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Existence of solution to a periodic boundary value problem for a nonlinear impulsive fractional differential equation

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Abstract. We study the existence of solution to a periodic boundary value problem for nonlinear impulsive fractional differential equations by using Schaeffer's fixed point theorem.

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1 Introduction

In this work, we consider the following periodic boundary value problem for a nonlinear impulsive fractional differential equation

$$D^{\delta}_{t_k+}u(t) - \lambda u(t) = f(t, u(t)), \;\; t \in (t_k, t_{k+1}), \;\; k = 0, \dots, p, \tag{1.1}$$

$$\lim_{t \to t_k^+} (t - t_k)^{1-\delta}(u(t) - u(t_k)) = I_k(u(t_k)), \;\; k = 1, \dots, p, \tag{1.2}$$

$$\lim_{t \to 0^+} t^{1-\delta} u(t) = u(1), \tag{1.3}$$

where $0 < \delta < 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, $D^{\delta}_{t_k+}$ represent the standard Riemann–Liouville fractional derivatives, $I_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, \dots, p$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, f is continuous at every point $(t, u) \in (t_k, t_{k+1}] \times \mathbb{R}$, $k = 0, \dots, p$, and satisfies the following restrictions concerning its behavior on the limit at $t = t_0$ and the impulse instants: for every $k = 0, \dots, p$

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and every function $v \in C(t_k, t_{k+1}]$ such that the limit $\lim_{t\to t_k^+} v(t)$ exists and is finite, then there exists the (finite) limit

$$\lim_{t\to t_k^+} f\big(t,(t-t_k)^{\delta-1}v(t)\big).$$

Note that condition (1.2) is equivalent to

$$\lim_{t\to t_k^+}(t-t_k)^{1-\delta}u(t) = I_k(u(t_k)),\ \ k=1,\ldots,p. \tag{1.4}$$

since $\lim_{t\to t_k^+}(t-t_k)^{1-\delta}u(t_k) = 0$. Thus the limit in (1.2) exists if and only if the limit in (1.4) exists and the value is the same.

The theory of impulsive differential equations has been emerging as an important area of investigations in recent years. For some general aspects of impulsive differential equations, see the classical monographs [14, 21], and Chapter 15 of [18]. From a mathematical point of view, the reader can see, for instance, [6, 8]. Differential equations involving impulsive effects occur in many applications: control theory [2, 9, 10], population dynamics [17], or chemotherapeutic treatment in medicine [13].

Fractional order models are, in some cases, more accurate than integer-order models, i.e., there are more degrees of freedom in the fractional order models. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a 'memory' term in a model. This memory term insures the history and its impact to the present and future. For more details, see [15].

Recently Belmekki *et al.* [5] investigated the existence and uniqueness of solution to the (nonimpulsive) problem

$$D^{\delta}u(t) - \lambda u(t) = f(t,u(t)),\ \ t\in J := (0,1],\ 0<\delta<1,$$

$$\lim_{t\to 0^+} t^{1-\delta}u(t) = u(1),$$

by using the fixed point theorem of Schaeffer and the Banach contraction principle. In [23], the authors consider a different impulsive problem and try to obtain existence and uniqueness results by using the Banach contraction principle. For $\delta = 1$, we refer the reader to the paper by Nieto *et al.* [16]. We cite [1, 7, 22] for some considerations on the concept and existence of solutions to fractional differential equations with impulses. The purpose of this paper is to study the existence of solution to the problem (1.1)–(1.3) by using Schaeffer's fixed point theorem. The results obtained extend in some sense those in [5, 16] and allow some conclusions about the problem studied in [23].

2 Preliminary results

In this section, we introduce the notations, definitions, and preliminary facts which are used throughout this paper.

Let $C(J)$ be the Banach space of all continuous real functions defined on J with the norm $\|f\| := \sup\{|f(t)| : t \in J\}$.

We also introduce the space $PC_r[a,b]$ for a general interval $[a,b]$, a sequence $a = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = b$ and a constant $0 < r < 1$, as follows:

$$PC_r[a,b] := \Big\{ f\colon [a,b] \longrightarrow \mathbb{R} \,:\, t^r f|_{[a,t_1]} \in C[a,t_1],\ (t-t_k)^r f|_{(t_k,t_{k+1}]} \in C(t_k,t_{k+1}],$$
$$k = 1,\ldots,p, \text{ and there exists } \lim_{t\to t_k^+} (t-t_k)^r f(t),\ k = 1,\ldots,p \Big\},$$

which obviously coincides with the set of functions $f\colon [a,b] \longrightarrow \mathbb{R}$ such that

$$f_{r,\{t_k\}}(t) = \begin{cases} t^r f(t), & t \in [a,t_1], \\ (t-t_k)^r f(t), & t \in (t_k,t_{k+1}],\ k = 1,\ldots,p \end{cases}$$

is piecewise continuous on $[a,b]$.

The definition of the space clearly depends on the sequence $\{t_k\}$, but we omit it in the notation for simplicity. Note also that $f(t_k^-) = f(t_k)$ for every $k = 1,\ldots,p$ and $f \in PC_r[a,b]$.

The space $PC_r[a,b]$ turns out to be a Banach space when it is endowed with the norm

$$\|f\|_r = \sup\Big\{ |f_{r,\{t_k\}}(t)| : t \in [a,b] \Big\} = \max_{k=0,\ldots,p} \Big\{ \sup\{(t-t_k)^r |f(t)| : t \in (t_k,t_{k+1}]\} \Big\}.$$

If $r = 0$, then $PC_r[a,b]$ is reduced to $PC[a,b]$ as defined in [4, 14].

Definition 2.1. ([19, 20]). The Riemann–Liouville fractional primitive of order $\delta > 0$ of a function $f : (0,1] \to \mathbb{R}$ is given by

$$I_0^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} f(\tau) d\tau,$$

provided that the right-hand side is pointwise defined on $(0,1]$. Here, Γ is the classical Gamma function.

For instance, I_0^δ exists for all $\delta > 0$, when $f \in C((0,1]) \cap L^1_{loc}(0,1]$; note also that when $f \in C[0,1]$, then $I_0^\delta f \in C[0,1]$ and moreover $I_0^\delta f(0) = 0$.

Recall that the law of composition $I^\delta I^\mu = I^{\delta+\mu}$ holds for all $\delta, \mu > 0$.

Definition 2.2. ([19, 20]). The Riemann–Liouville fractional derivative of order $0 < \delta < 1$ of a function $f\colon (0,1] \to \mathbb{R}$ is given by

$$\begin{aligned} D^\delta f(t) &= \frac{1}{\Gamma(1-\delta)} \frac{d}{dt} \int_0^t (t-\tau)^{-\delta} f(\tau) d\tau \\ &= \frac{d}{dt} I_0^{1-\delta} f(t), \end{aligned}$$

provided the right-hand side is pointwise defined on $(0,1]$.

We have $D^\delta I^\delta f = f$ for all $f \in C(0,1] \cap L^1_{loc}(0,1]$.

Lemma 2.3. ([3, 12]) *Let $0 < \delta < 1$. The fractional differential equation*

$$D^\delta u = 0,\ t \in [0,1]$$

has as solution $u(t) = ct^{\delta-1}$, where c is a real constant.

From this lemma, we deduce the following law of composition.

Proposition 2.4. *Assume that* $f \in C(0,1] \cap L^1_{loc}(0,1]$ *with a fractional derivative of order* $0 < \delta < 1$ *that belongs to* $C(0,1] \cap L^1_{loc}(0,1]$*. Then*

$$I^\delta D^\delta f(t) = f(t) + ct^{\delta-1}$$

for any $c \in \mathbb{R}$.

In this work, we also need the following concepts and properties of fractional primitives and derivatives.

Definition 2.5. ([12, 20]). The Riemann–Liouville fractional primitive of order $\delta > 0$ of a function $f\colon (0,1] \to \mathbb{R}$, $I^\delta_{a+}f$, where $0 \leq a < 1$, is given by

$$I^\delta_{a+}f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-\tau)^{\delta-1} f(\tau)d\tau,\ t > a,$$

provided that the right-hand side is pointwise defined on $(a,1]$.

Definition 2.6. ([12, 20]). The Riemann–Liouville fractional derivative of order $0 < \delta < 1$ of a function $f : (0,1] \to \mathbb{R}$, $D^\delta_{a+}f$, $0 \leq a < 1$, is given by

$$D^\delta_{a+}f(t) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dt} \int_a^t (t-\tau)^{-\delta} f(\tau)\, d\tau,\ t > a,$$

provided that the right-hand side is pointwise defined on $(a,1]$.

An issue which is interesting to our study is the behavior of the fractional primitives and derivatives over polynomials, deduced from the following properties.

Proposition 2.7. ([12, 20]). *If* $\delta \geq 0$ *and* $\beta > 0$*, then*

$$I^\delta_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\delta)}(t-a)^{\beta+\delta-1},\ (\delta > 0)$$

$$D^\delta_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\delta)}(t-a)^{\beta-\delta-1},\ (\delta \geq 0).$$

In particular, the fractional derivative of a constant function is not zero:

$$D^\delta_{a+}1 = \frac{1}{\Gamma(1-\delta)}(t-a)^{-\delta},\ (\delta \geq 0).$$

Moreover, for $j = 1, 2, \ldots, [\delta]+1$,

$$D^\delta_{a+}(t-a)^{\delta-j} = 0.$$

Concerning the impulsive problem of interest, the authors of [23] study the existence of solution to the problem

$$D^\delta u(t) - \lambda u(t) = f(t, u(t)),\ \ t \in J := (0,1],\ t \neq t_1,\ 0 < \delta < 1, \tag{2.1}$$

$$\lim_{t\to 0^+} t^{1-\delta}u(t) = u(1), \tag{2.2}$$

$$\lim_{t\to t_1^+} (t-t_1)^{1-\delta}(u(t) - u(t_1)) = I(u(t_1)), \tag{2.3}$$

where D^δ is the standard Riemann–Liouville fractional derivative, f is continuous at every point $(t,u) \in J_0 \times \mathbb{R}$, $J_0 = J \setminus \{t_1\}$, $0 < t_1 < 1$, $I \in C(\mathbb{R},\mathbb{R})$, $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. They provide an integral characterization of the solutions to problem (2.1)–(2.3) as the fixed points of the mapping $\mathcal{A}$ given by

$$(\mathcal{A}x)(t) = \int_0^1 G_{\lambda,\delta}(t,s) f(s,x(s))\, ds + \Gamma(\delta) G_{\lambda,\delta}(t,t_1) I(x(t_1)),$$

for a certain Green's function $G_{\lambda,\delta}$, and derive sufficient conditions for the existence of a unique solution. Although the approach of using integral formulations is important to the solvability of impulsive problems for fractional differential equations, the difficulty is that the solution to (2.1)–(2.3) is expected to be in the space $PC_{1-\delta}([0,1])$ and, hence, the use of the fractional derivative of u, $D_0^\delta u$, is combined with the possible existence of an 'infinite' jump of u at a point located inside the interval of interest. This produces that, for $x \in PC_{1-\delta}([0,1])$, the function $s \to f(s,x(s))$ is not necessarily continuous on $(0,1]$, due to the assumptions on the nonlinearity f.

In this paper, we propose a new formulation for the impulsive problem for fractional differential equations of Riemann–Liouville type, in terms of problem (1.1)–(1.3) and study, through a different procedure, the existence of solution to this new problem.

We remark that the assumptions imposed in this paper on function f, namely the continuity of f on $(t_k,t_{k+1}] \times \mathbb{R}$, $k = 0,\ldots,p$, and hypothesis

(H) for every $k = 0,\ldots,p$ and $v \in C(t_k,t_{k+1}]$ such that the limit $\lim_{t\to t_k^+} v(t)$ exists and is finite, then there exists the (finite) limit $\lim_{t\to t_k^+} f(t,(t-t_k)^{\delta-1}v(t))$,

guarantee the validity of the following property:

for every $u \in PC_{1-\delta}[0,1]$, the function $t \to f(t,u(t))$ belongs to $PC[0,1]$.

Indeed, for a fixed $u \in PC_{1-\delta}[0,1]$, the function $t^{1-\delta}u(t)$ is continuous on $(0,t_1]$, so that $u(t)$ is also continuous on $(0,t_1]$, thus the continuity of $f(t,u(t))$ on $(0,t_1]$ follows. On the other hand, for $k = 1,\ldots,p$, $(t-t_k)^{1-\delta}u(t)$ is continuous on $(t_k,t_{k+1}]$, hence $u(t)$ and $f(t,u(t))$ are continuous on $(t_k,t_{k+1}]$, by the continuity properties on f. Besides, for $k = 0,\ldots,p$, the limit

$$\lim_{t\to t_k^+} f(t,u(t)) = \lim_{t\to t_k^+} f(t,(t-t_k)^{\delta-1}(t-t_k)^{1-\delta}u(t))$$

exists and it is finite, due to the hypotheses on f and the finiteness of the limit

$$\lim_{t\to t_k^+} (t-t_k)^{1-\delta}u(t).$$

It is obvious that these restrictions on f are fulfilled for the nonlinearity in Example 4.1 [23].

3 Problem with a single impulse point

For simplicity, we focus our attention on the study of the problem

$$D_0^\delta u(t) - \lambda u(t) = f(t,u(t)), \ \ t \in (0,t_1), \tag{3.1}$$

$$D_{t_1+}^\delta u(t) - \lambda u(t) = f(t,u(t)), \ \ t \in (t_1,1), \tag{3.2}$$

$$\lim_{t\to t_1^+} (t-t_1)^{1-\delta}(u(t)-u(t_1)) = I(u(t_1)), \tag{3.3}$$

$$\lim_{t\to 0^+} t^{1-\delta}u(t) = u(1), \tag{3.4}$$

where $0<\delta<1$, $0=t_0<t_1<1$, $D_0^\delta = D_{0+}^\delta$, $D_{t_1+}^\delta$ represent the standard Riemann–Liouville fractional derivatives, $I \in C(\mathbb{R},\mathbb{R})$, $\lambda\in\mathbb{R}$, $\lambda\neq 0$ and f is continuous at every point $(t,u)\in (t_k,t_{k+1}]\times\mathbb{R}$, $k=0,1$, and satisfying the restriction (H) concerning its behavior on the limit at the instants $t=0$ and $t=t_1$, that is:

- for every function $v\in C(0,t_1]$ such that the limit $\lim_{t\to 0^+} v(t)$ exists and it is finite, then there exists the (finite) limit $\lim_{t\to 0^+} f(t,t^{\delta-1}v(t))$; and
- for every function $v\in C(t_1,1]$ such that the limit $\lim_{t\to t_1^+} v(t)$ exists and it is finite, then there exists the (finite) limit $\lim_{t\to t_1^+} f(t,(t-t_1)^{\delta-1}v(t))$.

The space of solutions will be the set $PC_{1-\delta}[0,1]$ of functions $f\colon [0,1]\longrightarrow\mathbb{R}$ such that $f_{1-\delta,t_1}$ is continuous except maybe at $t=t_1$, where it is left-continuous and has finite right-hand limit.

3.1 Some existence and characterization results

The following lemma is useful for the study of the solutions to (3.1)–(3.4).

Lemma 3.1. *Let $0<\delta<1$, $0\le a<b$, $\sigma\in C[a,b]$ and $c\in\mathbb{R}$. Then the unique solution to problem*

$$D_{a+}^\delta u(t) - \lambda u(t) = \sigma(t),\ \ t\in(a,b), \tag{3.5}$$

$$\lim_{t\to a^+}(t-a)^{1-\delta}(u(t)-u(a)) = c, \tag{3.6}$$

is given, for $t\in(a,b]$, by

$$u(t) = c\Gamma(\delta)(t-a)^{\delta-1}E_{\delta,\delta}(\lambda(t-a)^\delta) + \int_a^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^\delta\right)\sigma(s)\,ds. \tag{3.7}$$

Proof. Similar to the results in [5]. Obviously, these considerations provide the existence of solution to problem (3.5)–(3.6). Concerning the uniqueness of solution to (3.5)–(3.6), we refer to the results in [12], where the following general nonlinear fractional differential equation is considered

$$(D_{a+}^\delta y)(x) = f(x,y(x)),\ \Re(\delta)>0,\ x>a,$$

which admits the particular case $f\colon (a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$, $f(x,y)=\lambda y+\sigma(x)$. In this reference [12], the equivalence between the Cauchy type problem for the above-mentioned nonlinear differential equation and a Volterra integral equation is proved (see Theorems 3.1 and 3.10 [12] and also Theorems 1 and 2 [11]). This equivalence is used to prove the uniqueness of solution to the Cauchy problem by adding the Lipschitz type condition (see (3.2.15) [12])

$$|f(x,y_1)-f(x,y_2)|\le A|y_1-y_2|,\ x\in(a,b],\ y_1,y_2\in G,$$

where $A>0$ and G is an open set in $\mathbb{R}$, condition which is trivially fulfilled by $f(x,y)=\lambda y+\sigma(x)$. We refer to Theorems 3.3 and 3.11 [12] for these existence and uniqueness results. On the other hand, in [12, Section 3.3.3], the weighted Cauchy problem is considered for the case $0<\delta<1$, proving the existence and uniqueness of solution to the weighted Cauchy problem accordingly by using the Lipschitz condition (see [12, Theorem 3.12]). □

We also mention the monograph [19], where the fractional Green's function for a differential equation with fractional order and constant coefficients is obtained, getting an expression close to the solution to the homogeneous linear differential equation studied in [5] (see [5, Eq. (3.16)]).

Lemma 3.2. *Let $0 < \delta < 1$, $\sigma \in PC[0,1]$, and c_0, $c_1 \in \mathbb{R}$. Then the unique solution to problem*

$$D^{\delta}_{0+}u(t) - \lambda u(t) = \sigma(t), \;\; t \in (0, t_1), \tag{3.8}$$

$$D^{\delta}_{t_1+}u(t) - \lambda u(t) = \sigma(t), \;\; t \in (t_1, 1), \tag{3.9}$$

$$\lim_{t \to 0^+} t^{1-\delta}(u(t) - u(0)) = c_0, \tag{3.10}$$

$$\lim_{t \to t_1^+} (t - t_1)^{1-\delta}(u(t) - u(t_1)) = c_1, \tag{3.11}$$

is given, for $t \in (0, t_1]$, by

$$u(t) = c_0\Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^{\delta}) + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\sigma(s)\,ds \tag{3.12}$$

and, for $t \in (t_1, 1]$, by

$$u(t) = c_1\Gamma(\delta)(t-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(t-t_1)^{\delta}) + \int_{t_1}^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\sigma(s)\,ds. \tag{3.13}$$

Proof. Obvious from Lemma 3.1. □

Next, we consider the existence of solution to problem (3.1)–(3.4), for a function f which is independent of the second variable, that is, $f(t,u) = \sigma(t)$, as follows:

$$D^{\delta}_{0}u(t) - \lambda u(t) = \sigma(t), \;\; t \in (0, t_1), \tag{3.14}$$

$$D^{\delta}_{t_1+}u(t) - \lambda u(t) = \sigma(t), \;\; t \in (t_1, 1), \tag{3.15}$$

$$\lim_{t \to t_1^+} (t - t_1)^{1-\delta}(u(t) - u(t_1)) = I(u(t_1)), \tag{3.16}$$

$$\lim_{t \to 0^+} t^{1-\delta}u(t) = u(1), \tag{3.17}$$

where, for the rest of the paper, $\sigma \in PC[0,1]$ is piecewise continuous on $[0,1]$, and thus, allowing perhaps finite jump discontinuities at the impulse instants, in this case t_1.

Lemma 3.3. *Problem (3.14)–(3.16) joint to the condition*

$$\lim_{t \to 0^+} t^{1-\delta}u(t) = c_0 \tag{3.18}$$

has a unique solution $u(t)$ given by

$$u(t) = \begin{cases} c_0\Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^{\delta}) + \displaystyle\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\sigma(s)\,ds, & t \in (0, t_1], \\ I(u(t_1))\Gamma(\delta)(t-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(t-t_1)^{\delta}) & \\ \qquad + \displaystyle\int_{t_1}^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\sigma(s)\,ds, & t \in (t_1, 1]. \end{cases} \tag{3.19}$$

Proof. From the study in [5] (also Lemma 3.1 or Lemma 3.2), the solution to (3.14) joint to condition (3.18) is given by

$$u(t) = c_0\Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^{\delta}) + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\sigma(s)\,ds. \tag{3.20}$$

Hence $I(u(t_1)) = I\left(c_0\Gamma(\delta)t_1^{\delta-1}E_{\delta,\delta}(\lambda t_1^{\delta}) + \int_0^{t_1}(t_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t_1-s)^{\delta}\right)\sigma(s)\,ds\right)$. Next, the solution to the equation (3.15)–(3.16) is obtained by applying Lemma 3.1 (or Lemma 3.2). □

The integral characterization of the solution to the impulsive equation subject to an 'initial condition' given in Lemma 3.3 allows to obtain some conclusions for the periodic boundary value problem (3.14)–(3.17). In this sense, taking an appropriate 'initial value' c_0 for its replacement in expression (3.19), we can derive some immediate consequences concerning existence and uniqueness results for problem (3.14)–(3.17). The idea is to find which are the adequate numbers $c_0 \in \mathbb{R}$ for which the solution to problem (3.14)–(3.16) subject to the initial condition $\lim_{t\to 0^+} t^{1-\delta}u(t) = c_0$ satisfies that $u(1) = c_0$. These appropriate choices for c_0 are those which would make true the periodic boundary condition (3.17) and the corresponding solution can also be calculated by using (3.19).

Lemma 3.4. *Consider the function ϕ defined by*

$$c_0 \longrightarrow \phi(c_0) = \mathcal{R}(c_0)\Gamma(\delta)(1-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_1)^{\delta}) + \int_{t_1}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\sigma(s)\,ds,$$

where

$$\mathcal{R}(c_0) = I\left(c_0\Gamma(\delta)t_1^{\delta-1}E_{\delta,\delta}(\lambda t_1^{\delta}) + \int_0^{t_1}(t_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t_1-s)^{\delta}\right)\sigma(s)\,ds\right).$$

Problem (3.14)–(3.17) has solutions if and only if $Fix(\phi)$ is nonempty. In that case, the solutions to problem (3.14)–(3.17) are given by the expression (3.19), where $c_0 \in \mathbb{R}$ is any fixed point of the mapping ϕ.

Note that, in the previous lemma, $\phi(c_0)$ coincides with $u(1)$ for the solution u in (3.19). This way, the fixed points of ϕ are those 'initial conditions' for which $u(1) = c_0$. This way, to solve the periodic boundary value problem, we just write $u(1)$ as a function of c_0, which is possible by using the composition of several functions, $u(1) = \psi(I(\varphi(c_0)))$, being ψ, φ linear and I the impulse function.

Proposition 3.5. *If the impulse function I is linear, $I(x) = \mu x$, for some $\mu \in \mathbb{R}$, and*

$$K := 1 - \mu(\Gamma(\delta))^2 t_1^{\delta-1}E_{\delta,\delta}(\lambda t_1^{\delta})(1-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_1)^{\delta}) \neq 0,$$

then the periodic boundary value problem (3.14)–(3.17) has a unique solution given by (3.19), where

$$\begin{aligned} c_0 = {} & \frac{\mu}{K}\Gamma(\delta)(1-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_1)^{\delta})\int_0^{t_1}(t_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t_1-s)^{\delta}\right)\sigma(s)ds \\ & + \frac{1}{K}\int_{t_1}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\sigma(s)ds. \end{aligned}$$

Proof. It is deduced from the identity $\phi(c_0) = c_0$, where ϕ is given in Lemma 3.4, that is,

$$\begin{aligned} & I\left(c_0\Gamma(\delta)t_1^{\delta-1}E_{\delta,\delta}(\lambda t_1^{\delta}) + \int_0^{t_1}(t_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t_1-s)^{\delta}\right)\sigma(s)\,ds\right) \\ & \qquad \times\Gamma(\delta)(1-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_1)^{\delta}) + \int_{t_1}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\sigma(s)\,ds \\ & = c_0, \end{aligned}$$

which is equivalent to

$$c_0\left[1-\mu(\Gamma(\delta))^2 t_1^{\delta-1}E_{\delta,\delta}(\lambda t_1^{\delta})(1-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_1)^{\delta})\right]$$
$$=\mu\Gamma(\delta)(1-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_1)^{\delta})\int_0^{t_1}(t_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t_1-s)^{\delta}\right)\sigma(s)\,ds$$
$$+\int_{t_1}^{1}(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\sigma(s)\,ds$$

and, under the hypotheses imposed, the solution to the boundary value problem is uniquely determined. □

However, if $K=0$, then the boundary value problem is solvable if and only if the right-hand side in the previous expression is null, obtaining an infinite number of solutions corresponding to any value of $c_0 \in \mathbb{R}$. This is a problem at resonance and will be considered in the future.

Remark 3.6. The case $\mu = 1$ ($I(x) = x$, for every $x \in \mathbb{R}$) corresponds, in the ordinary case $\delta = 1$, to a nonimpulsive problem $u(t_1+) = u(t_1)$. The peculiarities of fractional differential equations force the non-continuous behavior of the solution at $t = t_1$, even for $\mu = 1$, since

$$\lim_{t\to t_1^+}(t-t_1)^{1-\delta}u(t)=u(t_1).$$

Remark 3.7. For I nonlinear, problem (3.14)–(3.17) is also nonlinear and, to deduce the existence of solution, we prove the existence of fixed points for function ϕ defined in Lemma 3.4 without obtaining their explicit expression.

Lemma 3.8. *If there exists $l > 0$ such that $|I(u) - I(v)| \le l|u - v|$, $\forall t \in [0, 1]$ and $u, v \in \mathbb{R}$ and, moreover,*

$$l\,(\Gamma(\delta))^2\, t_1^{\delta-1}\,|E_{\delta,\delta}(\lambda t_1^{\delta})|\,(1-t_1)^{\delta-1}|E_{\delta,\delta}(\lambda(1-t_1)^{\delta})|<1, \tag{3.21}$$

then problem (3.14)–(3.17) has a unique solution given by (3.19), for $c_0 \in \mathbb{R}$ the unique fixed point of the mapping ϕ defined in Lemma 3.4.

Proof. For $b_0, c_0 \in \mathbb{R}$, we get, from the definitions of ϕ, $\mathcal{R}$ in Lemma 3.4,

$$|\phi(b_0)-\phi(c_0)|=|\mathcal{R}(b_0)-\mathcal{R}(c_0)|\,\Gamma(\delta)(1-t_1)^{\delta-1}|E_{\delta,\delta}(\lambda(1-t_1)^{\delta})|$$
$$\le l|b_0-c_0|\,(\Gamma(\delta))^2\,t_1^{\delta-1}\,|E_{\delta,\delta}(\lambda t_1^{\delta})|\,(1-t_1)^{\delta-1}|E_{\delta,\delta}(\lambda(1-t_1)^{\delta})|$$

and the conclusion follows. □

Lemma 3.9. *If I is continuous and bounded, then problem (3.14)–(3.17) has at least one solution.*

Proof. Note that ϕ in Lemma 3.4 is a continuous mapping. Let $m > 0$ be such that $|I(u)| \le m$, $\forall\, u \in \mathbb{R}$ and choose $A > 0$ such that

$$A\ge m\Gamma(\delta)(1-t_1)^{\delta-1}|E_{\delta,\delta}(\lambda(1-t_1)^{\delta})|+\left|\int_{t_1}^{1}(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\sigma(s)\,ds\right|.$$

Then, the restriction of ϕ to the nonempty compact and convex set $[-A, A]$ takes values in $[-A, A]$ since, for $c_0 \in [-A, A]$,

$$|\phi(c_0)|\le|\mathcal{R}(c_0)|\Gamma(\delta)(1-t_1)^{\delta-1}|E_{\delta,\delta}(\lambda(1-t_1)^{\delta})|+\left|\int_{t_1}^{1}(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\sigma(s)\,ds\right|$$
$$\le m\Gamma(\delta)(1-t_1)^{\delta-1}|E_{\delta,\delta}(\lambda(1-t_1)^{\delta})|+\left|\int_{t_1}^{1}(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\sigma(s)\,ds\right|\le A.$$

In consequence, by Schauder's theorem, there exists a fixed point c_0 of ϕ in $[-A, A]$, which gives a solution to (3.14)–(3.17) through (3.19). □

Lemma 3.10. *If I is continuous and there exists $A > 0$ satisfying*

$$m\Gamma(\delta)(1-t_1)^{\delta-1}|E_{\delta,\delta}(\lambda(1-t_1)^{\delta})| + \left|\int_{t_1}^{1}(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\sigma(s)\,ds\right| \leq A,$$

where $m > 0$ is such that $\mathcal{R}([-A, A]) \subseteq [-m, m]$ ($\mathcal{R}$ given in Lemma 3.4), then problem (3.14)–(3.17) has at least one solution.

Proof. Note that, from the continuity of I, $\mathcal{R}([-B, B])$ is a compact set in $\mathbb{R}$, for every $B \in \mathbb{R}$. □

To deal with a problem where the nonlinearity depends on the second variable, we could extend Lemma 3.3 to the context of the more general type of right-hand side $f(t, u(t))$ in the equation, as follows:

Lemma 3.11. *Solutions to problem (3.1)–(3.3) joint to the condition (3.18) are the solutions of the integral equation*

$$u(t) = \begin{cases} c_0\Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^{\delta}) + \displaystyle\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)f(s,u(s))\,ds, & t \in (0, t_1], \\ I(u(t_1))\Gamma(\delta)(t-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(t-t_1)^{\delta}) & \\ \quad + \displaystyle\int_{t_1}^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)f(s,u(s))\,ds, & t \in (t_1, 1]. \end{cases} \tag{3.22}$$

However, the approach followed previously is not useful for equations with a general right-hand side depending on u, since it is not possible to avoid this dependence in the definition of the mapping ϕ.

For a different approach to the problem which will allow to deal with a nonlinearity f, we first consider the periodic boundary value problem

$$D_0^{\delta}u(t) - \lambda u(t) = \sigma(t), \ \ t \in (0, t_1), \tag{3.23}$$

$$D_{t_1+}^{\delta}u(t) - \lambda u(t) = \sigma(t), \ \ t \in (t_1, 1), \tag{3.24}$$

$$\lim_{t\to t_1^+}(t-t_1)^{1-\delta}u(t) = c_1, \tag{3.25}$$

$$\lim_{t\to 0^+} t^{1-\delta}u(t) = u(1), \tag{3.26}$$

whose solution is given by

$$u(t) = \begin{cases} c_0\Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^{\delta}) + \displaystyle\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\sigma(s)\,ds, & t \in (0, t_1], \\ c_1\Gamma(\delta)(t-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(t-t_1)^{\delta}) & \\ \quad + \displaystyle\int_{t_1}^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\sigma(s)\,ds, & t \in (t_1, 1], \end{cases} \tag{3.27}$$

where

$$c_0 = c_1\Gamma(\delta)(1-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_1)^{\delta}) + \int_{t_1}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\sigma(s)\,ds. \tag{3.28}$$

Next, we write the solution (3.27) in integral form, obtaining the Green's function associated to the boundary value problem (3.23)–(3.26).

Proposition 3.12. *The solution to* (3.23)–(3.26) *can be written as*

$$u(t) = c_1\Gamma(\delta)G_{\lambda,\delta}(t,t_1) + \int_0^1 G_{\lambda,\delta}(t,s)\sigma(s)\,ds, \quad t \in (0,1], \tag{3.29}$$

where $G_{\lambda,\delta}(t,s)$ *is defined, for* $(t,s) \in (0,1] \times [0,1]$, *by*

$$G_{\lambda,\delta}(t,s) = \begin{cases} \Gamma(\delta)t^{\delta-1}E_{\delta,\delta}\left(\lambda t^{\delta}\right)(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right), & \text{if } 0 < t \le t_1 \le s < 1, \\ (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right), & \text{if } 0 < t \le t_1,\ 0 \le s < t, \\ (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right), & \text{if } t_1 < t \le 1,\ t_1 \le s < t, \\ 0, & \text{otherwise.} \end{cases} \tag{3.30}$$

Proof. It is deduced from (3.27) and (3.28), taking into account that

$$G_{\lambda,\delta}(t,t_1) = \begin{cases} \Gamma(\delta)t^{\delta-1}E_{\delta,\delta}\left(\lambda t^{\delta}\right)(1-t_1)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-t_1)^{\delta}\right), & \text{if } 0 < t \le t_1, \\ (t-t_1)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-t_1)^{\delta}\right), & \text{if } t_1 < t \le 1. \end{cases}$$

□

Lemma 3.13. *The solutions to problem* (3.1)–(3.4) *are characterized by*

$$u(t) = I(u(t_1))\Gamma(\delta)G_{\lambda,\delta}(t,t_1) + \int_0^1 G_{\lambda,\delta}(t,s)f(s,u(s))\,ds, \quad t \in (0,1],$$

so that they are the fixed points of the mapping $\mathcal{B}$ *defined as*

$$[\mathcal{B}u](t) = I(u(t_1))\Gamma(\delta)G_{\lambda,\delta}(t,t_1) + \int_0^1 G_{\lambda,\delta}(t,s)f(s,u(s))\,ds, \quad t \in (0,1], \tag{3.31}$$

where $G_{\lambda,\delta}$ *is given by* (3.30).

Remark 3.14. The mapping $\mathcal{B}$ has an expression similar to the operator $\mathcal{A}$ defined in equation (3.2) [23], but the Green's function $G_{\lambda,\delta}$ is different.

Remark 3.15. From (3.27) and (3.28), it is clear that the expression of the mapping $\mathcal{B}$ can be expanded as

$$\begin{aligned} [\mathcal{B}u](t) = {} & I(u(t_1))(\Gamma(\delta))^2 t^{\delta-1}E_{\delta,\delta}(\lambda t^{\delta})(1-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_1)^{\delta}) \\ & + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right) f(s,u(s))\,ds \\ & + \Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^{\delta}) \int_{t_1}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right) f(s,u(s))\,ds, \ t \in (0,t_1], \end{aligned} \tag{3.32}$$

$$\begin{aligned} [\mathcal{B}u](t) = {} & I(u(t_1))\Gamma(\delta)(t-t_1)^{\delta-1}E_{\delta,\delta}(\lambda(t-t_1)^{\delta}) \\ & + \int_{t_1}^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right) f(s,u(s))\,ds, \ t \in (t_1,1]. \end{aligned} \tag{3.33}$$

3.2 Analysis of the nonlinear problem

In this section, we shall be concerned with the existence and uniqueness of solution to the nonlinear impulsive boundary value problem (3.1)–(3.4). To this end, we use the following fixed point theorem of Schaeffer.

Theorem 3.16. *Assume X to be a normed linear space, and let the operator $F\colon X \to X$ be compact. Then either*

i) the operator F has a fixed point in X, or

ii) the set $\mathcal{E} = \{u \in X : u = \mu F(u),\ \mu \in (0,1)\}$ is unbounded.

We define the operator $\mathcal{B} : PC_{1-\delta}[0,1] \to PC_{1-\delta}[0,1]$ by expression (3.31) in such a way that problem (3.1)–(3.4) has solutions if and only if the operator equation $\mathcal{B}u = u$ has fixed points.

Lemma 3.17. *Suppose that the following conditions hold:*

(H1) There exist positive constants M and m such that

$$|f(t,u)| \leq M,\ |I(u)| \leq m,\ \forall t \in [0,1],\ u \in \mathbb{R}. \tag{3.34}$$

(H2) There exist positive constants k and l such that

$$|f(t,u) - f(t,v)| \leq k|u-v|,\ |I(u) - I(v)| \leq l|u-v|,\ \forall t \in [0,1],\ u,v \in \mathbb{R}. \tag{3.35}$$

Then the operator $\mathcal{B}$ defined in Lemma 3.13 is well-defined, continuous and compact.

Proof. (a) First, we prove that the mapping $\mathcal{B}$ is well-defined, that is, $\mathcal{B}u \in PC_{1-\delta}[0,1]$, for every $u \in PC_{1-\delta}[0,1]$. We take $u \in PC_{1-\delta}[0,1]$ and prove that $t^{1-\delta}\mathcal{B}(u)(t)|_{[0,t_1]} \in C[0,t_1]$, $(t - t_1)^{1-\delta}\mathcal{B}(u)(t)|_{(t_1,1]} \in C(t_1,1]$ and the existence of the limit

$$\lim_{t\to t_1^+} (t-t_1)^{1-\delta}\mathcal{B}(u)(t).$$

Indeed, for any $0 < \tau_1 < \tau_2 \leq t_1$, we have

$$\left|\tau_1^{1-\delta}\mathcal{B}(u)(\tau_1) - \tau_2^{1-\delta}\mathcal{B}(u)(\tau_2)\right| \to 0, \text{ as } |\tau_1 - \tau_2| \to 0,$$

which is derived from (H1) and the inequality

$$\begin{aligned}
&\left|\tau_1^{1-\delta}\mathcal{B}(u)(\tau_1) - \tau_2^{1-\delta}\mathcal{B}(u)(\tau_2)\right| \\
&\quad = \left|\tau_1^{1-\delta} I(u(t_1))\Gamma(\delta)G_{\lambda,\delta}(\tau_1,t_1) + \tau_1^{1-\delta}\int_0^1 G_{\lambda,\delta}(\tau_1,s)f(s,u(s))\,ds\right. \\
&\qquad \left. - \tau_2^{1-\delta} I(u(t_1))\Gamma(\delta)G_{\lambda,\delta}(\tau_2,t_1) - \tau_2^{1-\delta}\int_0^1 G_{\lambda,\delta}(\tau_2,s)f(s,u(s))\,ds\right| \\
&\quad \leq m\Gamma(\delta)\left|\tau_1^{1-\delta}G_{\lambda,\delta}(\tau_1,t_1) - \tau_2^{1-\delta}G_{\lambda,\delta}(\tau_2,t_1)\right| \\
&\qquad + \left|\tau_1^{1-\delta}\int_0^1 G_{\lambda,\delta}(\tau_1,s)f(s,u(s))\,ds - \tau_2^{1-\delta}\int_0^1 G_{\lambda,\delta}(\tau_2,s)f(s,u(s))\,ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\le m(\Gamma(\delta))^2 \left|E_{\delta,\delta}\left(\lambda\tau_1^\delta\right) - E_{\delta,\delta}\left(\lambda\tau_2^\delta\right)\right| (1-t_1)^{\delta-1} \left|E_{\delta,\delta}\left(\lambda(1-t_1)^\delta\right)\right| \\
&\quad + \left|\tau_1^{1-\delta}\int_0^{\tau_1}(\tau_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_1-s)^\delta\right)f(s,u(s))\,ds\right. \\
&\qquad \left.-\tau_2^{1-\delta}\int_0^{\tau_1}(\tau_2-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_2-s)^\delta\right)f(s,u(s))\,ds\right| \\
&\quad + \left|\tau_2^{1-\delta}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_2-s)^\delta\right)f(s,u(s))\,ds\right| \\
&\quad +\Gamma(\delta)\left|\int_{t_1}^1 E_{\delta,\delta}\left(\lambda\tau_1^\delta\right)(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^\delta\right)f(s,u(s))\,ds\right. \\
&\qquad \left. - \int_{t_1}^1 E_{\delta,\delta}\left(\lambda\tau_2^\delta\right)(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^\delta\right)f(s,u(s))\,ds\right| \\
&\le m(\Gamma(\delta))^2 \left|E_{\delta,\delta}\left(\lambda\tau_1^\delta\right) - E_{\delta,\delta}\left(\lambda\tau_2^\delta\right)\right| (1-t_1)^{\delta-1} \left|E_{\delta,\delta}\left(\lambda(1-t_1)^\delta\right)\right| \\
&\quad +M\int_0^{\tau_1}\left|\tau_1^{1-\delta}(\tau_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_1-s)^\delta\right) - \tau_2^{1-\delta}(\tau_2-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_2-s)^\delta\right)\right|\,ds \\
&\quad +M\tau_2^{1-\delta}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(\tau_2-s)^\delta\right)\right|\,ds \\
&\quad +M\left|E_{\delta,\delta}\left(\lambda\tau_1^\delta\right) - E_{\delta,\delta}\left(\lambda\tau_2^\delta\right)\right|\Gamma(\delta)\int_{t_1}^1(1-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(1-s)^\delta\right)\right|\,ds.
\end{aligned}
$$

It is obvious that the first term on the right-hand side of the previous inequality tends to zero as $|\tau_1-\tau_2|\to 0$. From the calculations in section (a) of Lemma 4.1 proof in [5], the second term tends to zero as $|\tau_1-\tau_2|\to 0$ and, similarly to the calculations in (a) (proof of Lemma 4.1 in [5]) and those in [23], the two other terms also tend to zero as $|\tau_1-\tau_2|\to 0$ due to

$$\int_{t_1}^1(1-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(1-s)^\delta\right)\right|\,ds \le E_{\delta,\delta+1}(|\lambda|) \tag{3.36}$$

and

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(\tau_2-s)^\delta\right)\right|\,ds &\le \sum_{j=0}^{\infty}\frac{|\lambda|^j}{\Gamma(\delta j+\delta)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\delta j+\delta-1}\,ds \\
&=\sum_{j=0}^{\infty}\frac{|\lambda|^j}{\Gamma(\delta j+\delta)}\frac{(\tau_2-\tau_1)^{\delta j+\delta}}{\delta j+\delta} =(\tau_2-\tau_1)^\delta E_{\delta,\delta+1}\left(|\lambda|(\tau_2-\tau_1)^\delta\right).
\end{aligned} \tag{3.37}
$$

Besides, for $\tau_1,\ \tau_2\in(t_1,1]$ with $\tau_1<\tau_2$, we prove that

$$\left|(\tau_1-t_1)^{1-\delta}\mathcal{B}(u)(\tau_1) - (\tau_2-t_1)^{1-\delta}\mathcal{B}(u)(\tau_2)\right|$$

tends to 0 as $|\tau_1-\tau_2|\to 0$. Indeed, for $t_1<\tau_1<\tau_2\le 1$, from (H1) and following the calculations in Lemma 3.1 in [23], we get

$$
\begin{aligned}
&\left|(\tau_1-t_1)^{1-\delta}\mathcal{B}(u)(\tau_1) - (\tau_2-t_1)^{1-\delta}\mathcal{B}(u)(\tau_2)\right| \\
&\quad = \left|(\tau_1-t_1)^{1-\delta}I(u(t_1))\Gamma(\delta)G_{\lambda,\delta}(\tau_1,t_1) + (\tau_1-t_1)^{1-\delta}\int_0^1 G_{\lambda,\delta}(\tau_1,s)f(s,u(s))\,ds\right. \\
&\qquad \left. -(\tau_2-t_1)^{1-\delta}I(u(t_1))\Gamma(\delta)G_{\lambda,\delta}(\tau_2,t_1) - (\tau_2-t_1)^{1-\delta}\int_0^1 G_{\lambda,\delta}(\tau_2,s)f(s,u(s))\,ds\right|
\end{aligned}
$$

$$\begin{aligned}&\leq m\Gamma(\delta)\left|(\tau_1-t_1)^{1-\delta}G_{\lambda,\delta}(\tau_1,t_1)-(\tau_2-t_1)^{1-\delta}G_{\lambda,\delta}(\tau_2,t_1)\right|\\&\quad+M\int_0^1\left|(\tau_1-t_1)^{1-\delta}G_{\lambda,\delta}(\tau_1,s)-(\tau_2-t_1)^{1-\delta}G_{\lambda,\delta}(\tau_2,s)\right|ds.\end{aligned}$$

At this point, the calculations differ from [23], since the Green's function is defined in (3.30), then

$$\begin{aligned}&\left|(\tau_1-t_1)^{1-\delta}\mathcal{B}(u)(\tau_1)-(\tau_2-t_1)^{1-\delta}\mathcal{B}(u)(\tau_2)\right|\\&\leq m\Gamma(\delta)\left|E_{\delta,\delta}\left(\lambda(\tau_1-t_1)^{\delta}\right)-E_{\delta,\delta}\left(\lambda(\tau_2-t_1)^{\delta}\right)\right|\\&\quad+M\int_{t_1}^{\tau_1}\left|(\tau_1-t_1)^{1-\delta}(\tau_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_1-s)^{\delta}\right)\right.\\&\qquad\left.-(\tau_2-t_1)^{1-\delta}(\tau_2-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_2-s)^{\delta}\right)\right|ds\\&\quad+M(\tau_2-t_1)^{1-\delta}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(\tau_2-s)^{\delta}\right)\right|ds.\end{aligned}$$

Again, the first term in the right-hand side of the previous inequality tends to zero as $|\tau_1-\tau_2|\to 0$ and analogously with the last term due to (3.37). Finally, the integral term (multiplying M) is bounded by

$$\begin{aligned}&(\tau_1-t_1)^{1-\delta}\int_{t_1}^{\tau_1}\left|(\tau_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_1-s)^{\delta}\right)-(\tau_2-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_2-s)^{\delta}\right)\right|ds\\&\quad+\left|(\tau_1-t_1)^{1-\delta}-(\tau_2-t_1)^{1-\delta}\right|\int_{t_1}^{\tau_1}(\tau_2-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(\tau_2-s)^{\delta}\right)\right|ds.\end{aligned}$$

It is obvious that $\left|(\tau_1-t_1)^{1-\delta}-(\tau_2-t_1)^{1-\delta}\right|$ tends to zero as $|\tau_1-\tau_2|\to 0$ and, similarly to (3.37),

$$\begin{aligned}&\int_{t_1}^{\tau_1}(\tau_2-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(\tau_2-s)^{\delta}\right)\right|ds\\&\quad\leq\int_{t_1}^{\tau_2}(\tau_2-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(\tau_2-s)^{\delta}\right)\right|ds\leq(\tau_2-t_1)^{\delta}\,E_{\delta,\delta+1}\left(|\lambda|(\tau_2-t_1)^{\delta}\right)\\&\quad\leq(1-t_1)^{\delta}\,E_{\delta,\delta+1}\left(|\lambda|(1-t_1)^{\delta}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{t_1}^{\tau_1}\left|(\tau_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_1-s)^{\delta}\right)-(\tau_2-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_2-s)^{\delta}\right)\right|ds\\&\quad\leq\int_0^{\tau_1}\left|(\tau_1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_1-s)^{\delta}\right)-(\tau_2-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(\tau_2-s)^{\delta}\right)\right|ds\\&\quad=\int_0^{\tau_1}\left|\sum_{j=0}^{\infty}\frac{\lambda^j}{\Gamma(\delta j+\delta)}\left((\tau_1-s)^{\delta j+\delta-1}-(\tau_2-s)^{\delta j+\delta-1}\right)\right|ds\\&\quad\leq\sum_{j=0}^{\infty}\frac{|\lambda|^j}{\Gamma(\delta j+\delta)}\int_0^{\tau_1}\left|(\tau_1-s)^{\delta j+\delta-1}-(\tau_2-s)^{\delta j+\delta-1}\right|ds,\end{aligned}$$

as justified in [23], estimate which is of the type of the last term in equation (4.8) [5] and, in consequence, it has limit zero as $|\tau_1-\tau_2|\to 0$ due to inequalities (4.13) and (4.14) in [5] (analogously to the reasoning in [23]).

Moreover,

$$\begin{aligned}
&\lim_{t\to t_1^+}(t-t_1)^{1-\delta}\mathcal{B}(u)(t) \\
&\qquad = \lim_{t\to t_1^+}\Big(I(u(t_1))\Gamma(\delta)E_{\delta,\delta}(\lambda(t-t_1)^{\delta}) \\
&\qquad\qquad + (t-t_1)^{1-\delta}\int_{t_1}^{t}(t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)f(s,u(s))\,ds\Big).
\end{aligned}$$

Similarly to (3.37),

$$\begin{aligned}
\left|\int_{t_1}^{t}(t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)f(s,u(s))\,ds\right| &\le M\int_{t_1}^{t}(t-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\right|ds \\
\le M(t-t_1)^{\delta}\,E_{\delta,\delta+1}\left(|\lambda|(t-t_1)^{\delta}\right) &\le M(1-t_1)^{\delta}\,E_{\delta,\delta+1}\left(|\lambda|(1-t_1)^{\delta}\right)
\end{aligned}$$

and, in consequence,

$$\lim_{t\to t_1^+}(t-t_1)^{1-\delta}\mathcal{B}(u)(t) = I(u(t_1))\Gamma(\delta)\lim_{t\to t_1^+}E_{\delta,\delta}(\lambda(t-t_1)^{\delta}) = I(u(t_1)).$$

It is also clear that the limit $\lim_{t\to 0^+} t^{1-\delta}\mathcal{B}(u)(t)$ is finite (and equal to $\mathcal{B}(u)(1)$), due to (H1), the expressions (3.32) (and (3.33)) and the boundedness of the middle term in (3.32):

$$\left|\int_{0}^{t}(t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)f(s,u(s))\,ds\right| \le M\int_{0}^{t}(t-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\right|ds,$$

where, similarly to the calculations in (4.24) [5], we get, for $t\in(0,t_1]$,

$$\begin{aligned}
\int_{0}^{t}(t-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\right|ds &\le \sum_{j=0}^{\infty}\frac{|\lambda|^{j}}{\Gamma(\delta j+\delta)}\int_{0}^{t}(t-s)^{\delta j+\delta-1}\,ds \\
&= \sum_{j=0}^{\infty}\frac{|\lambda|^{j}}{\Gamma(\delta j+\delta)}\frac{t^{\delta j+\delta}}{\delta j+\delta} = t^{\delta}\sum_{j=0}^{\infty}\frac{(|\lambda|t^{\delta})^{j}}{\Gamma(\delta j+\delta+1)} \\
&= t^{\delta}\,E_{\delta,\delta+1}\left(|\lambda|t^{\delta}\right) \le t_1^{\delta}\,E_{\delta,\delta+1}\left(|\lambda|t_1^{\delta}\right) \le E_{\delta,\delta+1}\left(|\lambda|\right).
\end{aligned} \tag{3.38}$$

Therefore, $\mathcal{B}$ is well-defined.

(b) Now, we prove that $\mathcal{B}$ is continuous. Similarly to the expression (3.11) in [23], for $t\in(0,t_1]$,

$$\begin{aligned}
&t^{1-\delta}|\mathcal{B}(u)(t)-\mathcal{B}(v)(t)| \\
&\quad \le \|u-v\|_{1-\delta}\Big[lt^{1-\delta}t_1^{\delta-1}\Gamma(\delta)|G_{\lambda,\delta}(t,t_1)| + kt^{1-\delta}\int_{0}^{t_1}|G_{\lambda,\delta}(t,s)|s^{\delta-1}\,ds \\
&\qquad\qquad + kt^{1-\delta}\int_{t_1}^{1}|G_{\lambda,\delta}(t,s)|(s-t_1)^{\delta-1}\,ds\Big].
\end{aligned}$$

However, the Green's function is different from that in [23]. Note that, from the expression of $G_{\lambda,\delta}$, we get, for $t\in(0,t_1]$,

$$\begin{aligned}
t^{1-\delta}|G_{\lambda,\delta}(t,t_1)| &= \Gamma(\delta)\left|E_{\delta,\delta}\left(\lambda t^{\delta}\right)\right|(1-t_1)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(1-t_1)^{\delta}\right)\right| \\
&\le \Gamma(\delta)(1-t_1)^{\delta-1}E_{\delta,\delta}\left(|\lambda|t_1^{\delta}\right)E_{\delta,\delta}\left(|\lambda|(1-t_1)^{\delta}\right) = F_0.
\end{aligned} \tag{3.39}$$

Moreover, for $t \in (0, t_1]$,

$$t^{1-\delta} \int_0^{t_1} |G_{\lambda,\delta}(t,s)| s^{\delta-1}\, ds = t^{1-\delta} \int_0^t (t-s)^{\delta-1} \left| E_{\delta,\delta}\left(\lambda (t-s)^\delta\right)\right| s^{\delta-1}\, ds$$
$$\leq t^{1-\delta} \sum_{j=0}^{\infty} \frac{|\lambda|^j}{\Gamma(\delta j+\delta)} \int_0^t s^{\delta-1}(t-s)^{\delta j+\delta-1}\, ds = t^{1-\delta} \sum_{j=0}^{\infty} |\lambda|^j t^{\delta j+2\delta-1} \frac{\Gamma(\delta)}{\Gamma(\delta j+2\delta)}$$
$$= t^\delta \Gamma(\delta) \sum_{j=0}^{\infty} \frac{(|\lambda| t^\delta)^j}{\Gamma(\delta j+2\delta)} \leq t_1^\delta \Gamma(\delta)\, E_{\delta,2\delta}\left(|\lambda| t^\delta\right) \leq t_1^\delta \Gamma(\delta)\, E_{\delta,2\delta}\left(|\lambda| t_1^\delta\right) = F_1$$

and

$$t^{1-\delta} \int_{t_1}^1 |G_{\lambda,\delta}(t,s)| (s-t_1)^{\delta-1}\, ds$$
$$= \Gamma(\delta) \left| E_{\delta,\delta}\left(\lambda t^\delta\right)\right| \int_{t_1}^1 (1-s)^{\delta-1} \left| E_{\delta,\delta}\left(\lambda (1-s)^\delta\right)\right| (s-t_1)^{\delta-1}\, ds$$
$$\leq \Gamma(\delta)\, E_{\delta,\delta}\left(|\lambda|\right) E_{\delta,\delta}\left(|\lambda| (1-t_1)^\delta\right) \int_{t_1}^1 (1-s)^{\delta-1} (s-t_1)^{\delta-1}\, ds$$
$$= (1-t_1)^{2\delta-1} \frac{(\Gamma(\delta))^3}{\Gamma(2\delta)}\, E_{\delta,\delta}\left(|\lambda|\right) E_{\delta,\delta}\left(|\lambda| (1-t_1)^\delta\right) = F_2,$$

where we have used (4.21) in [5] or (3.14) in [23]. Therefore, for $t \in (0, t_1]$,

$$t^{1-\delta} |\mathcal{B}(u)(t) - \mathcal{B}(v)(t)| \leq \|u-v\|_{1-\delta} \left[l\, t_1^{\delta-1} \Gamma(\delta) F_0 + k F_1 + k F_2 \right].$$

Similarly to the procedure in [23] but attending to the particularities of the Green's function, we get for $t \in (t_1, 1]$,

$$(t-t_1)^{1-\delta} |\mathcal{B}(u)(t) - \mathcal{B}(v)(t)|$$
$$\leq (t-t_1)^{1-\delta} |I(u(t_1)) - I(v(t_1))| \Gamma(\delta) |G_{\lambda,\delta}(t,t_1)|$$
$$+ (t-t_1)^{1-\delta} \int_0^1 |G_{\lambda,\delta}(t,s)|\, |f(s,u(s)) - f(s,v(s))|\, ds$$
$$\leq (t-t_1)^{1-\delta} l |u(t_1) - v(t_1)| \Gamma(\delta) |G_{\lambda,\delta}(t,t_1)| + (t-t_1)^{1-\delta} \int_{t_1}^t |G_{\lambda,\delta}(t,s)|\, k |u(s) - v(s)|\, ds$$
$$\leq \|u-v\|_{1-\delta} \left[(t-t_1)^{1-\delta} l t_1^{\delta-1} \Gamma(\delta) |G_{\lambda,\delta}(t,t_1)| + k (t-t_1)^{1-\delta} \int_{t_1}^t |G_{\lambda,\delta}(t,s)|\, (s-t_1)^{\delta-1}\, ds \right].$$

Now, we have, for $t \in (t_1, 1]$,

$$(t-t_1)^{1-\delta} |G_{\lambda,\delta}(t,t_1)| = |E_{\delta,\delta}\left(\lambda (t-t_1)^\delta\right)| \leq E_{\delta,\delta}\left(|\lambda| (1-t_1)^\delta\right)$$

and

$$(t-t_1)^{1-\delta} \int_{t_1}^t |G_{\lambda,\delta}(t,s)|\, (s-t_1)^{\delta-1}\, ds$$
$$= (t-t_1)^{1-\delta} \int_{t_1}^t (t-s)^{\delta-1} \left| E_{\delta,\delta}\left(\lambda (t-s)^\delta\right)\right| (s-t_1)^{\delta-1}\, ds$$
$$\leq (t-t_1)^{1-\delta} E_{\delta,\delta}\left(|\lambda| (t-t_1)^\delta\right) \int_{t_1}^t (t-s)^{\delta-1} (s-t_1)^{\delta-1}\, ds$$
$$= (t-t_1)^\delta \frac{(\Gamma(\delta))^2}{\Gamma(2\delta)}\, E_{\delta,\delta}\left(|\lambda| (t-t_1)^\delta\right) \leq (1-t_1)^\delta \frac{(\Gamma(\delta))^2}{\Gamma(2\delta)}\, E_{\delta,\delta}\left(|\lambda| (1-t_1)^\delta\right),$$

which implies that

$$\sup_{t\in(t_1,1]} (t-t_1)^{1-\delta}|\mathcal{B}(u)(t)-\mathcal{B}(v)(t)| \\ \leq \|u-v\|_{1-\delta}\left[l\, t_1^{\delta-1}+k(1-t_1)^{\delta}\frac{\Gamma(\delta)}{\Gamma(2\delta)}\right]\Gamma(\delta)E_{\delta,\delta}\left(|\lambda|(1-t_1)^{\delta}\right).$$

This proves that

$$\|\mathcal{B}(u)-\mathcal{B}(v)\|_{1-\delta}\leq \mathcal{L}\|u-v\|_{1-\delta},\quad u,v\in PC_{1-\delta}[0,1],$$

where

$$\mathcal{L}=\max\left\{l\, t_1^{\delta-1}\Gamma(\delta)F_0+kF_1+kF_2,\left[l\, t_1^{\delta-1}+k(1-t_1)^{\delta}\frac{\Gamma(\delta)}{\Gamma(2\delta)}\right]\Gamma(\delta)E_{\delta,\delta}\left(|\lambda|(1-t_1)^{\delta}\right)\right\} \tag{3.40}$$

and, in particular, $\mathcal{B}$ is continuous.

(c) Next, we prove that $\mathcal{B}$ is a compact mapping. Let D be a bounded set in $PC_{1-\delta}[0,1]$.

(i) First, we check that $\{\mathcal{B}(u):\ u\in D\}$ is a bounded set in $PC_{1-\delta}[0,1]$.
Indeed, for $t\in(0,t_1]$ and using $(H1)$, we have

$$t^{1-\delta}|\mathcal{B}(u)(t)|\leq t^{1-\delta}|I(u(t_1))|\Gamma(\delta)|G_{\lambda,\delta}(t,t_1)|+t^{1-\delta}\int_0^1|G_{\lambda,\delta}(t,s)||f(s,u(s))|\,ds \\ \leq t^{1-\delta}m\Gamma(\delta)|G_{\lambda,\delta}(t,t_1)|+Mt^{1-\delta}\int_0^1|G_{\lambda,\delta}(t,s)|\,ds.$$

Note that, from (3.39), for $t\in(0,t_1]$,

$$t^{1-\delta}|G_{\lambda,\delta}(t,t_1)|\leq\Gamma(\delta)(1-t_1)^{\delta-1}E_{\delta,\delta}\left(|\lambda|t_1^{\delta}\right)E_{\delta,\delta}\left(|\lambda|(1-t_1)^{\delta}\right)=F_0$$

and, moreover,

$$t^{1-\delta}\int_0^1|G_{\lambda,\delta}(t,s)|\,ds=t^{1-\delta}\int_0^t(t-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\right|\,ds \\ +\Gamma(\delta)\left|E_{\delta,\delta}\left(\lambda t^{\delta}\right)\right|\int_{t_1}^1(1-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(1-s)^{\delta}\right)\right|\,ds, \tag{3.41}$$

where, from (3.38),

$$t^{1-\delta}\int_0^t(t-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\right|\,ds \\ \leq t\,E_{\delta,\delta+1}\left(|\lambda|t^{\delta}\right)\leq t_1\,E_{\delta,\delta+1}\left(|\lambda|t_1^{\delta}\right)\leq E_{\delta,\delta+1}\left(|\lambda|\right)$$

and an estimate can be provided for the second term in (3.41) by (3.36). Hence

$$\sup_{t\in(0,t_1]}t^{1-\delta}|\mathcal{B}(u)(t)| \\ \leq m\Gamma(\delta)F_0+Mt_1\,E_{\delta,\delta+1}\left(|\lambda|t_1^{\delta}\right)+M\Gamma(\delta)\,E_{\delta,\delta}\left(|\lambda|t_1^{\delta}\right)E_{\delta,\delta+1}\left(|\lambda|\right)=M_1. \tag{3.42}$$

On the other hand, for $t \in (t_1, 1]$, we have, by $(H1)$,

$$\begin{aligned}
(t-t_1)^{1-\delta}|\mathcal{B}(u)(t)| &\le (t-t_1)^{1-\delta}|I(u(t_1))|\,\Gamma(\delta)\,|G_{\lambda,\delta}(t,t_1)| \\
&\quad + (t-t_1)^{1-\delta}\int_0^1 |G_{\lambda,\delta}(t,s)||f(s,u(s))|\,ds \\
&\le m\Gamma(\delta)(t-t_1)^{1-\delta}\,|G_{\lambda,\delta}(t,t_1)| \\
&\quad + M(t-t_1)^{1-\delta}\int_0^1 |G_{\lambda,\delta}(t,s)|\,ds \\
&\le m\Gamma(\delta)E_{\delta,\delta}\left(|\lambda|(1-t_1)^{\delta}\right) \\
&\quad + M(t-t_1)^{1-\delta}\int_{t_1}^t (t-s)^{\delta-1}\left|E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)\right|\,ds,
\end{aligned}$$

where $(t-t_1)^{1-\delta}\int_{t_1}^t (t-s)^{\delta-1}|E_{\delta,\delta}\left(\lambda(t-s)^{\delta}\right)|\,ds$ is bounded by

$$\begin{aligned}
&(t-t_1)^{1-\delta}\sum_{j=0}^{\infty}\frac{|\lambda|^j}{\Gamma(\delta j+\delta)}\int_{t_1}^t (t-s)^{\delta j+\delta-1}\,ds = (t-t_1)^{1-\delta}\sum_{j=0}^{\infty}\frac{|\lambda|^j}{\Gamma(\delta j+\delta)}\frac{(t-t_1)^{\delta j+\delta}}{\delta j+\delta} \\
&= (t-t_1)\,E_{\delta,\delta+1}\left(|\lambda|(t-t_1)^{\delta}\right) \le (1-t_1)E_{\delta,\delta+1}\left(|\lambda|(1-t_1)^{\delta}\right) \le E_{\delta,\delta+1}\left(|\lambda|\right)
\end{aligned}$$

and, hence,

$$\begin{aligned}
&\sup_{t\in(t_1,1]}(t-t_1)^{1-\delta}|\mathcal{B}(u)(t)| \\
&\quad \le m\Gamma(\delta)E_{\delta,\delta}\left(|\lambda|(1-t_1)^{\delta}\right) + M(1-t_1)E_{\delta,\delta+1}\left(|\lambda|(1-t_1)^{\delta}\right) = M_2.
\end{aligned} \tag{3.43}$$

In consequence, for all $u \in D$,

$$\|\mathcal{B}(u)\|_{1-\delta} \le \max\left\{\sup_{t\in(0,t_1]} t^{1-\delta}|u(t)|,\ \sup_{t\in(t_1,1]}(t-t_1)^{1-\delta}|u(t)|\right\} = \max\{M_1,\, M_2\} < \infty.$$

(ii) Now, we obtain that $\{\mathcal{B}(u) : \ u \in D\}$ is an equicontinuous set in $PC_{1-\delta}[0,1]$, which can be deduced from the calculations in (a).

□

Theorem 3.18. *Assume that conditions $(H1)$ and $(H2)$ hold. Then the problem (3.1)–(3.4) has at least one solution in $PC_{1-\delta}[0,1]$.*

Proof. Consider the set $\mathcal{E} = \{u \in PC_{1-\delta}[0,1] : u = \mu\mathcal{B}(u),\ \mu \in (0,1)\}$.
Let u be any element of $\mathcal{E}$, then $u = \mu\mathcal{B}(u)$ for some $\mu \in (0,1)$. Thus, for each $t \in (0, t_1]$, we have

$$t^{1-\delta}|u(t)| = t^{1-\delta}\mu\,|\mathcal{B}(u)(t)| \le M_1 < \infty,$$

by (3.42) and, using (3.43), we get, for $t \in (t_1, 1]$,

$$(t-t_1)^{1-\delta}|u(t)| = (t-t_1)^{1-\delta}\mu\,|\mathcal{B}(u)(t)| \le M_2 < \infty,$$

which implies that

$$\|u\|_{1-\delta} \le \|\mathcal{B}(u)\|_{1-\delta} \le \max\{M_1,\, M_2\} < \infty$$

and the set $\mathcal{E}$ is bounded independently of $\mu \in (0,1)$. Using Lemma 3.17 and Theorem 3.16, we obtain that the operator $\mathcal{B}$ has at least one fixed point. □

Remark 3.19. Under the hypotheses of Theorem 3.18, if $\mathcal{L}$ given in (3.40) is less than 1, then there exists a unique solution to problem (3.1)–(3.4). This comes from the Contraction Mapping Theorem applied to the mapping $\mathcal{B}$ which is proved to be Lipschitzian in section (b) of Lemma 3.17, with Lipschitz constant $\mathcal{L}$.

4 Problem with finitely many impulses

In this section, we study the existence of solution to the general problem (1.1)–(1.3), where $0 < \delta < 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, \ldots, p$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and function $f \in C((t_k, t_{k+1}] \times \mathbb{R})$, for every $k = 0, \ldots, p$, satisfies restriction (H) concerning its behavior on the limit at $t = t_0$ and the impulse instants, that is, for every $k = 0, \ldots, p$ and every function $v \in C(t_k, t_{k+1}]$ such that the limit $\lim_{t \to t_k^+} v(t)$ exists and is finite, then there exists the (finite) limit

$$\lim_{t \to t_k^+} f(t, (t - t_k)^{\delta - 1} v(t)).$$

Lemmas 3.2 and 3.3 can be easily extended to the case of a multi impulsive problem, as follows.

Lemma 4.1. *Let $0 < \delta < 1$, $k \in \{0, 1, \ldots, p\}$, $\sigma \in C[t_k, t_{k+1}]$ and $c_k \in \mathbb{R}$. Then the unique solution to problem*

$$D^{\delta}_{t_k+} u(t) - \lambda u(t) = \sigma(t), \ \ t \in (t_k, t_{k+1}), \tag{4.1}$$

$$\lim_{t \to t_k^+} (t - t_k)^{1-\delta} (u(t) - u(t_k)) = c_k, \tag{4.2}$$

is given, for $t \in (t_k, t_{k+1}]$, by

$$u(t) = c_k \Gamma(\delta)(t - t_k)^{\delta - 1} E_{\delta,\delta}(\lambda (t - t_k)^{\delta}) + \int_{t_k}^{t} (t - s)^{\delta - 1} E_{\delta,\delta} \left(\lambda (t - s)^{\delta} \right) \sigma(s) ds. \tag{4.3}$$

Proof. Similar to the results in [5]. See also the proof of Lemma 3.1. □

Lemma 4.2. *The solution to*

$$D^{\delta}_{t_k+} u(t) - \lambda u(t) = \sigma(t), \ \ t \in (t_k, t_{k+1}), \ \ k = 1, \ldots, p, \tag{4.4}$$

$$\lim_{t \to t_k^+} (t - t_k)^{1-\delta} (u(t) - u(t_k)) = I_k(u(t_k)), \ \ k = 1, \ldots, p, \tag{4.5}$$

$$\lim_{t \to 0^+} t^{1-\delta} u(t) = c_0 \tag{4.6}$$

is given by

$$\begin{aligned} u(t) &= c_0 \Gamma(\delta) t^{\delta - 1} E_{\delta,\delta}(\lambda t^{\delta}) + \int_0^t (t - s)^{\delta - 1} E_{\delta,\delta} \left(\lambda (t - s)^{\delta} \right) \sigma(s) \, ds, \ t \in (0, t_1], \\ u(t) &= I_k(u(t_k)) \Gamma(\delta)(t - t_k)^{\delta - 1} E_{\delta,\delta}(\lambda (t - t_k)^{\delta}) \\ &\quad + \int_{t_k}^{t} (t - s)^{\delta - 1} E_{\delta,\delta} \left(\lambda (t - s)^{\delta} \right) \sigma(s) ds, \ t \in (t_k, t_{k+1}], \quad k = 1, \ldots, p. \end{aligned} \tag{4.7}$$

Lemma 4.3. *The solutions to the problem (4.4)–(4.5) subject to the boundary condition (3.17) are given by (4.7), for $c_0 \in \mathbb{R}$ a fixed point of the mapping $\tilde{\phi}$*

$$c_0 \longrightarrow \tilde{\phi}(c_0) = \tilde{v}_p(c_0)\Gamma(\delta)(1-t_p)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_p)^\delta) + \int_{t_p}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^\delta\right)\sigma(s)\,ds,$$

where function $\tilde{v}_p$ is defined through the recursive formula

$$\tilde{v}_j(c_0) = I_j\bigg(\tilde{v}_{j-1}(c_0)\Gamma(\delta)(t_j-t_{j-1})^{\delta-1}E_{\delta,\delta}(\lambda(t_j-t_{j-1})^\delta) + \int_{t_{j-1}}^{t_j}(t_j-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t_j-s)^\delta\right)\sigma(s)\,ds\bigg),$$

for $c_0 \in \mathbb{R}$, $j = 1,\ldots,p$ and $\tilde{v}_0(c_0) = c_0$.

In consequence, if there exist $l_1, l_2, \ldots, l_p$ positive constants such that

$$|I_j(u) - I_j(v)| \le l_j|u-v| \text{ for all } u,v \in \mathbb{R} \text{ and } j = 1,\ldots,p$$

and, moreover,

$$(\Gamma(\delta))^{p+1}\prod_{k=1}^{p} l_k \prod_{k=1}^{p+1}\left[(t_k-t_{k-1})^{\delta-1}\,|E_{\delta,\delta}(\lambda(t_k-t_{k-1})^\delta)|\right] < 1, \tag{4.8}$$

then there exists a unique solution to the problem (4.4)–(4.5) subject to the boundary condition (3.17).

Proof. We prove, by induction, that for $b_0, c_0 \in \mathbb{R}$ and $j = 1,\ldots,p$,

$$|\tilde{v}_j(b_0) - \tilde{v}_j(c_0)| \le (\Gamma(\delta))^j \prod_{k=1}^{j} l_k \prod_{k=1}^{j}\left[(t_k-t_{k-1})^{\delta-1}\,|E_{\delta,\delta}(\lambda(t_k-t_{k-1})^\delta)|\right]|b_0-c_0|. \tag{4.9}$$

Indeed, take $b_0, c_0 \in \mathbb{R}$. For $j = 1$, we have

$$|\tilde{v}_1(b_0) - \tilde{v}_1(c_0)| \le \Gamma(\delta)\, l_1\, t_1^{\delta-1}\,|E_{\delta,\delta}(\lambda t_1^\delta)|\,|b_0-c_0|,$$

which coincides with (4.9) for $j = 1$. Next, we suppose that (4.9) is true for $j-1$, that is,

$$|\tilde{v}_{j-1}(b_0) - \tilde{v}_{j-1}(c_0)| \le (\Gamma(\delta))^{j-1}\prod_{k=1}^{j-1} l_k \prod_{k=1}^{j-1}\left[(t_k-t_{k-1})^{\delta-1}\,|E_{\delta,\delta}(\lambda(t_k-t_{k-1})^\delta)|\right]\,|b_0-c_0|$$

and we deduce that

$$\begin{aligned}|\tilde{v}_j(b_0) - \tilde{v}_j(c_0)| &\le l_j\,|\tilde{v}_{j-1}(b_0) - \tilde{v}_{j-1}(c_0)|\,\Gamma(\delta)(t_j-t_{j-1})^{\delta-1}\,|E_{\delta,\delta}(\lambda(t_j-t_{j-1})^\delta)| \\ &\le (\Gamma(\delta))^j\prod_{k=1}^{j} l_k \prod_{k=1}^{j}\left[(t_k-t_{k-1})^{\delta-1}\,|E_{\delta,\delta}(\lambda(t_k-t_{k-1})^\delta)|\right]\,|b_0-c_0|.\end{aligned}$$

Therefore,

$$\begin{aligned}|\tilde{\phi}(b_0) - \tilde{\phi}(c_0)| &= |\tilde{v}_p(b_0) - \tilde{v}_p(c_0)|\Gamma(\delta)\,(1-t_p)^{\delta-1}\,|E_{\delta,\delta}(\lambda(1-t_p)^\delta)| \\ &\le (\Gamma(\delta))^{p+1}\prod_{k=1}^{p} l_k \prod_{k=1}^{p}\left[(t_k-t_{k-1})^{\delta-1}\,|E_{\delta,\delta}(\lambda(t_k-t_{k-1})^\delta)|\right] \\ &\quad \times (1-t_p)^{\delta-1}\,|E_{\delta,\delta}(\lambda(1-t_p)^\delta)|\,|b_0-c_0|.\end{aligned}$$

Taking into account that $t_{p+1} = 1$, we write

$$|\tilde{\phi}(b_0) - \tilde{\phi}(c_0)| \leq (\Gamma(\delta))^{p+1} \prod_{k=1}^{p} l_k \prod_{k=1}^{p+1} \left[(t_k - t_{k-1})^{\delta-1} \left| E_{\delta,\delta}(\lambda(t_k - t_{k-1})^{\delta}) \right| \right] |b_0 - c_0|,$$

and, by (4.8), the existence of a unique fixed point is justified. □

Remark 4.4. Taking $p = 1$ in condition (4.8), we get condition (3.21).

Lemma 4.5. *If $I_1, I_2, \ldots, I_p$ are continuous and I_p is bounded, then problem (4.4)–(4.5) subject to the boundary condition (3.17) has at least one solution.*

Proof. In this case, $\tilde{\phi}$ in Lemma 4.3 is a continuous mapping. Let $m > 0$ be such that $|I_p(u)| \leq m$, $\forall\, u \in \mathbb{R}$ and choose $A > 0$ such that

$$A \geq m\Gamma(\delta)(1 - t_p)^{\delta-1} |E_{\delta,\delta}(\lambda(1 - t_p)^{\delta})| + \left| \int_{t_p}^{1} (1 - s)^{\delta-1} E_{\delta,\delta} \left(\lambda(1 - s)^{\delta} \right) \sigma(s)\, ds \right|.$$

Thus, the restriction of $\tilde{\phi}$ to the nonempty compact and convex set $[-A, A]$ takes values in $[-A, A]$ since, for $c_0 \in [-A, A]$,

$$\begin{aligned} |\tilde{\phi}(c_0)| &\leq |\tilde{v}_p(c_0)| \Gamma(\delta)(1 - t_p)^{\delta-1} |E_{\delta,\delta}(\lambda(1 - t_p)^{\delta})| + \left| \int_{t_p}^{1} (1 - s)^{\delta-1} E_{\delta,\delta} \left(\lambda(1 - s)^{\delta} \right) \sigma(s)\, ds \right| \\ &\leq m\Gamma(\delta)(1 - t_p)^{\delta-1} |E_{\delta,\delta}(\lambda(1 - t_p)^{\delta})| + \left| \int_{t_p}^{1} (1 - s)^{\delta-1} E_{\delta,\delta} \left(\lambda(1 - s)^{\delta} \right) \sigma(s)\, ds \right| \leq A. \end{aligned}$$

Hence, by Schauder's theorem, there exists a fixed point c_0 of $\tilde{\phi}$ in $[-A, A]$, which provides a solution to (4.4)–(4.5) subject to the boundary condition (3.17) through (4.7). □

Lemma 4.6. *If $I_1, I_2, \ldots, I_p$ are continuous and there exists $A > 0$ satisfying*

$$m\Gamma(\delta)(1 - t_p)^{\delta-1} |E_{\delta,\delta}(\lambda(1 - t_p)^{\delta})| + \left| \int_{t_p}^{1} (1 - s)^{\delta-1} E_{\delta,\delta} \left(\lambda(1 - s)^{\delta} \right) \sigma(s)\, ds \right| \leq A,$$

where $m > 0$ is such that $(\tilde{v}_p \circ \cdots \circ \tilde{v}_1)([-A, A]) \subseteq [-m, m]$ ($\tilde{v}_j$, $j = 1, \ldots, p$, given in Lemma 4.3), then problem (4.4)–(4.5) subject to the boundary condition (3.17) has at least one solution.

Proof. Again, the continuity of $I_1, \ldots, I_p$ guarantees that $(\tilde{v}_p \circ \cdots \circ \tilde{v}_1)([-B, B])$ is a compact set in $\mathbb{R}$, for every $B \in \mathbb{R}$. The result is a consequence of Schauder's theorem. □

Finally, to prove the existence of solution to the general problem (1.1)–(1.3), we consider the periodic boundary value problem

$$D^{\delta}_{t_k+} u(t) - \lambda u(t) = \sigma(t), \;\; t \in (t_k, t_{k+1}), \;\; k = 0, \ldots, p, \tag{4.10}$$

$$\lim_{t \to t_k^+} (t - t_k)^{1-\delta} (u(t) - u(t_k)) = c_k, \;\; k = 1, \ldots, p, \tag{4.11}$$

$$\lim_{t \to 0^+} t^{1-\delta} u(t) = u(1), \tag{4.12}$$

for $0 < \delta < 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $\sigma \in PC[0,1]$ and $c_k \in \mathbb{R}$, $k = 1, \dots, p$, whose solution (see (4.3)) is given by

$$u(t) = c_k\Gamma(\delta)(t-t_k)^{\delta-1}E_{\delta,\delta}(\lambda(t-t_k)^\delta) + \int_{t_k}^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^\delta\right)\sigma(s)\,ds, \tag{4.13}$$

for $t \in (t_k, t_{k+1}]$, $k = 0, 1, \dots, p$, where

$$c_0 = c_p\Gamma(\delta)(1-t_p)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_p)^\delta) + \int_{t_p}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^\delta\right)\sigma(s)\,ds. \tag{4.14}$$

In the following proposition, we provide the Green's function associated to the boundary value problem (4.10)–(4.12).

Proposition 4.7. *The solution to problem* (4.10)–(4.12) *can be written as*

$$u(t) = \begin{cases} c_p\Gamma(\delta)G_{\lambda,\delta}(t,t_p) + \displaystyle\int_0^1 G_{\lambda,\delta}(t,s)\sigma(s)\,ds, & t \in (0,t_1], \\ c_k\Gamma(\delta)G_{\lambda,\delta}(t,t_k) + \displaystyle\int_0^1 G_{\lambda,\delta}(t,s)\sigma(s)\,ds, & t \in (t_k,t_{k+1}],\ k = 1,\dots,p, \end{cases} \tag{4.15}$$

where $G_{\lambda,\delta}(t,s)$ is defined, for $(t,s) \in (0,1] \times [0,1]$, by

$$G_{\lambda,\delta}(t,s) = \begin{cases} \Gamma(\delta)t^{\delta-1}E_{\delta,\delta}\left(\lambda t^\delta\right)(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^\delta\right), & \textit{if } 0 < t \le t_1,\ t_p \le s < 1, \\ (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^\delta\right), & \textit{if } 0 < t \le t_1,\ 0 \le s < t, \\ (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^\delta\right), & \textit{if } t_k < t \le t_{k+1}, \\ & \quad t_k \le s < t, k = 1,\dots,p, \\ 0, & \textit{otherwise.} \end{cases} \tag{4.16}$$

Proof. Using (4.14) and writing the solution (4.13) in integral form, we obtain the Green's function associated to the boundary value problem (4.10)–(4.12). We remark that, for $t \in (0, t_1]$,

$$\begin{aligned} u(t) &= c_0\Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^\delta) + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^\delta\right)\sigma(s)\,ds \\ &= c_p\Gamma(\delta)(1-t_p)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_p)^\delta)\Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^\delta) \\ &\quad + \Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^\delta)\int_{t_p}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^\delta\right)\sigma(s)\,ds \\ &\quad + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^\delta\right)\sigma(s)\,ds. \end{aligned}$$

□

Remark 4.8. Note that, taking $p = 1$ in (4.15), we obtain the expression (3.29) in Proposition 3.12. Besides, the Green's function (4.16) extends that in (3.30). We also remark that (4.16) can also be shortened as

$$G_{\lambda,\delta}(t,s) = \begin{cases} \Gamma(\delta)t^{\delta-1}E_{\delta,\delta}\left(\lambda t^\delta\right)(1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^\delta\right), & \text{if } 0 < t \le t_1,\ t_p \le s < 1, \\ (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^\delta\right), & \text{if } t_k < t \le t_{k+1},\ t_k \le s < t, \\ & \quad k = 0,\dots,p, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, expression (4.15) allows to rewrite problem (1.1)–(1.3) as a fixed point formulation.

Lemma 4.9. *The solutions to problem (1.1)–(1.3) are the fixed points of the mapping $\mathcal{A}$ defined as*

$$[\mathcal{A}u](t) = \begin{cases} I_p(u(t_p))\Gamma(\delta)G_{\lambda,\delta}(t,t_p) + \int_0^1 G_{\lambda,\delta}(t,s)f(s,u(s))\,ds, & t \in (0,t_1], \\ I_k(u(t_k))\Gamma(\delta)G_{\lambda,\delta}(t,t_k) + \int_0^1 G_{\lambda,\delta}(t,s)f(s,u(s))\,ds, & t \in (t_k,t_{k+1}],\ k = 1,\dots,p, \end{cases}$$

where $G_{\lambda,\delta}$ is given by (4.16).

We remark that the operator $\mathcal{A}$ is an extension of (3.31). Due to the similarities of the expression of the mapping $\mathcal{A}$ with respect to the case $p = 1$ and analogously to the procedure followed in Lemma 3.17 and Theorem 3.18, it is possible to deduce the continuity and compactness of the mapping $\mathcal{A}$ and the existence of at least one solution to problem (1.1)–(1.3) in the space $PC_{1-\delta}[0,1]$, just by imposing the boundedness and the Lipschitzian character of the function f and the impulse functions $I_1,\dots,I_p$ and, of course, assuming (H).

Lemma 4.10. *Suppose that the following conditions hold:*

$(H1^*)$ *There exist positive constants M and m such that*

$$|f(t,u)| \le M,\ |I_k(u)| \le m,\ \forall t \in [0,1],\ u \in \mathbb{R},\ k = 1,\dots,p. \tag{4.17}$$

$(H2^*)$ *There exist positive constants K and l such that*

$$\begin{gathered} |f(t,u) - f(t,v)| \le K|u-v|, \quad |I_k(u) - I_k(v)| \le l|u-v|, \\ \forall t \in [0,1],\ u,v \in \mathbb{R},\ k = 1,\dots,p. \end{gathered} \tag{4.18}$$

Then the operator $\mathcal{A}$ defined in Lemma 4.9 is well-defined, continuous and compact.

Proof. The expression of the mapping $\mathcal{A}$ can be written as

$$\begin{aligned} [\mathcal{A}u](t) = {} & I_p(u(t_p))(\Gamma(\delta))^2 t^{\delta-1}E_{\delta,\delta}(\lambda t^\delta)(1-t_p)^{\delta-1}E_{\delta,\delta}(\lambda(1-t_p)^\delta) \\ & + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^\delta\right) f(s,u(s))\,ds \\ & + \Gamma(\delta)t^{\delta-1}E_{\delta,\delta}(\lambda t^\delta)\int_{t_p}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(1-s)^\delta\right) f(s,u(s))\,ds,\ t \in (0,t_1], \end{aligned}$$

$$\begin{aligned} [\mathcal{A}u](t) = {} & I_k(u(t_k))\Gamma(\delta)(t-t_k)^{\delta-1}E_{\delta,\delta}(\lambda(t-t_k)^\delta) \\ & + \int_{t_k}^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\lambda(t-s)^\delta\right) f(s,u(s))\,ds,\ t \in (t_k,t_{k+1}],\ k = 1,\dots,p. \end{aligned}$$

Comparing this with the expression of the mapping $\mathcal{B}$ given in Remark 3.15, it is clear that the proof can be completed similarly to the proof of Lemma 3.17. □

Theorem 4.11. *Assume that conditions $(H1^*)$ and $(H2^*)$ hold. Then the problem (1.1)–(1.3) has at least one solution in $PC_{1-\delta}[0,1]$.*

Proof. Analogously to the proof of Theorem 3.18, we consider the set

$$\mathcal{E}^* = \{u \in PC_{1-\delta}[0,1] : u = \mu\mathcal{A}(u),\ \mu \in (0,1)\}.$$

If $u = \mu\mathcal{A}(u)$ for some $\mu \in (0,1)$, then, for each $t \in (0, t_1]$, we have, similarly to (3.42),

$$\begin{aligned} t^{1-\delta}|u(t)| = t^{1-\delta}\mu\,|\mathcal{A}(u)(t)| &\le m(\Gamma(\delta))^2(1-t_p)^{\delta-1}E_{\delta,\delta}\left(|\lambda|t_1^\delta\right)E_{\delta,\delta}\left(|\lambda|(1-t_p)^\delta\right) \\ &+ Mt_1\,E_{\delta,\delta+1}\left(|\lambda|t_1^\delta\right) + M\Gamma(\delta)\,E_{\delta,\delta}\left(|\lambda|t_1^\delta\right)E_{\delta,\delta+1}\left(|\lambda|\right) = M_1^* < \infty \end{aligned}$$

and, for $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, p$, we get, similarly to (3.43),

$$\begin{aligned} (t-t_k)^{1-\delta}|u(t)| &= (t-t_k)^{1-\delta}\mu\,|\mathcal{A}(u)(t)| \\ &\le m\Gamma(\delta)E_{\delta,\delta}\left(|\lambda|(t_{k+1}-t_k)^\delta\right) + M(t_{k+1}-t_k)E_{\delta,\delta+1}\left(|\lambda|(t_{k+1}-t_k)^\delta\right) \\ &= M_{2,k} < \infty, \end{aligned}$$

hence $\|u\|_{1-\delta} \le \|\mathcal{A}(u)\|_{1-\delta} \le \max\{M_1^*,\ M_{2,1}, \ldots,\ M_{2,p}\} < \infty$ and the conclusion follows. □

Remark 4.12. In the proof of Lemma 4.10, we obtain some interesting inequalities. For instance, for $t \in (0, t_1]$ and u, $v \in PC_{1-\delta}[0,1]$,

$$\begin{aligned} &t^{1-\delta}|\mathcal{A}(u)(t) - \mathcal{A}(v)(t)| \\ &\le \|u-v\|_{1-\delta}\Big[l(t_p-t_{p-1})^{\delta-1}(\Gamma(\delta))^2E_{\delta,\delta}(|\lambda|t_1^\delta)(1-t_p)^{\delta-1}E_{\delta,\delta}(|\lambda|(1-t_p)^\delta) \\ &\qquad +Kt^{1-\delta}\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(|\lambda|(t-s)^\delta\right)s^{\delta-1}\,ds \\ &\qquad +K\Gamma(\delta)E_{\delta,\delta}(|\lambda|t^\delta)\int_{t_p}^1 (1-s)^{\delta-1}E_{\delta,\delta}\left(|\lambda|(1-s)^\delta\right)(s-t_p)^{\delta-1}\,ds\Big] \\ &\le \|u-v\|_{1-\delta}\Big[l(t_p-t_{p-1})^{\delta-1}(\Gamma(\delta))^2(1-t_p)^{\delta-1}E_{\delta,\delta}\left(|\lambda|t_1^\delta\right)E_{\delta,\delta}\left(|\lambda|(1-t_p)^\delta\right) \\ &\qquad +Kt_1^\delta\Gamma(\delta)\,E_{\delta,2\delta}\left(|\lambda|t_1^\delta\right) + K(1-t_p)^{2\delta-1}\frac{(\Gamma(\delta))^3}{\Gamma(2\delta)}\,E_{\delta,\delta}\left(|\lambda|\right)E_{\delta,\delta}\left(|\lambda|(1-t_p)^\delta\right)\Big]. \end{aligned}$$

On the other hand, for $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, p$, we have

$$\begin{aligned} &(t-t_k)^{1-\delta}|\mathcal{A}(u)(t) - \mathcal{A}(v)(t)| \\ &\le \|u-v\|_{1-\delta}\Big[l(t_k-t_{k-1})^{\delta-1}\Gamma(\delta)E_{\delta,\delta}(|\lambda|(t_{k+1}-t_k)^\delta) \\ &\qquad +K(t-t_k)^{1-\delta}\int_{t_k}^t (t-s)^{\delta-1}E_{\delta,\delta}\left(|\lambda|(t-s)^\delta\right)(s-t_k)^{\delta-1}\,ds\Big] \\ &\le \|u-v\|_{1-\delta}\Big[l(t_k-t_{k-1})^{\delta-1}\Gamma(\delta)E_{\delta,\delta}\left(|\lambda|(t_{k+1}-t_k)^\delta\right) \\ &\qquad +K(t-t_k)^{1-\delta}E_{\delta,\delta}\left(|\lambda|(t-t_k)^\delta\right)\int_{t_k}^t (t-s)^{\delta-1}\ (s-t_k)^{\delta-1}\,ds\Big] \\ &\le \|u-v\|_{1-\delta}\Big[l(t_k-t_{k-1})^{\delta-1}\Gamma(\delta)E_{\delta,\delta}\left(|\lambda|(t_{k+1}-t_k)^\delta\right) \\ &\qquad +K(t_{k+1}-t_k)^\delta\frac{(\Gamma(\delta))^2}{\Gamma(2\delta)}\,E_{\delta,\delta}\left(|\lambda|(t_{k+1}-t_k)^\delta\right)\Big]. \end{aligned}$$

In these inequalities, we have used (see [5, Eq. (4.21)]) that

$$\int_{t_k}^{t}(t-s)^{\delta-1}(s-t_k)^{\delta-1}\,ds=\int_0^{t-t_k}(t-t_k-u)^{\delta-1}u^{\delta-1}\,du=(t-t_k)^{2\delta-1}\frac{(\Gamma(\delta))^2}{\Gamma(2\delta)},$$

for $k=1,\ldots,p$ and $t\in(t_k,t_{k+1}]$.

Using these properties and under the hypotheses of Theorem 4.11, we can deduce the existence of a unique solution to problem (3.1)–(3.4), just by imposing the restriction

$$\max\{\mathcal{L}_1^*,\mathcal{L}_{2,1},\ldots,\mathcal{L}_{2,p}\}<1,$$

where

$$\begin{aligned}\mathcal{L}_1^*&=l(t_p-t_{p-1})^{\delta-1}(\Gamma(\delta))^2(1-t_p)^{\delta-1}E_{\delta,\delta}\left(|\lambda|t_1^\delta\right)E_{\delta,\delta}\left(|\lambda|(1-t_p)^\delta\right)\\&\quad+Kt_1^\delta\Gamma(\delta)\,E_{\delta,2\delta}\left(|\lambda|t_1^\delta\right)+K(1-t_p)^{2\delta-1}\frac{(\Gamma(\delta))^3}{\Gamma(2\delta)}E_{\delta,\delta}\left(|\lambda|\right)E_{\delta,\delta}\left(|\lambda|(1-t_p)^\delta\right),\end{aligned}$$

and, for $k=1,\ldots,p$,

$$\mathcal{L}_{2,k}=l(t_k-t_{k-1})^{\delta-1}\Gamma(\delta)E_{\delta,\delta}\left(|\lambda|(t_{k+1}-t_k)^\delta\right)+K(t_{k+1}-t_k)^\delta\frac{(\Gamma(\delta))^2}{\Gamma(2\delta)}E_{\delta,\delta}\left(|\lambda|(t_{k+1}-t_k)^\delta\right).$$

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