

# On the stability of some fractional-order non-autonomous systems

Sheren A. Abd El-Salam  
e.mail: shrnahmed@maktoob.com

Ahmed M. A. El-Sayed  
e.mail: amasayed@hotmail.com

Faculty of Science, Alexandria University, Alexandria, Egypt

## Abstract

The fractional calculus (integration and differentiation of fractional-order) is a one of the singular integral and integro-differential operators. In this work a class of fractional-order non-autonomous systems will be considered. The stability (and some other properties concerning the existence and uniqueness) of the solution will be proved.

**Key words:** Fractional calculus, fractional-order non-autonomous systems, stability, asymptotic stability.

## 1 Introduction

Let  $L_1[a, b]$  denotes the space of all Lebesgue integrable functions on the interval  $[a, b]$ ,  $0 \leq a < b < \infty$ .

**Definition 1.1** The fractional (arbitrary) order integral of the function  $f \in L_1[a, b]$  of order  $\beta \in R^+$  is defined by (see [2] and [4] - [6])

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds,$$

where  $\Gamma(\cdot)$  is the gamma function.

**Definition 1.2** The Riemann-Liouville fractional-order derivative of  $f(t)$  of order  $\alpha \in (0, 1)$  is defined as (see [2] and [4] - [6])

$${}_a^*D_a^\alpha f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t), \quad t \in [a, b].$$

**Definition 1.3** The (Caputo) fractional-order derivative  $D^\alpha$  of order  $\alpha \in (0, 1]$  of the function  $g(t)$  is defined as (see [4] - [6])

$$D_a^\alpha g(t) = I_a^{1-\alpha} \frac{d}{dt} g(t), \quad t \in [a, b].$$

Now consider the non-autonomous linear system:

$$x'(t) = A(t) x(t), \quad (1)$$

with the initial condition

$$x(t_0) = x^0, \quad t \geq t_0,$$

where  $A(t)$  is a continuous  $n$  by  $n$  matrix on the half-axis  $t \geq 0$ . We know that (see [1]) the solution of system (1) is given by:

$$x(t, t_0, x^0) = X(t) X^{-1}(t_0) x^0, \quad t \geq 0,$$

where  $X(t)$  is an arbitrary fundamental matrix of the system (1) defined on the whole half-axis  $t \geq 0$ .

Now we shall present the main definitions (see [1]) related to the concepts of stability of the solution  $x = 0$  of (1).

**Definition 1.4** The solution  $x = 0$  of (1) will be called stable if to any  $\varepsilon > 0$ ,  $t_0 \geq 0$  there corresponds  $\delta(\varepsilon, t_0) > 0$  such that  $\|x(t, t_0, x^0)\| < \varepsilon$  for  $t \geq t_0$  as soon as  $\|x^0\| < \delta$ .

**Definition 1.5** The solution  $x = 0$  of (1) will be called uniformly stable if  $\delta(\varepsilon, t_0)$  from definition 1.4 can be chosen independent of  $t_0$ :  $\delta(\varepsilon, t_0) \equiv \delta(\varepsilon)$ .

**Definition 1.6** The solution  $x = 0$  of (1) will be called asymptotically stable if it is stable in the sense of definition 1.4 and there exists  $\gamma(t_0) > 0$  such that  $\lim_{t \rightarrow \infty} \|x(t, t_0, x^0)\| = 0$  for every  $x(t, t_0, x^0)$  with  $\|x^0\| < \gamma$ .

**Definition 1.7** The solution  $x = 0$  of (1) will be called uniformly asymptotically stable if it is uniformly stable in the sense of definition 1.5 and moreover, for any  $\varepsilon > 0$  there exists  $T(\varepsilon) > 0$  such that  $\|x(t, t_0, x^0)\| < \varepsilon$  for every  $t \geq t_0 + T(\varepsilon)$  and all  $x^0$  with  $\|x^0\| < \gamma_0$ , where  $\gamma_0$  is independent of  $t_0$ .

In other words,  $x = 0$  of (1) will be called uniformly asymptotically stable if it is uniformly stable and  $\lim_{t \rightarrow \infty} \|x(t, t_0, x^0)\| = 0$  as  $t - t_0 \rightarrow +\infty$  uniformly with respect to  $(t_0, x^0)$ ,  $t_0 \geq 0$ ,  $\|x^0\| < \gamma_0$ .

### Theorem 1.1

Let  $X(t)$  be a fundamental matrix of the system (1). A necessary and sufficient condition for the stability of the solution  $x = 0$  is the boundedness of  $X(t)$  on  $t \geq 0$ :

$$\|X(t)\| \leq M, \quad t \geq 0.$$

A necessary and sufficient condition for the asymptotic stability of the solution  $x = 0$  is

$$\lim_{t \rightarrow \infty} \|X(t)\| = 0.$$

### Theorem 1.2

Let  $X(t)$  be a fundamental matrix of the system (1). A necessary and sufficient condition for the uniform stability of the solution  $x = 0$  is the existence of a number  $M > 0$  such that:

$$\|X(t) X^{-1}(t_0)\| \leq M, \quad t \geq t_0 \geq 0.$$

A necessary and sufficient condition for the uniform asymptotic stability of the solution  $x = 0$  is the existence of two positive numbers  $M$  and  $\eta$  such that:

$$\|X(t) X^{-1}(t_0)\| \leq M \exp[-\eta (t - t_0)], \quad t \geq t_0 \geq 0.$$

Here in this work, we study the stability (and some other properties concerning the existence and uniqueness) of the solutions of the non-autonomous linear systems:

$$D_{t_0}^\alpha x(t) = A(t) x(t) + f(t), \quad \alpha \in (0, 1]$$

and

$$x'(t) = A(t) \frac{d}{dt} I_{t_0}^\alpha x(t) + f(t), \quad \alpha \in (0, 1],$$

with the initial condition

$$x(t_0) = x^0.$$

Also the special cases:

$$D_{t_0}^\alpha x(t) = A x(t), \quad \alpha \in (0, 1], \quad x(t_0) = x^0$$

and

$$x'(t) = A \frac{d}{dt} I_{t_0}^\alpha x(t), \quad \alpha \in (0, 1], \quad x(t_0) = x^0$$

will be studied.

## 2 Existence of solution

Here the space  $B[t_0, T]$  denotes the space of all  $n$  vector functions  $y$  such that  $e^{-Nt}|y_i(t)| \in L_1[t_0, T]$ ,  $T < \infty$ ,  $N > 0$ , and the space  $C^*[t_0, T]$  denotes the space of all  $n$  vector functions  $x$  such that  $e^{-Nt}|x_i(t)| \in C[t_0, T]$ ,  $T < \infty$ ,  $N > 0$ , while the space  $AC^*[t_0, T]$  denotes the space of all  $n$  vector absolutely continuous functions, in addition the norm on  $B[t_0, T]$  will be denoted by  $\|\cdot\|_1$ , that is, for  $y \in B[t_0, T]$ ,  $\|y\|_1 = \sum_{i=1}^n \|y_i\|_1 = \sum_{i=1}^n \int_{t_0}^t e^{-Ns} |y_i(s)| ds$ , while the norm on  $C^*[t_0, T]$  will be denoted by  $\|\cdot\|_2$ , that is, for  $x \in C^*[t_0, T]$ ,  $\|x\|_2 = \sum_{i=1}^n \|x_i\|_2 = \sum_{i=1}^n \sup_t e^{-Nt} |x_i(t)|$ . Throughout this paper we define an  $n \times n$  matrix function  $A(t) = (a_{ij}(t))$ ,  $i, j = 1, 2, \dots, n$  such that  $A : [t_0, T] \rightarrow R$ ,  $T < \infty$ , also define

$$\|A^*\| := \|a_{ij}^*\| = \sup_t |a_{ij}(t)| \quad \text{and} \quad \|\tilde{A}\| := \|\tilde{a}_{ij}\| = \sup_t |a'_{ij}(t)|$$

Consider firstly the problem:

$$D_{t_0}^\alpha x(t) = A(t) x(t) + f(t), \quad \alpha \in (0, 1], \quad x(t_0) = x^0. \quad (2)$$

### Theorem 2.1

Let  $f(t) \in AC^*[t_0, T]$ . If  $A(t) \in AC^*[t_0, T]$ , then there exists a unique solution of problem (2).

**Proof.** Problem (2) is equivalent to the equation

$$y(t) = x^0 \frac{d}{dt} I_{t_0}^\alpha A(t) + \frac{d}{dt} I_{t_0}^\alpha A(t) I y(t) + \frac{d}{dt} I_{t_0}^\alpha f(t). \quad (3)$$

Indeed: let  $x(t)$  be a solution of (2) and take  $y(t) = x'(t) \Rightarrow x(t) = x^0 + I y(t)$ , then

$$I_{t_0}^{1-\alpha} y(t) = A(t) (x^0 + I y(t)) + f(t)$$

Operating by  $I_{t_0}^\alpha$  on both sides of the last equation, we obtain

$$I y(t) = x^0 I_{t_0}^\alpha A(t) + I_{t_0}^\alpha A(t) I y(t) + I_{t_0}^\alpha f(t),$$

differentiating both sides, we get (3). Conversely Let  $y(t)$  be a solution of (3), take  $y(t) = x'(t) \Rightarrow x(t) = x^0 + I y(t)$  and  $x(t_0) = x^0$ , then

$$\begin{aligned} x'(t) &= x^0 \frac{d}{dt} I_{t_0}^\alpha A(t) + \frac{d}{dt} I_{t_0}^\alpha A(t) (x(t) - x^0) + \frac{d}{dt} I_{t_0}^\alpha f(t) \\ &= \frac{d}{dt} I_{t_0}^\alpha A(t) x(t) + \frac{d}{dt} I_{t_0}^\alpha f(t). \end{aligned}$$

Operating by  $I_{t_0}^{1-\alpha}$  on both sides of the last equation, we obtain

$$\begin{aligned} I_{t_0}^{1-\alpha} x'(t) &= I_{t_0}^{1-\alpha} \frac{d}{dt} I_{t_0}^\alpha A(t) x(t) + I_{t_0}^{1-\alpha} \frac{d}{dt} I_{t_0}^\alpha f(t) \\ &= I_{t_0}^{1-\alpha} \left( A(t_0) x^0 \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + I_{t_0}^\alpha (A(t) x(t))' \right) + \frac{d}{dt} I_{t_0}^{1-\alpha} I_{t_0}^\alpha f(t) \\ &= A(t_0) x^0 + A(t) x(t) - A(t_0) x^0 + f(t), \end{aligned}$$

then

$$D_{t_0}^\alpha x(t) = A(t) x(t) + f(t).$$

Which proves the equivalence.

Now define the operator  $F : B \rightarrow B$  by

$$Fy(t) = x^0 \frac{d}{dt} I_{t_0}^\alpha A(t) + \frac{d}{dt} I_{t_0}^\alpha A(t) I y(t) + \frac{d}{dt} I_{t_0}^\alpha f(t). \quad (4)$$

Let  $y_i, z_i \in B$ , then

$$\begin{aligned} Fy_i(t) - Fz_i(t) &= \frac{d}{dt} I_{t_0}^\alpha a_{ij}(t) I (y_j(t) - z_j(t)) \\ &= I_{t_0}^\alpha a'_{ij}(t) I (y_j(t) - z_j(t)) + I_{t_0}^\alpha a_{ij}(t) (y_j(t) - z_j(t)), \\ e^{-Nt} |Fy_i(t) - Fz_i(t)| &\leq e^{-Nt} \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |a'_{ij}(s)| \int_{t_0}^s |y_j(\theta) - z_j(\theta)| d\theta ds \\ &+ e^{-Nt} \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |a_{ij}(s)| |y_j(s) - z_j(s)| ds \end{aligned}$$

$$\begin{aligned}
&\leq \|\tilde{a}_{ij}\| e^{-Nt} \int_{t_0}^t |y_j(\theta) - z_j(\theta)| \int_{\theta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds d\theta \\
&+ \|a_{ij}^*\| e^{-Nt} \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |y_j(s) - z_j(s)| ds \\
&\leq \|\tilde{a}_{ij}\| e^{-Nt} \int_{t_0}^t |y_j(\theta) - z_j(\theta)| \frac{(t-\theta)^\alpha}{\Gamma(1+\alpha)} d\theta \\
&+ \|a_{ij}^*\| e^{-Nt} \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |y_j(s) - z_j(s)| ds,
\end{aligned}$$

and

$$\begin{aligned}
\int_{t_0}^t e^{-Ns} |Fy_i(s) - Fz_i(s)| ds &\leq \|\tilde{a}_{ij}\| \int_{t_0}^t e^{-Ns} \int_{t_0}^s |y_j(\theta) - z_j(\theta)| \frac{(s-\theta)^\alpha}{\Gamma(1+\alpha)} d\theta ds \\
&+ \|a_{ij}^*\| \int_{t_0}^t e^{-Ns} \int_{t_0}^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} |y_j(\theta) - z_j(\theta)| d\theta ds \\
&\leq \|\tilde{a}_{ij}\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - z_j(\theta)| \int_{\theta}^t e^{-N(s-\theta)} \frac{(s-\theta)^\alpha}{\Gamma(1+\alpha)} ds d\theta \\
&+ \|a_{ij}^*\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - z_j(\theta)| \int_{\theta}^t e^{-N(s-\theta)} \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} ds d\theta \\
&\leq \|\tilde{a}_{ij}\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - z_j(\theta)| \int_0^{N(t-\theta)} e^{-u} \frac{u^\alpha}{N^\alpha \Gamma(1+\alpha)} \frac{du}{N} d\theta \\
&+ \|a_{ij}^*\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - z_j(\theta)| \int_0^{N(t-\theta)} e^{-u} \frac{u^{\alpha-1}}{N^{\alpha-1} \Gamma(\alpha)} \frac{du}{N} d\theta \\
&\leq \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} \|y_j - z_j\|_1 + \frac{\|a_{ij}^*\|}{N^\alpha} \|y_j - z_j\|_1 \\
&\leq \left( \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} + \frac{\|a_{ij}^*\|}{N^\alpha} \right) \|y_j - z_j\|_1,
\end{aligned}$$

therefore

$$\begin{aligned}
\|Fy_i - Fz_i\|_1 &\leq \left( \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} + \frac{\|a_{ij}^*\|}{N^\alpha} \right) \|y_j - z_j\|_1, \\
\sum_{i=1}^n \|Fy_i - Fz_i\|_1 &\leq \sum_{i=1}^n \left( \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} + \frac{\|a_{ij}^*\|}{N^\alpha} \right) \|y_j - z_j\|_1, \\
\|Fy - Fz\|_1 &\leq \left( \frac{\|\tilde{A}\|}{N^{1+\alpha}} + \frac{\|A^*\|}{N^\alpha} \right) \|y - z\|_1.
\end{aligned}$$

Now choose  $N$  large enough such that  $\frac{\|\tilde{A}\|}{N^{1+\alpha}} + \frac{\|A^*\|}{N^\alpha} < 1$ , then we get

$$\|Fy - Fz\|_1 < \|y - z\|_1,$$

therefore the map  $F : B \rightarrow B$  is contraction and (4) has a unique fixed point  $y \in B[t_0, T]$ , therefore we deduce that the problem (2) has a unique solution  $x \in AC^*[t_0, T]$ .

Consider secondly the problem

$$x'(t) = A(t) \frac{d}{dt} I_{t_0}^\alpha x(t) + f(t), \quad \alpha \in (0, 1], \quad x(t_0) = x^0. \quad (5)$$

**Theorem 2.2**

Let  $f(t) \in B[t_0, T]$ . If  $A(t)$  be an  $n \times n$  matrix function which is bounded and measurable, then there exists a unique solution of problem (5).

**Proof.** Problem (5) is equivalent to the equation

$$y(t) = x^0 \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} A(t) + A(t) I_{t_0}^\alpha y(t) + f(t), \quad (6)$$

Indeed: let  $x(t)$  be a solution of (5) and take  $y(t) = x'(t) \Rightarrow x(t) = x^0 + I y(t)$ , then

$$\begin{aligned} y(t) &= A(t) \frac{d}{dt} I_{t_0}^\alpha (x^0 + I y(t)) + f(t) \\ &= x^0 \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} A(t) + A(t) \frac{d}{dt} I I_{t_0}^\alpha y(t) + f(t) \\ &= x^0 \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} A(t) + A(t) I_{t_0}^\alpha y(t) + f(t). \end{aligned}$$

Conversely Let  $y(t)$  be a solution of (6) and take  $y(t) = x'(t) \Rightarrow x(t) = x^0 + I y(t)$  and  $x(t_0) = x^0$ , then

$$\begin{aligned} x'(t) &= x^0 \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} A(t) + A(t) I_{t_0}^\alpha x'(t) + f(t) \\ &= A(t) \frac{d}{dt} I_{t_0}^\alpha x(t) + f(t). \end{aligned}$$

Which proves the equivalence.

Now define the operator  $F : B \rightarrow B$  by

$$Fy(t) = x^0 \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} A(t) + A(t) I_{t_0}^\alpha y(t) + f(t). \quad (7)$$

Let  $y_i, z_i \in B$ , then

$$\begin{aligned} Fy_i(t) - Fz_i(t) &= a_{ij}(t) I_{t_0}^\alpha (y_j(t) - z_j(t)), \\ e^{-Nt} |Fy_i(t) - Fz_i(t)| &\leq e^{-Nt} |a_{ij}(t)| \int_{t_0}^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} |y_j(s) - z_j(s)| ds, \\ \int_{t_0}^t e^{-Ns} |Fy_i(s) - Fz_i(s)| ds &\leq \|a_{ij}^*\| \int_{t_0}^t e^{-Ns} \int_{t_0}^s \frac{(s - \theta)^{\alpha - 1}}{\Gamma(\alpha)} |y_j(\theta) - z_j(\theta)| d\theta ds \\ &\leq \|a_{ij}^*\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - z_j(\theta)| \int_{\theta}^t e^{-N(s-\theta)} \frac{(s - \theta)^{\alpha - 1}}{\Gamma(\alpha)} ds d\theta \\ &\leq \|a_{ij}^*\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - z_j(\theta)| \int_0^{N(t-\theta)} e^{-u} \frac{u^{\alpha - 1}}{N^{\alpha - 1} \Gamma(\alpha)} \frac{du}{N} d\theta \\ &\leq \frac{\|a_{ij}^*\|}{N^\alpha} \|y_j - z_j\|_1, \end{aligned}$$

therefore

$$\begin{aligned} \|Fy_i - Fz_i\|_1 &\leq \frac{\|a_{ij}^*\|}{N^\alpha} \|y_j - z_j\|_1, \\ \sum_{i=1}^n \|Fy_i - Fz_i\|_1 &\leq \sum_{i=1}^n \frac{\|a_{ij}^*\|}{N^\alpha} \|y_j - z_j\|_1, \\ \|Fy - Fz\|_1 &\leq \frac{\|A^*\|}{N^\alpha} \|y - z\|_1. \end{aligned}$$

Now choose  $N$  large enough such that  $\|A^*\| < N^\alpha$ , then we get

$$\|Fy - Fz\|_1 < \|y - z\|_1,$$

therefore the map  $F : B \rightarrow B$  is contraction and (7) has a unique fixed point  $y \in B[t_0, T]$ , therefore we deduce that the problem (5) has a unique solution  $x \in AC^*[t_0, T]$ .

### 3 Stability of non-autonomous systems

In this section we study the stability of the solution of the initial-value problems (2) and (5).

#### Theorem 3.1

The solution of the initial-value problem (2) is uniformly stable

**Proof.** Let  $y(t)$  be a solution of

$$\begin{aligned} y(t) &= x^0 \frac{d}{dt} I_{t_0}^\alpha A(t) + \frac{d}{dt} I_{t_0}^\alpha A(t) I y(t) + \frac{d}{dt} I_{t_0}^\alpha f(t) \\ &= A(t_0) x^0 \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + x^0 I_{t_0}^\alpha A'(t) + I_{t_0}^\alpha A'(t) I y(t) + I_{t_0}^\alpha A(t) y(t) + \frac{d}{dt} I_{t_0}^\alpha f(t), \end{aligned}$$

and let  $\tilde{y}(t)$  be a solution of the above linear system such that  $\tilde{y}(t_0) = \tilde{x}^0$ , then

$$\begin{aligned} y_i(t) - \tilde{y}_i(t) &= (x_j^0 - \tilde{x}_j^0) a_{ij}(t_0) \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + (x_j^0 - \tilde{x}_j^0) I_{t_0}^\alpha a'_{ij}(t) \\ &\quad + I_{t_0}^\alpha a'_{ij}(t) I (y_j(t) - \tilde{y}_j(t)) + I_{t_0}^\alpha a_{ij}(t) (y_j(t) - \tilde{y}_j(t)), \\ e^{-Nt} |y_i(t) - \tilde{y}_i(t)| &\leq e^{-Nt} |x_j^0 - \tilde{x}_j^0| |a_{ij}(t_0)| \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad + e^{-Nt} |x_j^0 - \tilde{x}_j^0| \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |a'_{ij}(s)| ds \\ &\quad + e^{-Nt} \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |a'_{ij}(s)| \int_{t_0}^s |y_j(\theta) - \tilde{y}_j(\theta)| d\theta ds \\ &\quad + e^{-Nt} \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |a_{ij}(s)| |y_j(s) - \tilde{y}_j(s)| ds \end{aligned}$$

$$\begin{aligned}
&\leq \|a_{ij}^*\| e^{-Nt} |x_j^0 - \tilde{x}_j^0| \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + \|\tilde{a}_{ij}\| e^{-Nt} |x_j^0 - \tilde{x}_j^0| \frac{(t-t_0)^\alpha}{\Gamma(1+\alpha)} \\
&+ \|\tilde{a}_{ij}\| e^{-Nt} \int_{t_0}^t |y_j(\theta) - \tilde{y}_j(\theta)| \int_{\theta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds d\theta \\
&+ \|a_{ij}^*\| e^{-Nt} \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |y_j(s) - \tilde{y}_j(s)| ds \\
&\leq \|a_{ij}^*\| e^{-Nt} |x_j^0 - \tilde{x}_j^0| \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + \|\tilde{a}_{ij}\| e^{-Nt} |x_j^0 - \tilde{x}_j^0| \frac{(t-t_0)^\alpha}{\Gamma(1+\alpha)} \\
&+ \|\tilde{a}_{ij}\| e^{-Nt} \int_{t_0}^t |y_j(\theta) - \tilde{y}_j(\theta)| \frac{(t-\theta)^\alpha}{\Gamma(1+\alpha)} d\theta \\
&+ \|a_{ij}^*\| e^{-Nt} \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |y_j(s) - \tilde{y}_j(s)| ds
\end{aligned}$$

and

$$\begin{aligned}
\int_{t_0}^t e^{-Ns} |y_i(s) - \tilde{y}_i(s)| ds &\leq \|a_{ij}^*\| e^{-Nt_0} |x_j^0 - \tilde{x}_j^0| \int_{t_0}^t e^{-N(s-t_0)} \frac{(s-t_0)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ \|\tilde{a}_{ij}\| e^{-Nt_0} |x_j^0 - \tilde{x}_j^0| \int_{t_0}^t e^{-N(s-t_0)} \frac{(s-t_0)^\alpha}{\Gamma(1+\alpha)} ds \\
&+ \|\tilde{a}_{ij}\| \int_{t_0}^t e^{-Ns} \int_{t_0}^s |y_j(\theta) - \tilde{y}_j(\theta)| \frac{(s-\theta)^\alpha}{\Gamma(1+\alpha)} d\theta ds \\
&+ \|a_{ij}^*\| \int_{t_0}^t e^{-Ns} \int_{t_0}^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} |y_j(\theta) - \tilde{y}_j(\theta)| d\theta ds \\
&\leq \|a_{ij}^*\| \|x_j^0 - \tilde{x}_j^0\|_2 \int_0^{N(t-t_0)} e^{-u} \frac{u^{\alpha-1}}{N^{\alpha-1} \Gamma(\alpha)} \frac{du}{N} \\
&+ \|\tilde{a}_{ij}\| \|x_j^0 - \tilde{x}_j^0\|_2 \int_0^{N(t-t_0)} e^{-u} \frac{u^\alpha}{N^\alpha \Gamma(1+\alpha)} \frac{du}{N} \\
&+ \|\tilde{a}_{ij}\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - \tilde{y}_j(\theta)| \int_{\theta}^t e^{-N(s-\theta)} \frac{(s-\theta)^\alpha}{\Gamma(1+\alpha)} ds d\theta \\
&+ \|a_{ij}^*\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - \tilde{y}_j(\theta)| \int_{\theta}^t e^{-N(s-\theta)} \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} ds d\theta \\
&\leq \frac{\|a_{ij}^*\|}{N^\alpha} \|x_j^0 - \tilde{x}_j^0\|_2 + \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} \|x_j^0 - \tilde{x}_j^0\|_2 \\
&+ \|\tilde{a}_{ij}\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - \tilde{y}_j(\theta)| \int_0^{N(t-\theta)} e^{-u} \frac{u^\alpha}{N^\alpha \Gamma(1+\alpha)} \frac{du}{N} d\theta \\
&+ \|a_{ij}^*\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - \tilde{y}_j(\theta)| \int_0^{N(t-\theta)} e^{-u} \frac{u^{\alpha-1}}{N^{\alpha-1} \Gamma(\alpha)} \frac{du}{N} d\theta,
\end{aligned}$$

then

$$\|y_i - \tilde{y}_i\|_1 \leq \frac{\|a_{ij}^*\|}{N^\alpha} \|x_j^0 - \tilde{x}_j^0\|_2 + \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} \|x_j^0 - \tilde{x}_j^0\|_2$$



$$\begin{aligned}
& + \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} \|y_j - \tilde{y}_j\|_1 + \frac{\|a_{ij}^*\|}{N^\alpha} \|y_j - \tilde{y}_j\|_1, \\
\left(1 - \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} - \frac{\|a_{ij}^*\|}{N^\alpha}\right) \|y_j - \tilde{y}_j\|_1 & \leq \left(\frac{\|a_{ij}^*\|}{N^\alpha} + \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}}\right) \|x_j^0 - \tilde{x}_j^0\|_2 \\
\Rightarrow \|y_i - \tilde{y}_i\|_1 & \leq \left(1 - \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} - \frac{\|a_{ij}^*\|}{N^\alpha}\right)^{-1} \left(\frac{\|a_{ij}^*\|}{N^\alpha} + \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}}\right) \|x_j^0 - \tilde{x}_j^0\|_2.
\end{aligned}$$

Now, since

$$x(t) = x^0 + I y(t),$$

then

$$\begin{aligned}
x_i(t) - \tilde{x}_i(t) & = x_i^0 - \tilde{x}_i^0 + \int_{t_0}^t (y_i(s) - \tilde{y}_i(s)) ds, \\
e^{-Nt} |x_i(t) - \tilde{x}_i(t)| & \leq e^{-Nt} |x_i^0 - \tilde{x}_i^0| + e^{-Nt} \int_{t_0}^t |y_i(s) - \tilde{y}_i(s)| ds \\
& \leq e^{-Nt_0} |x_i^0 - \tilde{x}_i^0| + e^{-Nt} \int_{t_0}^t e^{Ns} e^{-Ns} |y_i(s) - \tilde{y}_i(s)| ds, \\
\|x_i - \tilde{x}_i\|_2 & \leq \|x_i^0 - \tilde{x}_i^0\|_2 + \|y_i - \tilde{y}_i\|_1 \\
& \leq \|x_i^0 - \tilde{x}_i^0\|_2 \\
& + \left(1 - \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} - \frac{\|a_{ij}^*\|}{N^\alpha}\right)^{-1} \left(\frac{\|a_{ij}^*\|}{N^\alpha} + \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}}\right) \|x_j^0 - \tilde{x}_j^0\|_2 \\
& \leq \left(1 - \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} - \frac{\|a_{ij}^*\|}{N^\alpha}\right)^{-1} \|x_j^0 - \tilde{x}_j^0\|_2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=1}^n \|x_i - \tilde{x}_i\|_2 & \leq \sum_{i=1}^n \left(1 - \frac{\|\tilde{a}_{ij}\|}{N^{1+\alpha}} - \frac{\|a_{ij}^*\|}{N^\alpha}\right)^{-1} \|x_j^0 - \tilde{x}_j^0\|_2, \\
\|x - \tilde{x}\|_2 & \leq \left(1 - \frac{\|\tilde{A}\|}{N^{1+\alpha}} - \frac{\|A^*\|}{N^\alpha}\right)^{-1} \|x^0 - \tilde{x}^0\|_2
\end{aligned}$$

Therefore, if  $\|x^0 - \tilde{x}^0\|_2 < \delta(\varepsilon)$ , then  $\|x - \tilde{x}\|_2 < \varepsilon$ , which complete the proof of the theorem.

### Theorem 3.2

The solution of the initial-value problem (5) is uniformly stable.

**Proof.** Let  $y(t)$  be a solution of

$$y(t) = x^0 \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} A(t) + A(t) I_{t_0}^\alpha y(t) + f(t)$$

and let  $\tilde{y}(t)$  be a solution of the above linear system such that  $\tilde{y}(t_0) = \tilde{x}^0$ , then

$$\begin{aligned}
 y_i(t) - \tilde{y}_i(t) &= (x_j^0 - \tilde{x}_j^0) \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} a_{ij}(t) + a_{ij}(t) I_{t_0}^\alpha (y_j(t) - \tilde{y}_j(t)), \\
 e^{-Nt} |y_i(t) - \tilde{y}_i(t)| &\leq e^{-Nt} |x_j^0 - \tilde{x}_j^0| \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} |a_{ij}(t)| \\
 &\quad + |a_{ij}(t)| e^{-Nt} \int_{t_0}^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} |y_j(s) - \tilde{y}_j(s)| ds \\
 &\leq \|a_{ij}^*\| e^{-Nt} |x_j^0 - \tilde{x}_j^0| \frac{(t - t_0)^{\alpha - 1}}{\Gamma(\alpha)} \\
 &\quad + \|a_{ij}^*\| e^{-Nt} \int_{t_0}^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} |y_j(s) - \tilde{y}_j(s)| ds, \\
 \int_{t_0}^t e^{-Ns} |y_j(s) - \tilde{y}_j(s)| ds &\leq \|a_{ij}^*\| e^{-Nt_0} |x_j^0 - \tilde{x}_j^0| \int_{t_0}^t e^{-N(s-t_0)} \frac{(s - t_0)^{\alpha - 1}}{\Gamma(\alpha)} ds \\
 &\quad + \|a_{ij}^*\| \int_{t_0}^t e^{-Ns} \int_{t_0}^s \frac{(s - \theta)^{\alpha - 1}}{\Gamma(\alpha)} |y_j(\theta) - \tilde{y}_j(\theta)| d\theta ds \\
 &\leq \|a_{ij}^*\| \|x_j^0 - \tilde{x}_j^0\|_2 \int_0^{N(t-t_0)} e^{-u} \frac{u^{\alpha - 1}}{N^{\alpha - 1} \Gamma(\alpha)} \frac{du}{N} \\
 &\quad + \|a_{ij}^*\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - \tilde{y}_j(\theta)| \int_\theta^t e^{-N(s-\theta)} \frac{(s - \theta)^{\alpha - 1}}{\Gamma(\alpha)} ds d\theta \\
 &\leq \frac{\|a_{ij}^*\|}{N^\alpha} \|x_j^0 - \tilde{x}_j^0\|_2 \\
 &\quad + \|a_{ij}^*\| \int_{t_0}^t e^{-N\theta} |y_j(\theta) - \tilde{y}_j(\theta)| \int_0^{N(t-\theta)} e^{-u} \frac{u^{\alpha - 1}}{N^{\alpha - 1} \Gamma(\alpha)} \frac{du}{N} d\theta \\
 &\leq \frac{\|a_{ij}^*\|}{N^\alpha} \|x_j^0 - \tilde{x}_j^0\|_2 + \frac{\|a_{ij}^*\|}{N^\alpha} \|y_j - \tilde{y}_j\|_1,
 \end{aligned}$$

then

$$\begin{aligned}
 \|y_i - \tilde{y}_i\|_1 &\leq \frac{\|a_{ij}^*\|}{N^\alpha} \|x_j^0 - \tilde{x}_j^0\|_2 + \frac{\|a_{ij}^*\|}{N^\alpha} \|y_j - \tilde{y}_j\|_1, \\
 \left(1 - \frac{\|a_{ij}^*\|}{N^\alpha}\right) \|y_j - \tilde{y}_j\|_1 &\leq \frac{\|a_{ij}^*\|}{N^\alpha} \|x_j^0 - \tilde{x}_j^0\|_2 \\
 \Rightarrow \|y_i - \tilde{y}_i\|_1 &\leq \frac{\|a_{ij}^*\|}{N^\alpha} \left(1 - \frac{\|a_{ij}^*\|}{N^\alpha}\right)^{-1} \|x_j^0 - \tilde{x}_j^0\|_2 \\
 &\leq \left(\frac{\|a_{ij}^*\|}{N^\alpha - \|a_{ij}^*\|}\right) \|x_j^0 - \tilde{x}_j^0\|_2.
 \end{aligned}$$

Now, since

$$x(t) = x^0 + I y(t),$$

then

$$\|x_i - \tilde{x}_i\|_2 \leq \|x_i^0 - \tilde{x}_i^0\|_2 + \|y_i - \tilde{y}_i\|_1$$

$$\begin{aligned} &\leq \|x_i^0 - \tilde{x}_i^0\|_2 + \left( \frac{\|a_{ij}^*\|}{N^\alpha - \|a_{ij}^*\|} \right) \|x_j^0 - \tilde{x}_j^0\|_2 \\ &\leq \left( \frac{N^\alpha}{N^\alpha - \|a_{ij}^*\|} \right) \|x_j^0 - \tilde{x}_j^0\|_2. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=1}^n \|x_i - \tilde{x}_i\|_2 &\leq \sum_{i=1}^n \left( \frac{N^\alpha}{N^\alpha - \|a_{ij}^*\|} \right) \|x_j^0 - \tilde{x}_j^0\|_2, \\ \|x - \tilde{x}\|_2 &\leq \left( \frac{N^\alpha}{N^\alpha - \|A^*\|} \right) \|x^0 - \tilde{x}^0\|_2 \end{aligned}$$

Therefore, if  $\|x^0 - \tilde{x}^0\|_2 < \delta(\varepsilon)$ , then  $\|x - \tilde{x}\|_2 < \varepsilon$ , which complete the proof of the theorem.

## 4 Autonomous systems

Now we study the problems:

$$D_{t_0}^\alpha x(t) = A x(t), \quad \alpha \in (0, 1], \quad x(t_0) = x^0 \quad (8)$$

and

$$x'(t) = A \frac{d}{dt} I_{t_0}^\alpha x(t), \quad \alpha \in (0, 1], \quad x(t_0) = x^0 \quad (9)$$

which are the special cases of the initial-value problems (2) and (5) when  $A(t) = A$ , where  $A$  is a real constant matrix and  $f(t)$  is the zero vector.

### Theorem 4.1

The solution of the initial-value problem (8) or (9) is given by the formula

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} (A I_{t_0}^\alpha)^k x^0 \\ &= \sum_{k=0}^{\infty} \frac{A^k (t - t_0)^{k\alpha}}{\Gamma(1 + k\alpha)} x^0 \end{aligned}$$

**Proof.** Firstly we prove that the two problems are equivalent to each other, indeed: Let  $x(t)$  be a solution of (8), then

$$I_{t_0}^{1-\alpha} \frac{dx}{dt} = A x(t),$$

operating by  $I_{t_0}^\alpha$  on both sides of the last relation, we get

$$x(t) = x^0 + A I_{t_0}^\alpha x(t),$$

differentiating both sides, we obtain (9) and when  $t = t_0$ , we obtain  $x(t_0) = x^0$ .  
 Conversely let  $x(t)$  be a solution of (9), operating by  $I_{t_0}^{1-\alpha}$  on both sides of it, we get

$$\begin{aligned} I_{t_0}^{1-\alpha} \frac{dx}{dt} &= A I_{t_0}^{1-\alpha} \left( x^0 \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + I_{t_0}^\alpha x'(t) \right) \\ \Rightarrow D_{t_0}^\alpha x(t) &= A (x^0 + x(t) - x^0) = A x(t). \end{aligned}$$

Now since

$$\begin{aligned} e^{-Nt} |a_{ij} I_{t_0}^\alpha x_j(t)| &\leq |a_{ij}| e^{-Nt} \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s)| ds \\ &\leq |a_{ij}| \int_{t_0}^t e^{-N(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-Ns} |x_j(s)| ds, \\ \|a_{ij} I_{t_0}^\alpha x_j\|_2 &\leq |a_{ij}| \|x_j\|_2 \int_0^{N(t-t_0)} e^{-u} \frac{u^{\alpha-1}}{N^{\alpha-1} \Gamma(\alpha)} \frac{du}{N} \\ &\leq \frac{|a_{ij}|}{N^\alpha} \|x_j\|_2, \end{aligned}$$

then

$$\begin{aligned} \sum_{i=1}^n \|a_{ij} I_{t_0}^\alpha x_j\|_2 &\leq \frac{1}{N^\alpha} \sum_{i=1}^n |a_{ij}| \|x_j\|_2, \quad i = 1, 2, \dots, n, \\ \|A I_{t_0}^\alpha x\|_2 &\leq \frac{\|A\|}{N^\alpha} \|x\|_2 < \|x\|_2, \end{aligned}$$

where  $\|A\| < N^\alpha$ , it follows that  $\|A I_{t_0}^\alpha\|_2 < 1$ , then from Neumann expansion (see [3]) we complete the proof.

#### Theorem 4.2

If  $A$  is a real constant matrix with the characteristic roots all having negative real parts, then the solution of the initial-value problem (8) (or (9)) is uniformly asymptotically stable.

**Proof.**

$$\begin{aligned} |x_i(t) - \tilde{x}_i(t)| &= |x_i(t) - e^{a_{ij}(t-t_0)} x_j^0 - \tilde{x}_i(t) + e^{a_{ij}(t-t_0)} \tilde{x}_j^0 \\ &\quad + e^{a_{ij}(t-t_0)} x_j^0 - e^{a_{ij}(t-t_0)} \tilde{x}_j^0| \\ &= |(x_i(t) - e^{a_{ij}(t-t_0)} x_j^0) - (\tilde{x}_i(t) - e^{a_{ij}(t-t_0)} \tilde{x}_j^0) \\ &\quad + e^{a_{ij}(t-t_0)} (x_j^0 - \tilde{x}_j^0)| \\ &= \left| \left( \sum_{k=0}^{\infty} \frac{a_{ij}^k (t-t_0)^{k\alpha}}{\Gamma(1+k\alpha)} x_j^0 - e^{a_{ij}(t-t_0)} x_j^0 \right) \right. \\ &\quad \left. - \left( \sum_{k=0}^{\infty} \frac{a_{ij}^k (t-t_0)^{k\alpha}}{\Gamma(1+k\alpha)} \tilde{x}_j^0 - e^{a_{ij}(t-t_0)} \tilde{x}_j^0 \right) + e^{a_{ij}(t-t_0)} (x_j^0 - \tilde{x}_j^0) \right| \\ &\leq \left| \sum_{k=0}^{\infty} \frac{a_{ij}^k (t-t_0)^{k\alpha}}{\Gamma(1+k\alpha)} - \sum_{k=0}^{\infty} \frac{a_{ij}^k (t-t_0)^k}{\Gamma(1+k)} \right| |x_j^0 - \tilde{x}_j^0| \end{aligned}$$

$$\begin{aligned}
& + e^{a_{ij}(t-t_0)} |x_j^0 - \tilde{x}_j^0| \\
& = \left| - \sum_{k=0}^{\infty} \frac{a_{ij}^k}{\Gamma(1+k)} \left( 1 - \frac{\Gamma(1+k)}{\Gamma(1+k\alpha)} (t-t_0)^{k\alpha-k} \right) (t-t_0)^k |x_j^0 - \tilde{x}_j^0| \right. \\
& \left. + e^{a_{ij}(t-t_0)} |x_j^0 - \tilde{x}_j^0| \right.
\end{aligned}$$

Let  $k\alpha = k - \beta$ ,  $\alpha \in (0, 1]$ ,  $\beta > 0$ , then

$$\begin{aligned}
|x_i(t) - \tilde{x}_i(t)| & \leq \left| - \sum_{k=0}^{\infty} \frac{a_{ij}^k}{\Gamma(1+k)} \left( 1 - \frac{\Gamma(1+k)}{\Gamma(1+k-\beta)} (t-t_0)^{-\beta} \right) (t-t_0)^k |x_j^0 - \tilde{x}_j^0| \right. \\
& \left. + e^{a_{ij}(t-t_0)} |x_j^0 - \tilde{x}_j^0| \right. \\
& \leq \left| \sum_{k=0}^{\infty} \frac{a_{ij}^k (t-t_0)^k}{\Gamma(1+k)} |x_j^0 - \tilde{x}_j^0| + e^{a_{ij}(t-t_0)} |x_j^0 - \tilde{x}_j^0| \right. \\
& = e^{a_{ij}(t-t_0)} |x_j^0 - \tilde{x}_j^0| + e^{a_{ij}(t-t_0)} |x_j^0 - \tilde{x}_j^0| \\
& = 2 e^{a_{ij}(t-t_0)} |x_j^0 - \tilde{x}_j^0|.
\end{aligned}$$

Since  $A$  is a real constant matrix with the characteristic roots all having negative real parts, then there exists positive constants  $K$  and  $\sigma$  such that

$$e^{At} \leq K e^{-\sigma t},$$

therefore

$$\begin{aligned}
e^{-Nt} |x_i(t) - \tilde{x}_i(t)| & \leq 2 K e^{-\sigma(t-t_0)} e^{-Nt} |x_i^0 - \tilde{x}_i^0| \\
& \leq 2 K e^{-\sigma(t-t_0)} e^{-Nt_0} |x_i^0 - \tilde{x}_i^0|, \\
\|x_i - \tilde{x}_i\|_2 & \leq 2 K e^{-\sigma(t-t_0)} \|x_i^0 - \tilde{x}_i^0\|_2, \\
\sum_{i=1}^n \|x_i - \tilde{x}_i\|_2 & \leq 2 K e^{-\sigma(t-t_0)} \sum_{i=1}^n \|x_i^0 - \tilde{x}_i^0\|_2, \\
\|x - \tilde{x}\|_2 & \leq 2 K e^{-\sigma(t-t_0)} \|x^0 - \tilde{x}^0\|_2.
\end{aligned}$$

Therefore, we deduce that the solution is uniformly asymptotically stable.

## References

- [1] Corduneanu, C. *Principles of Differential and Integral equations*, Allyn and Bacon, Inc., Boston, Massachusetts (1971).
- [2] Miller, K. S. and Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York (1993).
- [3] Oden, J. T. *Applied Functional Analysis*, Prentice - Hall, Inc. Englewood Cliffs, New Jersey (1979).

- [4] Podlubny, I. and EL-Sayed, A. M. A. On two definitions of fractional calculus, *Preprint UEF 03-96 (ISBN 80-7099-252-2), Slovak Academy of Science-Institute of Experimental phys.* (1996).
- [5] Podlubny, I. *Fractional Differential Equations*, Acad. press, San Diego-New York-London (1999).
- [6] Samko, S., Kilbas, A. and Marichev, O. L. *Fractional Integrals and Derivatives*, Gordon and Breach Science Publisher, (1993).

(Received December 12, 2006)