$

We denote by $F(t) \in \mathcal{V}'$ the following element

$$\langle F(t), v \rangle_{\mathcal{V}' \times \mathcal{V}} = (f(t), v)_H + (f_0(t), \gamma v)_{L^2(\Gamma_2)^n} \quad \forall v \in \mathcal{V}, \quad t \in (0,T). \tag{2.30}$$

The use of (2.26) permits to verify that

$$F \in L^2(0,T;\mathcal{V}'). \tag{2.31}$$

We introduce the following continuous functionals

$$\mathfrak{a}_0 : V \times V \to \mathbb{R}, \quad \mathfrak{a}_0(\zeta,\xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi dx + \alpha \int_\Gamma \zeta\xi d\Gamma, \tag{2.32}$$

$$\mathfrak{a}_1 : V \times V \to \mathbb{R}, \quad \mathfrak{a}_1(\zeta,\xi) = k_1 \int_\Omega \nabla\zeta \cdot \nabla\xi dx. \tag{2.33}$$

Finally, we consider the adhesion functional $j : L^\infty(\Gamma_3) \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by

$$j(\beta, u, v) = \int_{\Gamma_3} p_\nu(u_\nu)\, v_\nu da + \int_{\Gamma_3} (-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + p_\tau(\beta) R_\tau(u_\tau).v_\tau) da. \tag{2.34}$$

Keeping in mind (2.22) and (2.23), we observe that integrals in (2.34) are well defined. Using standard arguments based on Green's formula (2.3), we can derive the following variational formulation of the frictionless problem with normal compliance (2.4)−(2.16) as follows.

Problem PV. Find the displacement field $u : [0,T] \to \mathbb{R}^n$, the stress field $\sigma : [0,T] \to \mathbb{S}_n$, the temperature $\theta : [0,T] \to \mathbb{R}$, the damage field $\varsigma : [0,T] \to \mathbb{R}$ and the adhesion field $\beta : [0,T] \to \mathbb{R}$ such that

$$\begin{gathered}\sigma(t) = \mathcal{A}\big(\varepsilon(\dot u(t))\big) + \mathcal{E}\big(\varepsilon(u(t))\big) + \textstyle\int_0^t \mathcal{G}\big(\sigma(s) - \mathcal{A}\big(\varepsilon(\dot u(s))\big), \varepsilon\big(u(s)\big), \theta(s), \varsigma(s)\big) ds \\ \text{a.e. } t \in (0,T),\end{gathered} \tag{2.35}$$

$$\begin{gathered}\langle \rho \ddot u(t), v\rangle_{\mathcal{V}' \times \mathcal{V}} + (\sigma(t), \varepsilon(v))_{\mathcal{H}} + j(\beta(t), u(t), v) = \langle F(t), v\rangle_{\mathcal{V}' \times \mathcal{V}} \\ \forall v \in \mathcal{V},\ \text{a.e. } t \in (0,T),\end{gathered} \tag{2.36}$$

$$\begin{gathered}\langle \rho \dot\theta(t), \omega\rangle_{V' \times V} + \mathfrak{a}_0(\theta(t), \omega) \\ = \big\langle \psi\big(\sigma(t) - \mathcal{A}\big(\varepsilon(\dot u(t))\big), \varepsilon\big(u(t)\big), \theta(t), \varsigma(t)\big), \omega\big\rangle_{V' \times V} + (q(t), \omega)_{L^2(\Omega)} \\ \forall \omega \in V,\ \text{a.e. } t \in (0,T),\end{gathered} \tag{2.37}$$

$$\begin{gathered}\langle \rho \dot\varsigma(t), \xi - \varsigma(t)\rangle_{V' \times V} + \mathfrak{a}_1(\varsigma(t), \xi - \varsigma(t)) \\ \geq \big\langle \phi\big(\sigma(t) - \mathcal{A}\big(\varepsilon(\dot u(t))\big), \varepsilon\big(u(t)\big), \theta(t), \varsigma(t)\big), \xi - \varsigma(t)\big\rangle_{V' \times V} \\ \forall \xi \in K,\ \text{a.e. } t \in (0,T), \varsigma(t) \in K,\end{gathered} \tag{2.38}$$

$$\begin{gathered}\dot\beta(t) = -\big(\beta(t)\big[\gamma_\nu (R_\nu(u_\nu(t)))^2 + \gamma_\tau |R_\tau\big(u_\tau(t)|^2\big)\big] - \varepsilon_a\big)_+ \\ \text{a.e. } t \in (0,T),\end{gathered} \tag{2.39}$$

$$u(0) = u_0, \quad \dot u(0) = w_0, \quad \theta(0) = \theta_0, \quad \varsigma(0) = \varsigma_0, \quad \beta(0) = \beta_0. \tag{2.40}$$

3. Main Results

The existence of the unique solution to Problem PV is proved in the next section. To this end, we consider the following remark which is used in different places of the paper.

Remark 3.1. We note that, in Problem P and in Problem PV, we do not need to impose explicitly the restriction $0 \leq \beta \leq 1$. Indeed, (2.39) guarantees that $\beta(x,t) \leq \beta_0(x)$ and, therefore, assumption (2.28) shows that $\beta(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$.
On the other hand, if $\beta(x,t_0) = 0$ at time t_0, then it follows from (2.39) that $\beta(x,t) = \beta_0(x)$ for all $t \geq t_0$, and therefore $\beta(x,t) = 0$ for all $t \geq t_0, x \in \Gamma_3$.
We conclude that $0 \leq \beta(x,t) \leq 1$ for all $t \geq t_0, x \in \Gamma_3$.

Theorem 3.2 (Existence and uniqueness)**.** *Under assumptions* (2.17)–(2.29)*, there exists a unique solution* $\{u, \sigma, \theta, \varsigma, \beta\}$ *to problem PV. Moreover, the solution has the regularity*

$$u \in \mathcal{C}^0(0,T;\mathcal{V}) \cap \mathcal{C}^1(0,T;H), \tag{3.1}$$

$$\dot u \in L^2(0,T;\mathcal{V}), \tag{3.2}$$

$$\ddot u \in L^2(0,T;\mathcal{V}'), \tag{3.3}$$

$$\sigma \in L^2(0,T;\mathcal{H}), \tag{3.4}$$

$$\theta \in L^2(0,T;V) \cap \mathcal{C}^0(0,T;L^2(\Omega)), \tag{3.5}$$

$$\dot{\theta} \in L^2(0,T;V'), \tag{3.6}$$

$$\varsigma \in L^2(0,T;V) \cap \mathcal{C}^0(0,T;L^2(\Omega)), \tag{3.7}$$

$$\dot{\varsigma} \in L^2(0,T;V'), \tag{3.8}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}. \tag{3.9}$$

A quintuple $(u, \sigma, \theta, \varsigma, \beta)$ which satisfies (2.35)–(2.40) is called a weak solution to the compliance contact Problem P. We conclude that under the stated assumptions, problem (2.4)–(2.16) has a unique weak solution satisfying (3.1)–(3.9).

We turn now to the proof of Theorem 3.2, which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments. To this end, we assume in the following that (2.17)–(2.29) hold. Below, C denotes a generic positive constant which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{A}$, $\mathcal{E}$, $\mathcal{G}$, ψ, ϕ, p_ν, p_τ, γ_ν, γ_τ, L and T but does not depend on t nor on the rest of input data, and whose value may change from place to place. Moreover, for the sake of simplicity we suppress in what follows the explicit dependence of various functions on $x \in \Omega \cup \Gamma$.

Let $\eta \in L^2(0,T;\mathcal{V}')$ be given. In the first step we consider the following variational problem.

Problem PV$_\eta$. Find the displacement field $u_\eta : [0,T] \to \mathbb{R}^n$, such that

$$\begin{gathered}\langle \rho \ddot{u}_\eta(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} + \big(\mathcal{A}\big(\varepsilon(\dot{u}_\eta(t))\big), \varepsilon(v)\big)_{\mathcal{H}} + \langle \eta(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} = \langle F(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} \\ \forall v \in \mathcal{V},\ \text{a.e. } t \in (0,T),\end{gathered} \tag{3.10}$$

$$u_\eta(0) = u_0, \quad \dot{u}_\eta(0) = w_0 \ in\ \Omega. \tag{3.11}$$

Lemma 3.3. *For all $\eta \in L^2(0,T;\mathcal{V}')$, there exists a unique solution u_η to the auxiliary problem PV_η satisfying* (3.1)–(3.3).

Proof. Let us introduce the operator $A : \mathcal{V} \to \mathcal{V}'$,

$$\langle Au, v\rangle_{\mathcal{V}'\times\mathcal{V}} = \big(\mathcal{A}\big(\varepsilon(u)\big), \varepsilon(v)\big)_{\mathcal{H}}. \tag{3.12}$$

Therefore, (3.10) can be rewritten as follows

$$\rho \ddot{u}_\eta(t) + A\big(\dot{u}_\eta(t)\big) = F_\eta(t) \quad \text{on } \mathcal{V}' \text{ a.e. } t \in (0,T), \tag{3.13}$$

where

$$F_\eta(t) = F(t) - \eta(t) \in \mathcal{V}'.$$

It follows from (2.1), (3.12) and hypothesis (2.17) that A is bounded, semi-continuous and coercive on $\mathcal{V}$. We recall that by (2.31) we have $F_\eta \in L^2(0,T;\mathcal{V}')$. Then by using classical arguments of functional analysis concerning parabolic equations [8, 24] we can easily prove the existence and uniqueness of w_η satisfying

$$w_\eta \in L^2(0,T;\mathcal{V}) \cap \mathcal{C}^0(0,T;H), \tag{3.14}$$

$$\dot{w}_\eta \in L^2(0,T;\mathcal{V}'), \tag{3.15}$$

$$\rho \dot{w}_\eta(t) + A(w_\eta(t)) = F_\eta(t) \ \text{ on } \ \mathcal{V}' \ \text{ a.e. } t \in (0,T), \tag{3.16}$$

$$w_\eta(0) = w_0. \tag{3.17}$$

Consider now the function $u_\eta : (0,T) \to \mathcal{V}$ defined by

$$u_\eta(t) = \int_0^t w_\eta(s)ds + u_0 \quad \forall t \in (0,T). \tag{3.18}$$

It follows from (3.16) and (3.17) that u_η is a solution of the equation (3.13) and it satisfies (3.1)–(3.3). $\square$

In the second step we use the displacement field u_η obtained in Lemma 3.3 and we consider the following initial value problem.

Problem $\mathbf{PV}_\beta$. Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot\beta_\eta(t) = -\big(\beta_\eta(t)\big[\gamma_\nu(R_\nu(u_{\eta\nu})(t))^2 + \gamma_\tau|R_\tau(u_{\eta\tau}(t))|^2\big] - \varepsilon_a\big)_+, \tag{3.19}$$

$$\beta_\eta(0) = \beta_0 \ \text{ in } \ \Omega. \tag{3.20}$$

Lemma 3.4. *There exists a unique solution $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$ to Problem PV_β.*

Proof. We use a version of the classical Cauchy–Lipschitz theorem given in [38, p. 60]. $\square$

Problem $\mathbf{PV}_\lambda$. Find the temperature $\theta_\lambda : [0,T] \to \mathbb{R}$ such that

$$\langle \rho\dot\theta_\lambda(t), \omega\rangle_{V'\times V} + \mathfrak{a}_0(\theta_\lambda(t),\omega) = \langle \lambda(t) + q(t), \omega\rangle_{V'\times V} \forall \omega \in V, \ \text{a.e. } t \in (0,T), \tag{3.21}$$

$$\theta_\lambda(0) = \theta_0 \ in \ \Omega. \tag{3.22}$$

Lemma 3.5. *For all $\lambda \in L^2(0,T;V')$, there exists a unique solution θ_λ to the auxiliary problem PV_λ satisfying* (3.5) *and* (3.6).

Proof. By an application of the Poincaré–Friedrichs inequality, we can find a constant $\alpha' > 0$ such that

$$\int_\Omega |\nabla\zeta|^2 dx + \frac{\alpha}{k_0}\int_\Gamma |\zeta|^2 d\gamma \geq \alpha' \int_\Omega |\zeta|^2 dx \quad \forall \zeta \in V.$$

Thus, we obtain

$$\mathfrak{a}_0(\zeta,\zeta) \geq C_1\|\zeta\|_V^2 \quad \forall \zeta \in V, \tag{3.23}$$

where $C_1 = k_0 \min(1,\alpha')/2$, which implies that $\mathfrak{a}_0$ is V-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (3.21) has a unique solution θ_λ satisfies (3.5) and (3.6). $\square$

Problem $\mathbf{PV}_\mu$. Find the damage field $\varsigma_\mu : [0,T] \to \mathbb{R}$ such that

$$\begin{aligned} &\langle \rho\dot\varsigma_\mu(t), \xi - \varsigma_\mu(t)\rangle_{V'\times V} + \mathfrak{a}_1(\varsigma_\mu(t), \xi - \varsigma_\mu(t)) \\ &\geq \langle \mu, \xi - \varsigma_\mu(t)\rangle_{V'\times V} \quad \forall \xi \in K, \ \text{a.e. } t \in (0,T), \ \varsigma_\mu(t) \in K, \end{aligned} \tag{3.24}$$

$$\varsigma_\mu(0) = \varsigma_0 \quad \text{in } \Omega. \tag{3.25}$$

Lemma 3.6. *For all $\mu \in L^2(0,T;V')$, there exists a unique solution ς_μ to the auxiliary problem PV_μ satisfying* (3.7)–(3.8).

Proof. We know that the form $\mathfrak{a}_1$ is not V-elliptic. To solve this problem we introduce the functions

$$\tilde\varsigma_\mu(t) = e^{-k_1 t}\varsigma_\mu(t), \quad \tilde\xi(t) = e^{-k_1 t}\xi(t).$$

We remark that if ς_μ, $\xi \in K$ then $\tilde\varsigma_\mu$, $\tilde\xi \in K$. Consequently, (3.24) is equivalent to the inequality

$$\begin{aligned} &\langle \rho\dot{\tilde\varsigma}_\mu(t), \tilde\xi - \tilde\varsigma_\mu(t)\rangle_{V'\times V} + \mathfrak{a}_1(\tilde\varsigma_\mu(t), \tilde\xi - \tilde\varsigma_\mu(t)) + k_1(\rho\tilde\varsigma_\mu, \tilde\xi - \tilde\varsigma_\mu(t))_{L^2(\Omega)} \\ &\geq \langle e^{-k_1 t}\mu, \tilde\xi - \tilde\varsigma_\mu(t)\rangle_{V'\times V} \quad \forall \tilde\xi \in K, \ \text{a.e. } t \in (0,T), \ \tilde\varsigma_\mu \in K. \end{aligned} \tag{3.26}$$

The fact that

$$\mathfrak{a}_1(\tilde{\xi},\tilde{\xi}) + k_1(\rho\tilde{\xi},\tilde{\xi})_{L^2(\Omega)} \geq k_1 \min(\rho^*,1)\|\tilde{\xi}\|_V^2 \quad \forall \tilde{\xi} \in V, \tag{3.27}$$

and using classical arguments of functional analysis concerning parabolic inequalities [8, 13], implies that (3.24) has a unique solution $\tilde{\varsigma}_\mu$ having the regularity (3.7) and (3.8). □

Let us consider now the auxiliary problem.

Problem PV$_{\eta,\lambda,\mu}$. Find the stress field $\sigma_{\eta,\lambda,\mu} : [0,T] \to \mathbb{S}_n$ which is a solution of the problem

$$\sigma_{\eta,\lambda,\mu}(t) = \mathcal{E}\big(\varepsilon(u_\eta(t))\big) + \int_0^t \mathcal{G}\big(\sigma_{\eta,\lambda,\mu}(s), \varepsilon\big(u_\eta(s)\big), \theta_\lambda(s), \varsigma_\mu(s)\big)ds \quad \forall t \in [0,T]. \tag{3.28}$$

Lemma 3.7. *There exists a unique solution of Problem $PV_{\eta,\lambda,\mu}$ and it satisfies* (3.4)*. Moreover, if $u_{\eta_i}, \theta_{\lambda_i}, \varsigma_{\mu_i}$ and $\sigma_{\eta_i,\lambda_i,\mu_i}$ represent the solutions of problems PV_{η_i}, PV_{λ_i}, PV_{μ_i} and $PV_{\eta_i,\lambda_i,\mu_i}$, respectively, for $i = 1,2$, then there exists $C > 0$ such that*

$$\begin{aligned}
&\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}}^2 \leq C\big(\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_{\mathcal{V}}^2 \\
&+ \int_0^t (\|u_{\eta_1}(s) - u_{\eta_2}(s)\|_{\mathcal{V}}^2 + \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|_V^2 + \|\varsigma_{\mu_1}(s) - \varsigma_{\mu_2}(s)\|_V^2)ds\big).
\end{aligned} \tag{3.29}$$

Proof. Let $\Sigma_{\eta,\lambda,\mu} : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ be the mapping given by

$$\Sigma_{\eta,\lambda,\mu}\sigma(t) = \mathcal{E}\big(\varepsilon(u_\eta(t))\big) + \int_0^t \mathcal{G}\big(\sigma(s), \varepsilon\big(u_\eta(s)\big), \theta_\lambda(s), \varsigma_\mu(s)\big)ds. \tag{3.30}$$

Let $\sigma_i \in L^2(0,T;\mathcal{H})$, $i = 1,2$ and $t_1 \in (0,T)$.
Using hypothesis (2.19) and Hölder's inequality, we find

$$\|\Sigma_{\eta,\lambda,\mu}\sigma_1(t_1) - \Sigma_{\eta,\lambda,\mu}\sigma_2(t_1)\|_{\mathcal{H}}^2 \leq L_{\mathcal{G}}^2 T \int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds. \tag{3.31}$$

By reapplication of mapping $\Sigma_{\eta,\lambda,\mu}$, it yields

$$\left\|\Sigma_{\eta,\lambda,\mu}^2\sigma_1(t_1 - \Sigma_{\eta,\lambda,\mu}^2\sigma_2(t_1)\right\|_{\mathcal{H}}^2 \leq L_{\mathcal{G}}^4 T^2 \int_0^{t_1}\int_0^{t_2} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2\, ds dt_2.$$

Reiterating this inequality m times leads to

$$\left\|\Sigma_{\eta,\lambda,\mu}^m\sigma_1(t_1) - \Sigma_{\eta,\lambda,\mu}^m\sigma_2(t_1)\right\|_{\mathcal{H}}^2 \leq L_{\mathcal{G}}^{2m} T^m \int_0^{t_1}\int_0^{t_2}...\int_0^{t_m} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2\, ds dt_m...dt_2.$$

Integration on the time interval $(0,T)$, it follows that

$$\left\|\Sigma_{\eta,\lambda,\mu}^m\sigma_1 - \Sigma_{\eta,\lambda,\mu}^m\sigma_2\right\|_{L^2(0,T;\mathcal{H})}^2 \leq \frac{L_{\mathcal{G}}^{2m} T^{2m}}{m!} \left\|\sigma_1 - \sigma_2\right\|_{L^2(0,T;\mathcal{H})}^2. \tag{3.32}$$

It follows from this inequality that for m large enough, a power m of the mapping $\Sigma_{\eta,\lambda,\mu}$ is a contraction on the space $L^2(0,T;\mathcal{H})$ and, therefore, from the Banach fixed point theorem, there exists a unique element $\sigma_{\eta,\lambda,\mu} \in L^2(0,T;\mathcal{H})$ such that $\Sigma_{\eta,\lambda,\mu}\sigma_{\eta,\lambda,\mu} = \sigma_{\eta,\lambda,\mu}$, which represents the unique solution of the problem PV$_{\eta,\lambda,\mu}$. Moreover, if $u_{\eta_i}, \theta_{\lambda_i}, \varsigma_{\mu_i}$ and $\sigma_{\eta_i,\lambda_i,\mu_i}$ represent the solutions of the problems

PV_{η_i}, PV_{λ_i} , PV_{μ_i} and $\mathrm{PV}_{\eta_i,\lambda_i,\mu_i}$, respectively, for $i = 1, 2$, then we use (2.1), (2.17)−(2.19) and Young's inequality to obtain

$$\begin{aligned}&\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|^2_{\mathcal{H}}\\ \leq& C\Big(\|u_{\eta_1}(t) - u_{\eta_2}(t)\|^2_{\mathcal{V}} + \int_0^t (\|\sigma_{\eta_1,\lambda_1,\mu_1}(s) - \sigma_{\eta_2,\lambda_2,\mu_2}(s)\|^2_{\mathcal{H}}\\ &+ \|u_{\eta_1}(s) - u_{\eta_2}(s)\|^2_{\mathcal{V}} + \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|^2_V + \|\varsigma_{\mu_1}(s) - \varsigma_{\mu_2}(s)\|^2_V)ds\Big).\end{aligned}$$

Which permits us to obtain, using Gronwall's lemma, the inequality (3.29).
Second step. Let us consider the mapping

$$\Lambda : L^2(0, T; \mathcal{V}' \times V' \times V') \to L^2(0, T; \mathcal{V}' \times V' \times V'),$$

defined by

$$\Lambda\big(\eta(t), \lambda(t), \mu(t)\big) = \big(\Lambda^0(\eta(t), \lambda(t), \mu(t)), \Lambda^1(\eta(t), \lambda(t), \mu(t)), \Lambda^2(\eta(t), \lambda(t), \mu(t))\big), \tag{3.33}$$

where the mappings Λ^0, Λ^1 and Λ^2 are given by

$$\begin{aligned}&\big\langle \Lambda^0\big(\eta(t), \lambda(t), \mu(t)\big), v\big\rangle_{\mathcal{V}'\times\mathcal{V}} = \big(\mathcal{E}\big(\varepsilon(u_\eta(t)), \varepsilon(v)\big)_{\mathcal{H}} + j(\beta_\eta(t), u_\eta(t), v)\\ &+\big(\textstyle\int_0^t \mathcal{G}\big(\sigma_{\eta,\lambda,\mu}(s), \varepsilon\big(u_\eta(s)\big), \theta_\lambda(s), \varsigma_\mu(s)\big)ds, \varepsilon(v)\big)_{\mathcal{H}}\ \forall v \in \mathcal{V},\end{aligned} \tag{3.34}$$

$$\Lambda^1(\eta(t), \lambda(t), \mu(t)) = \psi\big(\sigma_{\eta,\lambda,\mu}(t), \varepsilon(u_\eta(t)), \theta_\lambda(t), \varsigma_\mu(t)\big), \tag{3.35}$$

$$\Lambda^2(\eta(t), \lambda(t), \mu(t)) = \phi\big(\sigma_{\eta,\lambda,\mu}(t), \varepsilon(u_\eta(t)), \theta_\lambda(t), \varsigma_\mu(t)\big). \tag{3.36}$$

□

Lemma 3.8. *The mapping Λ has a fixed point*

$$(\eta^*, \lambda^*, \mu^*) \in L^2(0, T; \mathcal{V}' \times V' \times V').$$

Proof. Let $(\eta_1, \lambda_1, \mu_1), (\eta_2, \lambda_2, \mu_2) \in L^2(0, T; \mathcal{V}' \times V' \times V')$.

We use the notation $u_{\eta_i} = u_i$, $\dot u_{\eta_i} = \dot u_i$, $\ddot u_{\eta_i} = \ddot u_i$, $\beta_{\eta_i} = \beta_i$, $\theta_{\lambda_i} = \theta_i, \varsigma_{\mu_i} = \varsigma_i$ and $\sigma_{\eta_i,\lambda_i,\mu_i} = \sigma_i$, for $i = 1, 2$. Let us start by using (2.1), hypotheses (2.17)–(2.19), (2.21)–(2.23) and the definition of R_η, R_τ and Remark 3.1 we have

$$\begin{aligned}&\|\Lambda^0\big(\eta_1(t), \lambda_1(t), \mu_1(t)\big) - \Lambda^0\big(\eta_2(t), \lambda_2(t), \mu_2(t)\big)\|^2_{\mathcal{V}'} \leq \|\mathcal{E}\big(\varepsilon(u_1(t))\big) - \mathcal{E}\big(\varepsilon(u_2(t))\big)\|^2_{\mathcal{V}}\\ &+ \int_0^t \|\mathcal{G}\big(\sigma_1(s), \varepsilon\big(u_\eta(s)\big), \theta_1(s), \varsigma_1(s)\big) - \mathcal{G}\big(\sigma_2(s), \varepsilon\big(u_2(s)\big), \theta_2(s), \varsigma_2(s)\big)\|^2_{\mathcal{H}} ds\\ &+ C\big(\|p_\nu(u_{1\eta\nu}(t)) - p_\nu(u_{2\eta\nu}(t))\|^2_{L^2(\Gamma_3)}\big)\\ &+ C\big(\|\beta_1^2(t)R_\nu(u_{1\eta\nu}(t)) - \beta_2^2(t)R_\nu(u_{2\eta\nu}(t))\|^2_{L^2(\Gamma_3)}\big)\\ &+ C\big(\|p_\tau(\beta_1(t))R_\tau(u_{1\eta\tau}(t)) - p_\tau(\beta_2(t))R_\tau(u_{2\eta\tau}(t))\|^2_{L^2(\Gamma_3)}\big),\end{aligned}$$

so we obtain

$$\begin{aligned}&\|\Lambda^0(\eta_1(t), \lambda_1(t), \mu_1(t)) - \Lambda^0(\eta_2(t), \lambda_2(t), \mu_2(t))\|^2_{\mathcal{V}'}\\ \leq& C\Big(\int_0^t \big(\|\sigma_1(s) - \sigma_2(s)\|^2_{\mathcal{H}} + \|u_1(s) - u_2(s)\|^2_{\mathcal{V}} + \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)}\\ &+ \|\varsigma_1(s) - \varsigma_2(s)\|^2_{L^2(\Omega)}\big)ds + \|u_1(t) - u_2(t)\|^2_{\mathcal{V}} + \|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)}\Big).\end{aligned} \tag{3.37}$$

We use estimate (3.29) to obtain

$$\begin{aligned}&\|\Lambda^0\big(\eta_1(t),\lambda_1(t),\mu_1(t)\big)-\Lambda^0\big(\eta_2(t),\lambda_2(t),\mu_2(t)\big)\|^2_{\mathcal{V}'}\\ \le& C\Big(\int_0^t\big(\|u_1(s)-u_2(s)\|^2_{\mathcal{V}}+\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\\ &+\|\varsigma_1(s)-\varsigma_2(s)\|^2_{L^2(\Omega)}\big)ds+\|u_1(t)-u_2(t)\|^2_{\mathcal{V}}+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\Big).\end{aligned} \tag{3.38}$$

From the Cauchy problem (3.19)–(3.20) we can write

$$\beta_i(t)=\beta_0-\int_0^t\big(\beta_i(s)\big[\gamma_\nu(R_\nu(u_\nu(s)))^2+\gamma_\tau|R_\tau\big(u_\tau(s)|^2\big)\big]-\varepsilon_a\big)_+ds,$$

and then

$$\begin{aligned}\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\le& C\int_0^t\|\beta_1(s)\big(R_\nu(u_{1\nu}(s))\big)^2-\beta_2(s)\big(R_\nu(u_{2\nu}(s))\big)^2\|_{L^2(\Gamma_3)}ds\\ +C\int_0^t&\|\beta_1(s)|R_\tau(u_{1\tau}(s))|^2-\beta_2(s)|R_\tau(u_{2\tau}(s))|^2\|_{L^2(\Gamma_3)}ds.\end{aligned}$$

Using the definition of R_ν and R_τ and writing $\beta_1=\beta_1-\beta_2+\beta_2$, we get

$$\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\le C\Big(\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}ds+\int_0^t\|u_1(s)-u_2(s)\|_{L^2(\Gamma_3)^d}ds\Big).$$

Next, we apply Gronwall's inequality to deduce

$$\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\le C\int_0^t\|u_1(s)-u_2(s)\|_{L^2(\Gamma_3)^d}ds,$$

and from the relation (2.1) we obtain that

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\le C\int_0^t\|u_1(s)-u_2(s)\|^2_{\mathcal{V}}ds \tag{3.39}$$

holds. On the other hand, since $u_i(t)=u_0+\int_0^t\dot{u}_i(s)ds$, we know that for a.e. $t\in(0,T)$,

$$\|u_1(t)-u_2(t)\|_{\mathcal{V}}\le\int_0^t\|\dot{u}_1(s)-\dot{u}_2(s)\|_{\mathcal{V}}ds. \tag{3.40}$$

Applying Young's and Hölder's inequalities, (3.38) becomes, via (3.39) and (3.40)

$$\begin{aligned}&\|\Lambda^0\big(\eta_1(t),\lambda_1(t),\mu_1(t)\big)-\Lambda^0\big(\eta_2(t),\lambda_2(t),\mu_2(t)\big)\|^2_{\mathcal{V}'}\\ \le& C\Big(\int_0^t\big(\|\dot{u}_1(s)-\dot{u}_2(s)\|^2_{\mathcal{V}}+\|u_1(s)-u_2(s)\|^2_{\mathcal{V}}\\ &+\|\theta_1(s)-\theta_2(s)\|^2_V+\|\varsigma_1(s)-\varsigma_2(s)\|^2_V\big)ds\Big)\quad\text{a.e. } t\in(0,T).\end{aligned} \tag{3.41}$$

Furthermore, we find by taking the substitution $\eta=\eta_1$, $\eta=\eta_2$ in (3.10) and choosing $v=\dot{u}_1-\dot{u}_2$ as test function

$$\begin{aligned}&\langle\rho(\ddot{u}_1(t)-\ddot{u}_2(t))+A\dot{u}_1(t)-A\dot{u}_2(t),\dot{u}_1(t)-\dot{u}_2(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\\ =&\langle\eta_2(t)-\eta_1(t),\dot{u}_1(t)-\dot{u}_2(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\quad\text{a.e. } t\in(0,T).\end{aligned}$$

By virtue of (2.17) and (2.24), this equation becomes

$$\frac{(\rho^*)^2}{2}\frac{d}{dt}\|\dot{u}_1(t)-\dot{u}_2(t)\|^2_H+m_{\mathcal{A}}\|\dot{u}_1(t)-\dot{u}_2(t)\|^2_{\mathcal{V}}\le\|\eta_2(t)-\eta_1(t)\|_{\mathcal{V}'}\|\dot{u}_1(t)-\dot{u}_2(t)\|_{\mathcal{V}}.$$

Integrating this inequality over the interval time variable $(0,t)$, Young's inequality leads to

$$(\rho^*)^2\|\dot u_1(t)-\dot u_2(t)\|_H^2+m_{\mathcal A}\int_0^t\|\dot u_1(s)-\dot u_2(s)\|_{\mathcal V}^2ds\le\frac{2}{m_{\mathcal A}}\int_0^t\|\eta_1(s)-\eta_2(s)\|_{\mathcal V'}^2ds.$$

Consequently,

$$\int_0^t\|\dot u_1(s)-\dot u_2(s)\|_{\mathcal V}^2ds\le C\int_0^t\|\eta_1(s)-\eta_2(s)\|_{\mathcal V'}^2ds\quad\text{a.e. } t\in(0,T),\tag{3.42}$$

which also implies, using a variant of (3.40), that

$$\|u_1(t)-u_2(t)\|_{\mathcal V}^2\le C\int_0^t\|\eta_1(s)-\eta_2(s)\|_{\mathcal V'}^2ds\quad\text{a.e. } t\in(0,T).\tag{3.43}$$

Moreover, if we take the substitution $\lambda=\lambda_1$, $\lambda=\lambda_2$ in (3.21) and subtracting the two obtained equations, we deduce by choosing $\omega=\theta_{\lambda_1}-\theta_{\lambda_2}$ as test function

$$\begin{aligned}&\frac{(\rho^*)^2}{2}\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2+C_1\int_0^t\|\theta_1(s)-\theta_2(s)\|_V^2ds\\ \le&\int_0^t\|\lambda_1(s)-\lambda_2(s)\|_{V'}\|\theta_1(s)-\theta_2(s)\|_Vds\quad\text{a.e. } t\in(0,T).\end{aligned}$$

Employing Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}&\|\theta_{\lambda_1}(t)-\theta_{\lambda_2}(t)\|_{L^2(\Omega)}^2+\int_0^t\|\theta_{\lambda_1}(s)-\theta_{\lambda_2}(s)\|_V^2ds\\ \le&C\int_0^t\|\lambda_1(s)-\lambda_2(s)\|_{V'}^2ds\quad\text{a.e. } t\in(0,T).\end{aligned}\tag{3.44}$$

Substituting now $\{\mu=\mu_1,\xi=\tilde\varsigma_{\mu_1}\}$, $\{\mu=\mu_2,\xi=\tilde\varsigma_{\mu_2}\}$ in (3.26) and subtracting the two inequalities, we obtain

$$\begin{aligned}&\|\tilde\varsigma_1(t)-\tilde\varsigma_2(t)\|_{L^2(\Omega)}^2+\int_0^t\|\tilde\varsigma_1(s)-\tilde\varsigma_2(s)\|_V^2ds\\ \le&C\int_0^t\|e^{-k_1t}(\mu_1(s)-\mu_2(s))\|_{V'}^2ds\quad\text{a.e. } t\in(0,T),\end{aligned}$$

from which also follows that

$$\begin{aligned}&\|\varsigma_1(t)-\varsigma_2(t)\|_{L^2(\Omega)}^2+\int_0^t\|\varsigma_1(s)-\varsigma_2(s)\|_V^2ds\\ \le&C\int_0^t\|\mu_1(s)-\mu_2(s)\|_{V'}^2ds\quad\text{a.e. } t\in(0,T).\end{aligned}\tag{3.45}$$

We can infer, using (3.41)–(3.45), that

$$\begin{aligned}&\|\Lambda^0(\eta_1(t),\lambda_1(t),\mu_1(t))-\Lambda^0(\eta_2(t),\lambda_2(t),\mu_2(t))\|_{\mathcal V'}^2\\ \le&C\big(\|\eta_1(t)-\eta_2(t)\|_{\mathcal V'}^2+\|\lambda_1(t)-\lambda_2(t)\|_{V'}^2+\|\mu_1(t)-\mu_2(t)\|_{V'}^2\big).\end{aligned}\tag{3.46}$$

From hypothesis (2.20), (3.29) and (2.21) it follows

$$\begin{aligned}&\|\Lambda^1(\eta_1(t),\lambda_1(t),\mu_1(t))-\Lambda^1(\eta_2(t),\lambda_2(t),\mu_2(t))\|_{V'}^2\\ =&\|\psi\big(\sigma_1(t),\varepsilon(u_1(t)),\theta_1(t),\varsigma_1(t)\big)-\psi\big(\sigma_2(t),\varepsilon(u_2(t)),\theta_2(t),\varsigma_2(t)\big)\|_{V'}^2\\ \le&C\big(\|u_1(t)-u_2(t)\|_{\mathcal V}^2+\|\theta_1(t)-\theta_2(t)\|_V^2+\|\varsigma_1(t)-\varsigma_2(t)\|_V^2\big)\quad\text{a.e. } t\in(0,T).\end{aligned}$$

This permits us to deduce, via (3.42), (3.44) and (3.45), that

$$\begin{aligned}&\|\Lambda^1(\eta_1(t),\lambda_1(t),\mu_1(t))-\Lambda^1(\eta_2(t),\lambda_2(t),\mu_2(t))\|^2_{V'}\\ \le& C\big(\|\eta_1(t)-\eta_2(t)\|^2_{\mathcal{V}'}+\|\lambda_1(t)-\lambda_2(t)\|^2_{V'}+\|\mu_1(t)-\mu_2(t)\|^2_{V'}\big).\end{aligned} \tag{3.47}$$

Similarly, using (3.29), (3.43), (3.44) and (3.45), we obtain the following estimate for Λ^2

$$\begin{aligned}&\|\Lambda^2(\eta_1(t),\lambda_1(t),\mu_1(t))-\Lambda^2(\eta_2(t),\lambda_2(t),\mu_2(t))\|^2_{V'}\\ =&\|\phi\big(\sigma_1(t),\varepsilon(u_1(t)),\theta_1(t),\varsigma_1(t)\big)-\phi\big(\sigma_2(t),\varepsilon(u_2(t)),\theta_2(t),\varsigma_2(t)\big)\|^2_{V'}\\ \le& C\big(\|\eta_1(t)-\eta_2(t)\|^2_{\mathcal{V}'}+\|\lambda_1(t)-\lambda_2(t)\|^2_{V'}+\|\mu_1(t)-\mu_2(t)\|^2_{V'}\big).\end{aligned} \tag{3.48}$$

From (3.46), (3.47) and (3.48), we conclude that there exists a positive constant $C>0$ verifying

$$\|\Lambda(\eta_1,\lambda_1,\mu_1)-\Lambda(\eta_2,\lambda_2,\mu_2)\|^2_{\mathcal{V}'\times V'\times V'}\le C\|(\eta_1-\eta_2,\lambda_1-\lambda_2,\mu_1-\mu_2)\|^2_{\mathcal{V}'\times V'\times V'}. \tag{3.49}$$

We generalize this procedure by recurrence on m. Then we obtain the formula

$$\begin{aligned}&\|\Lambda^m(\eta_1,\lambda_1,\mu_1)-\Lambda^m(\eta_2,\lambda_2,\mu_2)\|^2_{L^2(0,T;\mathcal{V}'\times V'\times V')}\\ \le&\frac{C^mT^m}{m!}\|(\eta_1-\eta_2,\lambda_1-\lambda_2,\mu_1-\mu_2)\|^2_{L^2(0,T;\mathcal{V}'\times V'\times V')}.\end{aligned} \tag{3.50}$$

Thus, for m sufficiently large, Λ^m is a contraction on $L^2(0,T;\mathcal{V}'\times V'\times V')$. Hence, Banach's fixed point theorem shows that Λ admits a unique fixed point $(\eta^*,\lambda^*,\mu^*)\in L^2(0,T;\mathcal{V}'\times V'\times V')$. □

Now, we have all the ingredients to prove Theorem (3.2).

Proof. Let $(\eta^*,\lambda^*,\mu^*)\in L^2(0,T;\mathcal{V}'\times V'\times V')$ be the fixed point of Λ defined by (3.33)–(3.36) and denote by

$$(a)\ u=u_{\eta^*},\quad (b)\ \theta=\theta_{\lambda^*},\quad (c)\ \varsigma=\varsigma_{\mu^*}, \tag{3.51}$$

$$(a)\ \sigma=\mathcal{A}\varepsilon(\dot u)+\sigma_{\eta^*\lambda^*\mu^*}\ ,\ (b)\ \beta=\beta_{\eta^*}. \tag{3.52}$$

We prove that $(u,\sigma,\theta,\varsigma,\beta)$ satisfies (2.35)–(2.40) and (3.1)–(3.9). Indeed, we write (3.28) for $\eta=\eta^*$, $\lambda=\lambda^*$ and $\mu=\mu^*$ using (3.51) and (3.52)(a) to obtain that (2.35) is satisfied. Now we consider (3.10) for $\eta=\eta^*$ and using (3.51)(a) to find

$$\begin{gathered}\langle\rho\ddot u,v\rangle_{V'\times V}+(\mathcal{A}(\varepsilon(\dot u(t))),\varepsilon(v))_{\mathcal{H}}+\langle\eta^*(t),v\rangle_{V'\times V}=\langle F(t),v\rangle_{V'\times V}\\ \forall v\in V,\text{a.e.}t\in(0,T).\end{gathered} \tag{3.53}$$

Equalities $\Lambda^1(\eta^*,\lambda^*,\mu^*)=\eta^*$, $\Lambda^2(\eta^*,\lambda^*,\mu^*)=\lambda^*$ and $\Lambda^2(\eta^*,\lambda^*,\mu^*)=\mu^*$ combined with (3.34)–(3.36),(3.51) and (3.52) show that

$$\begin{aligned}&\langle\eta^*(t),v\rangle_{\mathcal{V}'\times\mathcal{V}}=\big(\mathcal{E}(\varepsilon(u(t))),\varepsilon(v)\big)_{\mathcal{H}}+j(\beta(t),u(t),v)\\ &+\big(\textstyle\int_0^t\mathcal{G}\big(\sigma(s)-\mathcal{A}(\varepsilon(\dot u(s))),\varepsilon(u(s)),\varsigma(s)\big)ds,\varepsilon(v)\big)_{\mathcal{H}}\ \ \forall v\in\mathcal{V},\end{aligned} \tag{3.54}$$

$$\lambda^*(t)=\psi\big(\sigma(t)-\mathcal{A}(\varepsilon(\dot u(t))),\varepsilon(u(t)),\theta(t),\varsigma(t)\big), \tag{3.55}$$

$$\mu^*(t)=\phi\big(\sigma(t)-\mathcal{A}(\varepsilon(\dot u(t))),\varepsilon(u(t)),\theta(t),\varsigma(t)\big). \tag{3.56}$$

Now we substitute (3.54) in (3.53) and use (2.35) to see that (2.36) is satisfied. We write (3.21) for $\lambda=\lambda^*$ and use (3.51)(b) and (3.55) to find that (2.37) is satisfied, also we write (3.23) for $\mu=\mu^*$ and using (3.51)(c) and (3.56) to find that (2.38) is satisfied. We consider now (3.19) for $\eta=\eta^*$ and use (3.51)(a) and (3.52)(b) to obtain that (2.39) is satisfied. Next (2.40) and the regularities (3.1)–(3.3), (3.5)–(3.9) follow lemmas (3.3), (3.4), (3.5) and (3.6). The regularity (3.4) follows from lemma (3.7). The uniqueness part of theorem (3.2) is a consequence of the uniqueness of the fixed point of the operator Λ defined (3.34)–(3.36) and the unique solvability of the problems PV_η, PV_β, PV_λ, PV_μ and $PV_{\eta,\lambda,\mu}$ which completes the proof. $\square$

Acknowledgements
The authors would like to thank the referee for his useful suggestions.

References

[1] A. Amassad and C. Fabre; *Existence for Viscoplastic Contact with Coulomb Friction Problems*, Int. J. Math. Math. Sci. 32 (2002), 411–437.

[2] A. Amassad, C. Fabre and M. Sofonea; *A Quasistatic Viscoplastic Contact Problem with Normal Compliance and Friction*, IMA Journal of Applied Mathematics 69 (2004), 463–482.

[3] K. T. Andrews, A. Klarbring, M. Shillor and S. Wright; *A Dynamic Thermoviscoelastic Contact Problem with Friction and Wear*, Int. J. Eng. Sci, Vol 35, No 14, 1291–1309. (1997).

[4] Y. Ayyad and M. Sofonea; *Analysis of Two Dynamic Frictionless Contact Problems for Elastic-Visco-Plastic Materials*, Electronic Journal of Differential Equations, Vol. 2007(2007), No. 55, pp.1–17.

[5] A. Azeb Ahmed, S. Boutechebak; *Analysis of a dynamic electro-elastic-viscoplastic contact problem.* Wulfenia Journal Klagenfurt Austria, Vol 20, No. 3; pp. 43–63; Mars 2013.

[6] C. Baiocchi and A. Capelo; *Variational and Quasivariational Inequalities: Application to Free Boundary Problems,* Wiley-Interscience, Chichester-New York; 1984.

[7] E. Bonetti and G. Bonfanti; *Existence and Uniqueness of the Solution to 3D Thermoviscoelastic System,* Electronic Journal of Differential Equations, Vol. 2003, No. 50, pp. 1–15.

[8] H. Brézis; *Equations et Inéquations Non Linéaires dans les Espaces en Dualité,* Annale de l'Institut Fourier, Tome 18, n°1, (1968), p. 115–175.

[9] O. Chau, J. R. Fernández, M. Shillor, M. Sofonea; *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion* , J. Comput. Appl. Math, 159, pp. 431–465, 2003.

[10] O. Chau, M. Shillor, M. Sofonea; *Dynamic frictionless contact with adhesion*, Z. Angew. Math.Phys., 55, pp. 32–47, 2004.

[11] O. Chau, J.R. Fernández, W. Han, M. Sofonea, *A frictionless contact problem for elastic-visco-plastic materials with normal compliance and damage*, Comput. Methods Appl. Mech. Eng., 191, pp. 5007–5026, 2002

[12] N. Cristescu and I. Suliciu; *Viscoplasticity,* Martinus Nijhoff Publishers, Editura Tehnica, Bucharest, (1982).

[13] G. Duvaut and J. L. Lions; *Les Inéquations en Mécanique et en Physique*, Dunod (1976).

[14] J. R. Fernández-García, M. Sofonea and J. M. Viaño; *A Frictionless Contact Problem for Elastic-Viscoplastic Materials with Normal Compliance*, Numerische Mathematik 90 (2002), 689–719.

[15] M. Frémond; *Adhérence des solides, J. Mécanique Théorique et Appliquée*, 6, pp. 383–407, 1987.

[16] M. Frémond; *Equilibre des structures qui adhèrent à leur support* C. R. Acad. Sci. Paris, Sér. II, 295, pp. 913–916, 1982.

[17] M. Frémond, B. Nedjar; *Damage in concrete: The unilateral phenomenon*, Nucl. Eng. Des. 156, pp. 323–335, 1995.

[18] M. Frémond and B. Nedjar; *Damage, Gradient of Damage and Principle of Virtual Work*, Int. J. Solids Structures, 33 (8),1083–1103. (1996).

[19] M. Frémond, K. L. Kuttler, B. Nedjar, M. Shillor; *One-dimensional models of damage*, Adv. Math. Sci. Appl., 8(2), pp. 541-570, 1998.

[20] P. Germain; *Cours de Mécanique des Milieux Continus,* Masson et Cie, Paris, (1973).

[21] I. R. Ionescu and M. Sofonea; *Functional and Numerical Methods in Viscoplasticity*, Oxford University Press, Oxford, 1994.

[22] S. Kobayashi and N. Robelo; *A Coupled Analysis of Viscoplastic Deformation and Heat Transfer: I Theoretical Consideration, II Applications,* Int, J. of Mech. Sci, 22, 699–705, 707–718, (1980).

[23] K. L. Kuttler; *Dynamic Frictional Contact for Elastic Viscoplastic Material*, Electronic Journal of Differential Equations, Vol. 2007(2007), No. 75, pp. 1–20.

[24] J. L. Lions; *Quelques Méthodes de Résolution des Problèmes Aux Limites Non Linéaires*, Dunod (1969).

[25] J. L. Lions and E. Magénes; *Problèmes aux Limites Non Homogènes et Applications*, Volume I, Dunod (1968).

[26] J. Nečas and J. Kratochvil; *On Existence of the Solution Boundary Value Problems for Elastic-Inelastic Solids*, Comment. Math. Univ. Carolinea, 14, 755–760, (1973).

[27] F. Messelmi and B. Merouani; *Quasi-Static Evolution of Damage in Thermo-Viscoplastic Materials,* Analele Universităţii Oradea, Fasc. Mathematica, Tome XVII (2010), Issue No. 2, 133–148.

[28] A. B. Merouani, F. Messelmi; *Dynamic Evolution of Damage in Elastic-Thermo-Viscoplastic Materials.* Electronic Journal of Differential Equations, Vol. 2010(2010), No. 129, pp. 1–15.

[29] F. Messelmi, B. Merouani and M. Meflah; *Nonlinear Thermoelasticity Problem*, Analele Universităţii Oradea, Fasc. Mathematica, Tome XV (2008), 207–217.

[30] F. Messelmi, B. Merouani and F. Bouzeghaya; *Steady-State Thermal Herschel-Bulkley Flow with Tresca's Friction Law,* Electronic Journal of Differential Equations, Vol. 2010(2010), No. 46, pp. 1–14.

[31] M. Raous, L. Cangémi, M. Cocu; *A consistent model coupling adhesion, friction, and unilateral contact*, Comput. Methods Appl. Mech. Eng., 177, pp. 383–399, 1999.

[32] M. Rochdi, M. Shillor and M. Sofonea; *A Quasistatic Viscoelastic Contact Problem with Normal Compliance and Friction,* Journal of Elasticity 51 (1998), 105–126.

[33] J. Rojek, J. J. Telega; *Contact problems with friction, adhesion and wear in orthopaedic biomechanics.* I: General developments, J. Theor. Appl. Mech., 39, pp. 655–677, 2001.

[34] M. Selmani, L. Selmani; *Analysis of a frictionless contact problem for elastic-viscoplastic material.* Nonlinear Analysis: Modelling and Control, 2012, Vol. 17, No. 1, 99–117

[35] M. Sofonea; *Quasistatic Processes for Elastic-Viscoplastic Materials with Internal State Variables*, Annales Scientifiques de l'Université Clermont-Ferrand 2, Tome 94, Série Mathématiques, n° 25. p. 47–60, (1989).

[36] M. Sofonea; *Functional Methods in Thermo-Elasto-Visco-Plasticity,* Ph. D. Thesis, Univ of Bucharest, (1988) (in Romanian).

[37] M. Sofonea, W. Han, M. Shillor; *Analysis and Approximation of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics, Vol. 276, Chapman, Hall/CRC Press, New York, 2006.

[38] P. Suquet, *Plasticité et homogénéisation*, Ph.D. thesis, Université Pierre et Marie Curie, Paris 6, 1982.

(Received June 22, 2013)