Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **27**, 1–14; http://www.math.u-szeged.hu/ejqtde/

Oscillation results for even order functional dynamic equations on time scales

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Received 8 January 2014, appeared 6 June 2014
Communicated by John R. Graef

Abstract. By employing a generalized Riccati type transformation and the Taylor monomials, some new oscillation criteria for the even order functional dynamic equation

$$\left(r(t)\left|x^{\Delta^{n-1}}(t)\right|^{\alpha-1}x^{\Delta^{n-1}}(t)\right)^{\Delta}+F(t,x(t),x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t)))=0, t\in[t_0,\infty)_{\mathbb{T}},$$

are established. Several examples are also considered to illustrate the main results.

Keywords: oscillation, even order, delay dynamic equations.

2010 Mathematics Subject Classification: 34K11, 39A10, 34C10.

1 Introduction

In this paper, we consider the oscillatory behavior of solutions of the even order functional dynamic equation

$$\left(r(t)\left|x^{\Delta^{n-1}}(t)\right|^{\alpha-1}x^{\Delta^{n-1}}(t)\right)^{\Delta}+F\big(t,x(t),x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t))\big)=0,\quad t\in[t_0,\infty)_{\mathbb{T}},\tag{1.1}$$

where $n\geq 2$ is an even integer, $\alpha>0$ is a constant, $t_0\in\mathbb{T}$ and $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$ denotes a time scale interval with $\sup\mathbb{T}=\infty$. Throughout this paper, we assume that the following conditions hold:

(C1) $r\colon[t_0,\infty)_{\mathbb{T}}\to(0,\infty)$ is a real valued rd-continuous function with $r^{\Delta}(t)\geq 0$ on $[t_0,\infty)_{\mathbb{T}}$ and

$$\int_{t_0}^{\infty}\frac{\Delta s}{r^{1/\alpha}(s)}=\infty;\tag{1.2}$$

(C2) $\tau\colon\mathbb{T}\to\mathbb{T}$ is real valued rd-continuous function such that $\tau(t)\leq t$ and $\lim_{t\to\infty}\tau(t)=\infty$;

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(C3) $\operatorname{sgn} F(t,x,u,v,w) = \operatorname{sgn} x$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $x, u, v, w \in \mathbb{R}$;

(C4) $F\colon [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function, and there exists a positive rd-continuous function $q(t)$ defined on $[t_0,\infty)_{\mathbb{T}}$ such that

$$\frac{F(t,x,u,v,w)}{|u|^{\alpha-1}u} \geq q(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}},\ x,\ u \in \mathbb{R}\backslash\{0\},\ v,\ w \in \mathbb{R}; \tag{1.3}$$

(C5) $\tau^{\Delta}(t) > 0$ is rd-continuous on $\mathbb{T}$, $\widetilde{\mathbb{T}} := \tau(\mathbb{T}) = \{\tau(t) : t \in \mathbb{T}\} \subset \mathbb{T}$ is a time scale, and $(\tau^{\sigma})(t) = (\sigma \circ \tau)(t)$ for all $t \in \mathbb{T}$, where $\sigma(t)$ is the forward jump operator on $\mathbb{T}$ and $(\tau^{\sigma})(t) := (\tau \circ \sigma)(t)$.

By a solution of (1.1), we mean a non-trivial function $x\colon [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}$, $t_* \geq t_0$, such that $x \in C_{rd}^{n-1}([t_*,\infty)_{\mathbb{T}},\mathbb{R})$, $r(t)\left|x^{\Delta^{n-1}}(t)\right|^{\alpha-1}x^{\Delta^{n-1}}(t) \in C_{rd}^{1}([t_*,\infty)_{\mathbb{T}},\mathbb{R})$ and $x(t)$ satisfies equation (1.1) on $[t_*,\infty)_{\mathbb{T}}$. Our attention is restricted to those solutions of (1.1) which exist on $[t_*,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 \geq t_*$. A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory. Readers not familiar with time scale calculus and related concepts are referred to the books [1,2].

Since Hilger [17] introduced the theory of time scales in order to unify continuous and discrete analysis, there has been an increasing interest in studying the oscillation and nonoscillation of solutions of different calsses of dynamic equations on time scales. For interested readers we refer to the papers [3–14,16,18–24] and the references quoted therein. However, most of the results obtained were centered around second-order dynamic equations on time scales, and there are very few results dealing with the qualitative behavior of solutions of higher-order dynamic equations on time scales. Regarding higher-order dynamic equations, Grace et al. [9] considered the even order linear dynamic equation

$$x^{\Delta^{n}}(t) + q(t)x = 0, \quad \text{for } t \in \mathbb{T}, \tag{1.4}$$

and established some sufficient conditions for oscillation of (1.4). Grace [13] considered the even order dynamic equation

$$\left(a(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} + q(t)(x^{\sigma}(t))^{\lambda} = 0, \quad \text{for } t \in \mathbb{T}, \tag{1.5}$$

and gave some oscillation results where α and λ are the ratios of positive odd integers. Chen and Qu [5] considered the even order advanced type dynamic equation with mixed nonlinearities of the form

$$\left[r(t)\Phi_{\alpha}\left(x^{\Delta^{n-1}}(t)\right)\right]^{\Delta} + p(t)\Phi_{\alpha}(x(\delta(t))) + \sum_{i=1}^{k} p_i(t)\Phi_{\alpha i}(x(\delta(t))) = 0, \quad \text{for } t \in \mathbb{T}, \tag{1.6}$$

where $\Phi_*(u) = |u|^{*-1}u$ and $\delta(t) \geq t$, and obtained some sufficient conditions for the oscillation of the equation (1.6) that extend and supplement some results in the relevant literature.

Motivated by the works of Grace et al. [9], Grace [13,14], Chen [4], and Chen and Qu [5], using Riccati type transformations and the Taylor monomials we establish some sufficient conditions guaranteeing the oscillation of solutions of equation (1.1). Here, the results obtained extend and supplement some results in [9,13,14]. We also want to emphasize that the results in this work can be applied on the time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$. At the end, some examples are given to illustrate the theoretical analysis of this work.

2 Main results

In order to prove our main results, we shall employ the following lemmas.

Lemma 2.1 ([15]). *If X and Y are nonnegative and $\lambda > 1$, then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda - 1) Y^{\lambda},$$

where equality holds if and only if $X = Y$.

Lemma 2.2 ([4, Lemma 2.3]). *Suppose that (C5) holds. Let $x\colon \mathbb{T} \to \mathbb{R}$. If $x^{\Delta}(t)$ exists for all sufficiently large $t \in \mathbb{T}$, then $(x \circ \tau)^{\Delta}(t) = \left(x^{\Delta} \circ \tau\right)(t)\tau^{\Delta}(t)$ for all sufficiently large $t \in \mathbb{T}$.*

Lemma 2.3 ([9, Lemma 2.1]). *Let $\sup \mathbb{T} = \infty$ and $x \in C^{m}_{rd}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$. If $x^{\Delta^m}(t)$ is of constant sign on $[t_0,\infty)_{\mathbb{T}}$ and not identically zero on $[t_1,\infty)_{\mathbb{T}}$ for any $t_1 \ge t_0$, then there exist a $t_x \ge t_1$ and an integer l, $0 \le l \le m$ with $m + l$ even for $x^{\Delta^m}(t) \ge 0$ or $m + l$ odd for $x^{\Delta^m}(t) \le 0$ such that*

$$l > 0 \quad \textit{implies} \quad x^{\Delta^k}(t) > 0 \quad \textit{for} \quad t \ge t_x, \quad k \in \{0, 1, 2, \dots, l-1\} \tag{2.1}$$

and

$$l \le m - 1 \quad \textit{implies} \quad (-1)^{l+k} x^{\Delta^k}(t) > 0 \quad \textit{for} \quad t \ge t_x, \quad k \in \{l, l+1, \dots, m-1\} \tag{2.2}$$

Lemma 2.4 ([1, Theorem 1.90]). *Assume that $x(t)$ is Δ-differentiable and eventually positive or eventually negative, then*

$$(x^{\alpha}(t))^{\Delta} = \alpha \left\{ \int_0^1 [(1-h)x(t) + h x(\sigma(t))]^{\alpha-1} \, dh \right\} x^{\Delta}(t) \tag{2.3}$$

It will be convenient to employ the Taylor monomials (see [1, Sec. 1.6]) $\{h_n(t,s)\}_{n=0}^{\infty}$ which are defined recursively as follows

$$h_0(t,s) = 1, \quad h_{n+1}(t,s) = \int_s^t h_n(u,s)\Delta u, \quad t, s \in \mathbb{T} \text{ and } n \ge 0.$$

It follows that $h_1(t,s) = t - s$ for any time-scale, but simple formulas in general do not hold for $n \ge 2$.

Lemma 2.5 ([18, Corollary 1]). *Assume that $n \in \mathbb{N}$, $s, t \in \mathbb{T}$ and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then*

$$\int_s^t \int_{\eta_{n+1}}^t \cdots \int_{\eta_2}^t f(\eta_1)\Delta\eta_1\Delta\eta_2 \dots \Delta\eta_{n+1} = (-1)^n \int_s^t h_n(s, \sigma(\eta)) f(\eta)\Delta\eta$$

Now, we present some sufficient conditions for the oscillation of all solutions of equation (1.1). We begin with the following result.

Theorem 2.6. *Let (C1)–(C5) hold, and*

$$\int_{t_0}^{\infty} h_1(\sigma(s), t_0) \left(\frac{1}{r(s)} \int_s^{\infty} q(u)\Delta u \right)^{1/\alpha} \Delta s = \infty. \tag{2.4}$$

Assume also that for all sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$, there exist $T_1 > T$, and a positive non-decreasing $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $\tau(T_1) > T$ and

$$\limsup_{t\to\infty}\int_{T_1}^{t}\left[\delta(s)q(s)-r(s)\delta^{\Delta}(s)h_{n-1}^{-\alpha}(\tau(s),T)\right]\Delta s=\infty. \tag{2.5}$$

Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. Then, from (1.1) and (C3), we have, for $t \geq t_1$,

$$\left(r(t)\left|x^{\Delta^{n-1}}(t)\right|^{\alpha-1}x^{\Delta^{n-1}}(t)\right)^{\Delta}=-F(t,x(t),x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t)))<0, \tag{2.6}$$

so $r(t)\left|x^{\Delta^{n-1}}(t)\right|^{\alpha-1}x^{\Delta^{n-1}}(t)$ is eventually decreasing on $[t_1,\infty)_{\mathbb{T}}$, say for $t \in [t_2,\infty)_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}$. We now claim that

$$x^{\Delta^{n-1}}(t)>0 \quad \text{for} \quad t\geq t_2. \tag{2.7}$$

If this is not the case, then there exists $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that $x^{\Delta^{n-1}}(t_3) \leq 0$. In view of (2.6), there is a $t_4 \geq t_3$ such that

$$r(t)\left|x^{\Delta^{n-1}}(t)\right|^{\alpha-1}x^{\Delta^{n-1}}(t)\leq r(t_4)\left|x^{\Delta^{n-1}}(t_4)\right|^{\alpha-1}x^{\Delta^{n-1}}(t_4):=c<0 \quad \text{for} \quad t\in[t_4,\infty)_{\mathbb{T}}.$$

From the last inequality, we obtain

$$x^{\Delta^{n-1}}(t)\leq-(-c)^{1/\alpha}\frac{1}{r^{1/\alpha}(t)} \quad \text{for} \quad t\geq t_4. \tag{2.8}$$

Integrating (2.8) from t_4 to t, we get

$$x^{\Delta^{n-2}}(t)\leq x^{\Delta^{n-2}}(t_4)-(-c)^{1/\alpha}\int_{t_4}^{t}\frac{\Delta s}{r^{1/\alpha}(s)}$$

which gives by (1.2) that $\lim_{t\to\infty}x^{\Delta^{n-2}}(t)=-\infty$. Similarly, we can prove

$$\lim_{t\to\infty}x^{\Delta^{n-3}}(t)=\lim_{t\to\infty}x^{\Delta^{n-4}}(t)=\cdots=\lim_{t\to\infty}x^{\Delta}(t)=\lim_{t\to\infty}x(t)=-\infty,$$

which contradicts the fact that $x(t) > 0$ for $t \geq t_1$. Hence, (2.7) holds. Thus, from (1.1), (C4) and (2.7), we see that

$$\left(r(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}\leq-q(t)x^{\alpha}(\tau(t))<0 \quad \text{for} \quad t\geq t_2, \tag{2.9}$$

and so $r(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}$ is decreasing on $[t_2,\infty)_{\mathbb{T}}$. Since $\left(r(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} < 0$ on $[t_2,\infty)_{\mathbb{T}}$ and $r^{\Delta}(t) \geq 0$ on $[t_0,\infty)_{\mathbb{T}}$, we have after differentiation that

$$r^{\Delta}(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}+r^{\sigma}(t)\left(\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}<0,$$

which implies

$$\left(\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}<0 \quad \text{for} \quad t\geq t_2.$$

From this and (2.3), we obtain

$$0 > \left(\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} = \alpha x^{\Delta^{n}}(t) \int_{0}^{1} \left[(1-h)x^{\Delta^{n-1}}(t) + hx^{\Delta^{n-1}}(\sigma(t))\right]^{\alpha-1} dh. \tag{2.10}$$

Since $x^{\Delta^{n-1}}(t) > 0$ for $t \geq t_2$, we get from (2.10) that

$$x^{\Delta^{n}}(t) < 0 \quad \text{for} \quad t \geq t_2. \tag{2.11}$$

Thus, from Lemma 2.3, there exists an integer $l \in \{1, 3, \ldots, n-1\}$ such that (2.1) and (2.2) hold for all $t \geq t_2$, and so

$$x^{\Delta}(t) > 0 \quad \text{for} \quad t \geq t_2. \tag{2.12}$$

From (2.12), there exists a constant $c > 0$ such that

$$x(t) \geq x(t_2) = c \quad \text{for} \quad t \geq t_2.$$

Since $\lim_{t\to\infty} \tau(t) = \infty$, we can choose $t_3 \geq t_2$ such that $\tau(t) \geq t_2$ for all $t \geq t_3$, and so

$$x(\tau(t)) \geq x(t_2) = c > 0 \quad \text{for} \quad t \geq t_3. \tag{2.13}$$

We now claim that $l = n-1$. To this end, we suppose that

$$x^{\Delta^{n-2}}(t) < 0 \quad \text{and} \quad x^{\Delta^{n-3}}(t) > 0 \quad \text{for} \quad t \geq t_3.$$

Integrating (2.9) from $t \geq t_3$ to $u \geq t$, letting $u \to \infty$ and using (2.13), we obtain

$$x^{\Delta^{n-1}}(t) \geq c\left(\frac{1}{r(t)} \int_{t}^{\infty} q(s)\Delta s\right)^{1/\alpha} \quad \text{for} \quad t \geq t_3. \tag{2.14}$$

Integrating (2.14) from t to v, and letting $v \to \infty$, we get

$$x^{\Delta^{n-2}}(t) \leq -c \int_{t}^{\infty} \left(\frac{1}{r(u)} \int_{u}^{\infty} q(s)\Delta s\right)^{1/\alpha} \Delta u.$$

Integrating this inequality from t_3 to t, and using Lemma 2.5, we obtain

$$\begin{aligned}
x^{\Delta^{n-3}}(t) &\leq x^{\Delta^{n-3}}(t_3) - c \int_{t_3}^{t} \int_{v}^{\infty} \left(\frac{1}{r(u)} \int_{u}^{\infty} q(s)\Delta s\right)^{1/\alpha} \Delta u \Delta v \\
&= x^{\Delta^{n-3}}(t_3) + c \int_{t_3}^{t} h_1(t_3, \sigma(v)) \left(\frac{1}{r(v)} \int_{v}^{\infty} q(s)\Delta s\right)^{1/\alpha} \Delta v \\
&= x^{\Delta^{n-3}}(t_3) - c \int_{t_3}^{t} h_1(\sigma(v), t_3) \left(\frac{1}{r(v)} \int_{v}^{\infty} q(s)\Delta s\right)^{1/\alpha} \Delta v,
\end{aligned}$$

and so

$$\int_{t_3}^{\infty} h_1(\sigma(v), t_3) \left(\frac{1}{r(v)} \int_{v}^{\infty} q(s)\Delta s\right)^{1/\alpha} \Delta v \leq \frac{1}{c} x^{\Delta^{n-3}}(t_3) < \infty,$$

which contradicts (2.4). Therefore, we have $l = n-1$.

Since $l = n-1$, we have by Lemma 2.3 that

$$x^{\Delta^n}(t) < 0 \quad \text{and} \quad x^{\Delta^i}(t) > 0, \quad i = 0, 1, \ldots, n-1 \quad \text{for } t \geq t_3. \tag{2.15}$$

In view of the facts that $x^{\Delta^{n-1}}(t)$ is decreasing on $[t_3, \infty)_{\mathbb{T}}$, $x^{\Delta^{n-2}}(t) > 0$ for $t \geq t_3$, and

$$x^{\Delta^{n-2}}(t) = x^{\Delta^{n-2}}(t_3) + \int_{t_3}^{t} x^{\Delta^{n-1}}(s)\Delta s,$$

we obtain

$$x^{\Delta^{n-2}}(t) \geq h_1(t, t_3) x^{\Delta^{n-1}}(t) \quad \text{for} \quad t \geq t_3. \tag{2.16}$$

Integrating (2.16) $(n-3)$ times from t_3 to t, we obtain

$$x^{\Delta}(t) \geq h_{n-2}(t, t_3) x^{\Delta^{n-1}}(t) \quad \text{for} \quad t \geq t_3. \tag{2.17}$$

Integrating (2.17) from t_3 to t, we find

$$x(t) \geq h_{n-1}(t, t_3) x^{\Delta^{n-1}}(t) \quad \text{for} \quad t \geq t_3. \tag{2.18}$$

Now, consider the generalized Riccati substitution

$$w(t) = \delta(t) \frac{r(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}}{x^{\alpha}(\tau(t))} \quad \text{for} \quad t \geq t_3. \tag{2.19}$$

Clearly, $w(t) > 0$, and

$$\begin{aligned} w^{\Delta}(t) &= \frac{\delta(t)}{x^{\alpha}(\tau(t))}\left(r(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} + \left(r(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\sigma}(t)\left[\frac{\delta(t)}{x^{\alpha}(\tau(t))}\right]^{\Delta} \\ &\leq -\delta(t)q(t) + \left(r(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\sigma}(t)\left[\frac{\delta^{\Delta}(t)}{x^{\alpha}(\tau(\sigma(t)))} - \frac{\delta(t)\left(x^{\alpha}(\tau(t))\right)^{\Delta}}{x^{\alpha}(\tau(t))x^{\alpha}(\tau(\sigma(t)))}\right] \\ &= -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\delta(t)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)\left(x^{\alpha}(\tau(t))\right)^{\Delta}}{x^{\alpha}(\tau(t))x^{\alpha}(\tau(\sigma(t)))}. \end{aligned} \tag{2.20}$$

By Lemmas 2.2 and 2.4, we get

$$\begin{aligned} \left(x^{\alpha}(\tau(t))\right)^{\Delta} &= \alpha\left\{\int_0^1 \left[(1-h)x(\tau(t)) + hx(\tau(\sigma(t)))\right]^{\alpha-1} dh\right\}\left(x(\tau(t))\right)^{\Delta} \\ &\geq \begin{cases} \alpha\left(x(\tau(t))\right)^{\alpha-1} x^{\Delta}(\tau(t))\tau^{\Delta}(t), & \alpha \geq 1 \\ \alpha\left(x(\tau(\sigma(t)))\right)^{\alpha-1} x^{\Delta}(\tau(t))\tau^{\Delta}(t), & 0 < \alpha < 1. \end{cases} \end{aligned} \tag{2.21}$$

If $0 < \alpha < 1$, then (2.20) and (2.21) imply

$$\begin{aligned} w^{\Delta}(t) &\leq -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha\delta(t)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)\left(x(\tau(\sigma(t)))\right)^{\alpha-1} x^{\Delta}(\tau(t))\tau^{\Delta}(t)}{x^{\alpha}(\tau(t))x^{\alpha}(\tau(\sigma(t)))} \\ &= -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha\delta(t)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)x^{\Delta}(\tau(t))\tau^{\Delta}(t)}{x^{\alpha+1}(\tau(\sigma(t)))}\frac{\left(x(\tau(\sigma(t)))\right)^{\alpha}}{x^{\alpha}(\tau(t))}. \end{aligned} \tag{2.22}$$

If $\alpha \geq 1$, then (2.20) and (2.21) imply

$$w^{\Delta}(t) \leq -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha\delta(t)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)\left(x(\tau(t))\right)^{\alpha-1}x^{\Delta}(\tau(t))\tau^{\Delta}(t)}{x^{\alpha}(\tau(t))x^{\alpha}(\tau(\sigma(t)))}$$
$$= -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha\delta(t)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)x^{\Delta}(\tau(t))\tau^{\Delta}(t)}{x^{\alpha+1}(\tau(\sigma(t)))}\frac{x(\tau(\sigma(t)))}{x(\tau(t))}. \quad (2.23)$$

Since $t \leq \sigma(t)$, $\tau^{\Delta}(t) > 0$, and $x(t)$ is increasing on $[t_3,\infty)_{\mathbb{T}}$, we get $x(\tau(t)) \leq x(\tau(\sigma(t)))$. Therefore, (2.22) and (2.23) yield

$$w^{\Delta}(t) \leq -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha\delta(t)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)x^{\Delta}(\tau(t))\tau^{\Delta}(t)}{x^{\alpha+1}(\tau(\sigma(t)))}, \quad (2.24)$$

on $[t_3,\infty)_{\mathbb{T}}$ for $\alpha > 0$. From (2.17) and (2.18) , we have

$$x^{\Delta}(\tau(t)) \geq h_{n-2}(\tau(t),t_3)x^{\Delta^{n-1}}(\tau(t)) \quad \text{for} \quad t \geq t_4 \geq t_3 \quad (2.25)$$

and

$$x(\tau(t)) \geq h_{n-1}(\tau(t),t_3)x^{\Delta^{n-1}}(\tau(t)) \quad \text{for} \quad t \geq t_4 \geq t_3, \quad (2.26)$$

respectively, where we assume that $\tau(t) \geq t_3$ for $t \geq t_4$. Using (2.25) in (2.24) , we obtain

$$w^{\Delta}(t) \leq -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha\delta(t)h_{n-2}(\tau(t),t_3)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)x^{\Delta^{n-1}}(\tau(t))\tau^{\Delta}(t)}{x^{\alpha+1}(\tau(\sigma(t)))}$$
$$\leq -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha\delta(t)h_{n-2}(\tau(t),t_3)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)x^{\Delta^{n-1}}(t)\tau^{\Delta}(t)}{x^{\alpha+1}(\tau(\sigma(t)))} \quad (2.27)$$

for $t \geq t_4$. In view of (2.9), (2.15) and $\tau^{\Delta}(t) > 0$, (2.27) yields

$$w^{\Delta}(t) \leq -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)$$
$$\leq -\delta(t)q(t) + \delta^{\Delta}(t)\frac{\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)}{x^{\alpha}(\tau(\sigma(t)))}$$
$$\leq -\delta(t)q(t) + \delta^{\Delta}(t)\frac{r(t)\left(x^{\Delta^{n-1}}\right)^{\alpha}(t)}{x^{\alpha}(\tau(\sigma(t)))}$$
$$\leq -\delta(t)q(t) + \delta^{\Delta}(t)\frac{r(t)\left(x^{\Delta^{n-1}}\right)^{\alpha}(\tau(t))}{x^{\alpha}(\tau(t))}$$
$$= -\delta(t)q(t) + r(t)\delta^{\Delta}(t)\left(\frac{x^{\Delta^{n-1}}(\tau(t))}{x(\tau(t))}\right)^{\alpha}. \quad (2.28)$$

Using (2.26) in (2.28) , we see that

$$w^{\Delta}(t) \leq -\delta(t)q(t) + r(t)\delta^{\Delta}(t)h_{n-1}^{-\alpha}(\tau(t),t_3) \quad \text{for} \quad t \geq t_4. \quad (2.29)$$

Integrating (2.29) from t_4 ($\tau(t) > t_3$ for $t \geq t_4$) to t, we obtain

$$w(t) \leq w(t_4) - \int_{t_4}^{t} \left[\delta(s)q(s) - r(s)\delta^{\Delta}(s)h_{n-1}^{-\alpha}(\tau(s), t_3)\right] \Delta s. \tag{2.30}$$

Taking the limit superior of both sides of the inequality (2.30) as $t \to \infty$ and using (2.5) we obtain a contradiction to the fact that $w(t) > 0$ on $[t_3, \infty)_{\mathbb{T}}$. This completes the proof. □

Theorem 2.7. *Let (C1)–(C5) and (2.4) hold. Assume also that for all sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, there exist $T_1 > T$, and a positive non-decreasing $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $\tau(T_1) > T$ and*

$$\limsup_{t\to\infty} \int_{T_1}^{t} \left[\delta(s)q(s) - \frac{1}{(\alpha+1)^{\alpha+1}} r(s)\left(\delta^{\Delta}(s)\right)^{\alpha+1}\left(\delta(s)\tau^{\Delta}(s)h_{n-2}\left(\tau(s), T\right)\right)^{-\alpha}\right] \Delta s = \infty. \tag{2.31}$$

Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \geq t_1$. Proceeding as in the proof of Theorem 2.6, we obtain $l = n-1$, (2.17), (2.18), (2.25) and (2.26). Define the function $w(t)$ by (2.19). Then as in the proof of Theorem 2.6, we arrive at (2.27) which can be rewritten as

$$\begin{aligned} w^{\Delta}(t) \leq & -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} w^{\sigma}(t) \\ & - \frac{\alpha\delta(t)h_{n-2}(\tau(t), t_3)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{1/\alpha}(t)\tau^{\Delta}(t)}{r^{1/\alpha}(t)x^{\alpha+1}(\tau(\sigma(t)))}. \end{aligned} \tag{2.32}$$

Since $t \leq \sigma(t)$, we have $\left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)(t) \geq \left(r\left(x^{\Delta^{n-1}}\right)^{\alpha}\right)^{\sigma}(t)$. Using this in (2.32), we obtain

$$w^{\Delta}(t) \leq -\delta(t)q(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} w^{\sigma}(t) - \frac{\alpha\delta(t)h_{n-2}(\tau(t), t_3)\tau^{\Delta}(t)}{r^{1/\alpha}(t)\left(\delta^{\sigma}(t)\right)^{1+1/\alpha}}\left(w^{\sigma}(t)\right)^{1+1/\alpha} \quad \text{for} \quad t \geq t_4. \tag{2.33}$$

Letting

$$X = \left(\frac{\alpha\delta(t)\tau^{\Delta}(t)h_{n-2}(\tau(t), t_3)}{r^{1/\alpha}(t)}\right)^{\alpha/(\alpha+1)} \frac{w^{\sigma}(t)}{\delta^{\sigma}(t)}, \quad \lambda = 1 + 1/\alpha$$

and

$$Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\left(\delta^{\Delta}(t)\right)^{\alpha}\left(\left[\frac{r^{1/\alpha}(t)}{\alpha\delta(t)\tau^{\Delta}(t)h_{n-2}(\tau(t), t_3)}\right]^{\alpha/(\alpha+1)}\right)^{\alpha},$$

in Lemma 2.1, (2.33) implies

$$w^{\Delta}(t) \leq -\delta(t)q(t) + \frac{r(t)\left(\delta^{\Delta}(t)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left[\delta(t)\tau^{\Delta}(t)h_{n-2}(\tau(t), t_3)\right]^{\alpha}} \quad \text{on} \quad [t_4, \infty)_{\mathbb{T}}. \tag{2.34}$$

Integrating (2.34) from t_4 ($\tau(t) > t_3$ for $t \geq t_4$) to t, we get

$$w(t) \leq w(t_4) - \int_{t_4}^{t} \left[\delta(s)q(s) - \frac{r(s)\left(\delta^{\Delta}(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left[\delta(s)\tau^{\Delta}(s)h_{n-2}(\tau(s), t_3)\right]^{\alpha}}\right] \Delta s. \tag{2.35}$$

Taking the limit superior of both sides of the inequality (2.35) as $t \to \infty$ and using (2.31), we get a contradiction to the fact that $w(t) > 0$ on $[t_3, \infty)_{\mathbb{T}}$. This completes the proof. □

Theorem 2.8. *Let $\alpha \ge 1$, conditions (C1)–(C5) and (2.4) hold. Assume also that for all sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$, there exist $T_1 > T$ and a positive non-decreasing $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $\tau(T_1) > T$ and*

$$\limsup_{t\to\infty}\int_{T_1}^{t}\left[\delta(s)q(s)-\frac{r^{\sigma}(s)\left(\delta^{\Delta}(s)\right)^2}{4\alpha\delta(s)\tau^{\Delta}(s)h_{n-2}\left(\tau(s),T\right)\left(h_{n-1}\left(\tau(s),T\right)\right)^{\alpha-1}}\right]\Delta s=\infty. \tag{2.36}$$

Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \ge t_1$. Proceeding as in the proof of Theorem 2.7, we arrive at (2.33) which for $t \ge t_4$ can be rewritten as

$$w^{\Delta}(t)\le-\delta(t)q(t)+\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-\frac{\alpha\delta(t)h_{n-2}(\tau(t),t_3)\tau^{\Delta}(t)}{r^{1/\alpha}(t)\left(\delta^{\sigma}(t)\right)^{1+1/\alpha}}\left(w^{\sigma}(t)\right)^{\frac{1}{\alpha}-1}\left(w^{\sigma}(t)\right)^2. \tag{2.37}$$

From (2.18), we have

$$\frac{x(t)}{x^{\Delta^{n-1}}(t)}h_{n-1}(t,t_3)\quad\text{for}\quad t\ge t_3. \tag{2.38}$$

Thus,

$$\begin{aligned}w^{\frac{1}{\alpha}-1}(t)&=\delta^{\frac{1}{\alpha}-1}(t)r^{\frac{1}{\alpha}-1}(t)\left(\frac{x^{\Delta^{n-1}}(t)}{x(\tau(t))}\right)^{1-\alpha}\\&=\delta^{\frac{1}{\alpha}-1}(t)r^{\frac{1}{\alpha}-1}(t)\left(\frac{x(\tau(t))}{x^{\Delta^{n-1}}(t)}\right)^{\alpha-1}.\end{aligned} \tag{2.39}$$

Using the fact that $x^{\Delta^{n-1}}(t)$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$, we have $x^{\Delta^{n-1}}(\tau(t)) \ge x^{\Delta^{n-1}}(t)$. From this, (2.38) and (2.39), there exists $t_4 \ge t_3$ such that

$$\begin{aligned}w^{\frac{1}{\alpha}-1}(t)&\ge\delta^{\frac{1}{\alpha}-1}(t)r^{\frac{1}{\alpha}-1}(t)\left(\frac{x(\tau(t))}{x^{\Delta^{n-1}}(\tau(t))}\right)^{\alpha-1}\\&\ge\delta^{\frac{1}{\alpha}-1}(t)r^{\frac{1}{\alpha}-1}(t)\left(h_{n-1}(\tau(t),t_3)\right)^{\alpha-1}\quad\text{for } t\ge t_4.\end{aligned} \tag{2.40}$$

Using (2.40) in (2.37), we obtain

$$\begin{aligned}w^{\Delta}(t)\le&-\delta(t)q(t)+\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-\frac{\alpha\delta(t)\tau^{\Delta}(t)h_{n-2}(\tau(t),t_3)\left(h_{n-1}(\tau(\sigma(t)),t_3)\right)^{\alpha-1}}{r^{\sigma}(t)\left(\delta^{\sigma}(t)\right)^2}\left(w^{\sigma}(t)\right)^2\\\le&-\delta(t)q(t)+\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-\frac{\alpha\delta(t)\tau^{\Delta}(t)h_{n-2}(\tau(t),t_3)\left(h_{n-1}(\tau(t),t_3)\right)^{\alpha-1}}{r^{\sigma}(t)\left(\delta^{\sigma}(t)\right)^2}\left(w^{\sigma}(t)\right)^2\\\le&-\delta(t)q(t)-\left[\frac{\sqrt{\alpha\delta(t)\tau^{\Delta}(t)h_{n-2}(\tau(t),t_3)\left(h_{n-1}(\tau(t),t_3)\right)^{\alpha-1}}}{\delta^{\sigma}(t)\sqrt{r^{\sigma}(t)}}w^{\sigma}(t)\right.\\&\left.-\frac{\delta^{\Delta}(t)\sqrt{r^{\sigma}(t)}}{2\sqrt{\alpha\delta(t)\tau^{\Delta}(t)h_{n-2}(\tau(t),t_3)\left(h_{n-1}(\tau(t),t_3)\right)^{\alpha-1}}}\right]^2\\&+\frac{r^{\sigma}(t)\left(\delta^{\Delta}(t)\right)^2}{4\alpha\delta(t)\tau^{\Delta}(t)h_{n-2}(\tau(t),t_3)\left(h_{n-1}(\tau(t),t_3)\right)^{\alpha-1}}\\\le&-\delta(t)q(t)+\frac{r^{\sigma}(t)\left(\delta^{\Delta}(t)\right)^2}{4\alpha\delta(t)\tau^{\Delta}(t)h_{n-2}(\tau(t),t_3)\left(h_{n-1}(\tau(t),t_3)\right)^{\alpha-1}}.\end{aligned} \tag{2.41}$$

Integrating (2.41) from t_4 ($\tau(t) > t_3$ for $t \geq t_4$) to t, we obtain

$$w(t) \leq w(t_4) - \int_{t_4}^{t} \left[\delta(s)q(s) - \frac{r^{\sigma}(s)\left(\delta^{\Delta}(s)\right)^2}{4\alpha\delta(s)\tau^{\Delta}(s)h_{n-2}(\tau(s),t_3)\left(h_{n-1}(\tau(s),t_3)\right)^{\alpha-1}} \right] \Delta s.$$

Taking the limit superior of both sides of the last inequality as $t \to \infty$ and using (2.36), we obtain a contradiction to the fact that $w(t) > 0$ on $[t_3, \infty)_{\mathbb{T}}$. This completes the proof. □

3 Examples

In this section, we give some examples to illustrate our main results.

Example 3.1. Consider the dynamic equation

$$\left(\left| x^{\Delta^3}(t) \right|^{-1/2} x^{\Delta^3}(t) \right)^{\Delta} + \left(t^3 + t\right) |x(t-1)|^{-1/2} x(t-1) \left(1 + x^2(t) + \left(x^{\Delta}(t)\right)^2 \right) = 0, \quad (3.1)$$

for $t \in [1, \infty)_{\mathbb{T}}$, where $n = 4$, $\alpha = 1/2$, $\tau(t) = t - 1 \leq t$, $r(t) = 1$, $q(t) = t^3 + t$, and $\mathbb{T} = \mathbb{Z}$.

Then

$$\int_{t_0}^{\infty} \frac{\Delta s}{r^{1/\alpha}(s)} = \int_{1}^{\infty} \Delta s = \infty,$$

and so (1.2) holds. Since $\int_s^{\infty} (u^3 + u)\Delta u = \infty$ for $s \geq 1$, and $\sigma(t) = t + 1$, we obtain

$$\begin{aligned} \int_{t_0}^{\infty} h_1(\sigma(s), t_0) \left(\frac{1}{r(s)} \int_s^{\infty} q(u)\Delta u \right)^{1/\alpha} \Delta s &= \int_1^{\infty} h_1(s+1, 1) \left(\int_s^{\infty} \left(u^3 + u\right) \Delta u \right)^2 \Delta s \\ &= \int_1^{\infty} s \left(\int_s^{\infty} \left(u^3 + u\right) \Delta u \right)^2 \Delta s = \infty, \end{aligned}$$

so (2.4) holds.

With $\delta(t) = 1$, and for all $T_1 \geq \tau(T_1) > T \geq 1$,

$$\limsup_{t \to \infty} \int_{T_1}^{t} \left[\delta(s)q(s) - r(s)\delta^{\Delta}(s)h_{n-1}^{-\alpha}(\tau(s), T) \right] \Delta s = \limsup_{t \to \infty} \int_{T_1}^{t} \left(s^3 + s\right) \Delta s = \infty,$$

which implies that (2.5) holds. Therefore, equation (3.1) is oscillatory by Theorem 2.6.

Example 3.2. Consider the dynamic equation

$$\left(t^{2/3} \left| x^{\Delta}(t) \right| x^{\Delta}(t) \right)^{\Delta} + \frac{1}{t^2} \left| x\left(\frac{t}{2}\right) \right| x\left(\frac{t}{2}\right) \left(1 + \frac{1}{1 + \left(x^{\Delta}(t)\right)^2} \right) = 0 \qquad (3.2)$$

for $t \in [1, \infty)_{\mathbb{T}}$, where $n = 2$, $r(t) = t^{2/3}$, $q(t) = 1/t^2$, $\tau(t) = t/2 \leq t$, $\alpha = 2$ and $t \in \mathbb{T} := \overline{2^{\mathbb{Z}}} = \left\{2^k : k \in \mathbb{Z}\right\} \cup \{0\}$. Then, $\sigma(t) = 2t$,

$$\int_{t_0}^{\infty} \frac{\Delta s}{r^{1/\alpha}(s)} = \int_{1}^{\infty} \frac{\Delta s}{\left(s^{2/3}\right)^{1/2}} = \int_{1}^{\infty} \frac{\Delta s}{s^{1/3}} = \infty,$$

and

$$\int_{t_0}^{\infty} h_1(\sigma(s),t_0)\left(\frac{1}{r(s)}\int_s^{\infty} q(u)\Delta u\right)^{1/\alpha}\Delta s = \int_1^{\infty} h_1(2s,1)\left(\frac{1}{s^{2/3}}\int_s^{\infty}\frac{1}{u^2}\Delta u\right)^{1/2}\Delta s$$
$$= \int_1^{\infty}(2s-1)\left(\frac{2}{s^{5/3}}\right)^{1/2}\Delta s$$
$$= \sqrt{2}\int_1^{\infty}\frac{2s-1}{s^{5/6}}\Delta s \geq \int_1^{\infty}\frac{\sqrt{2}}{s^{5/6}}\Delta s = \infty,$$

so (1.2) and (2.4) are satisfied.

For $\delta(t) = t$ and for all $T_1 \geq \tau(T_1) > T \geq 1$, condition (2.31) becomes

$$\limsup_{t\to\infty}\int_{T_1}^{t}\left[\delta(s)q(s) - \frac{1}{(\alpha+1)^{\alpha+1}}r(s)\left(\delta^{\Delta}(s)\right)^{\alpha+1}\left(\delta(s)\tau^{\Delta}(s)h_{n-2}(\tau(s),T)\right)^{-\alpha}\right]\Delta s$$
$$= \limsup_{t\to\infty}\int_{T_1}^{t}\left[\frac{1}{s} - \frac{4}{27}\frac{s^{2/3}}{s^2(h_0(\frac{s}{2},T))}\right]\Delta s$$
$$= \limsup_{t\to\infty}\int_{T_1}^{t}\left[\frac{1}{s} - \frac{4}{27}\frac{1}{s^{4/3}}\right]\Delta s. \tag{3.3}$$

Since

$$\lim_{s\to\infty}\left(\frac{1}{s} - \frac{4}{27s^{4/3}}\right)\Big/\left(\frac{1}{s}\right) = 1 > 0 \quad \text{and} \quad \int_{T_1}^{\infty}\frac{1}{s}\Delta s = \infty,$$

we have

$$\int_{T_1}^{\infty}\left[\frac{1}{s} - \frac{4}{27}\frac{1}{s^{4/3}}\right]\Delta s = \infty.$$

Thus, from (3.3), we get

$$\limsup_{t\to\infty}\int_{T_1}^{t}\left[\frac{1}{s} - \frac{4}{27}\frac{1}{s^{4/3}}\right]\Delta s = \infty,$$

which implies that (2.31) holds. Therefore, equation (3.2) is oscillatory by Theorem 2.7.

Example 3.3. Consider the dynamic equation

$$\left(t\left|x^{\Delta^5}(t)\right|^2 x^{\Delta^5}(t)\right)^{\Delta} + t^{-3}\left|x\left(\frac{t}{4}\right)\right|^2 x\left(\frac{t}{4}\right)\left(1 + (x(t))^2 + \frac{1}{1+(x^{\Delta}(t))^2}\right) = 0 \tag{3.4}$$

for $t \in [2,\infty)_{\mathbb{T}}$, where $n = 6$, $r(t) = t$, $q(t) = t^{-3}$, $\tau(t) = t/4 \leq t$, $\alpha = 3$ and $\mathbb{T} = \mathbb{R}$. Then, $\sigma(t) = t$. Thus,

$$\int_{t_0}^{\infty}\frac{\Delta s}{r^{1/\alpha}(s)} = \int_2^{\infty}\frac{ds}{s^{1/3}} = \infty,$$

and

$$\int_{t_0}^{\infty} h_1(\sigma(s),t_0)\left(\frac{1}{r(s)}\int_{s}^{\infty} q(u)\Delta u\right)^{1/\alpha}\Delta s = \int_{2}^{\infty} h_1(s,1)\left(\frac{1}{s}\int_{s}^{\infty} u^{-3}du\right)^{1/3} ds$$
$$= \int_{2}^{\infty}(s-1)\left(\frac{1}{2s^3}\right)^{1/3} ds$$
$$= \frac{1}{2^{1/3}}\int_{2}^{\infty}\frac{s-1}{s}ds = \infty,$$

so (1.2) and (2.4) hold.

For $\delta(t) = t^2$ and $T_1 \geq \tau(T_1) > T \geq 2$, condition (2.36) becomes

$$\limsup_{t\to\infty}\int_{T_1}^{t}\left[\delta(s)q(s) - \frac{r^{\sigma}(s)\left(\delta^{\Delta}(s)\right)^2}{4\alpha\delta(s)\tau^{\Delta}(s)h_{n-2}(\tau(s),T)\left(h_{n-1}(\tau(s),T)\right)^{\alpha-1}}\right]\Delta s$$
$$= \limsup_{t\to\infty}\int_{T_1}^{t}\left[\frac{1}{s} - \frac{4s^3}{3s^2 h_4(\frac{s}{4},T)(h_5(\frac{t}{4},T))^2}\right] ds$$
$$= \limsup_{t\to\infty}\int_{T_1}^{t}\left[\frac{1}{s} - \frac{11520s}{3(\frac{s}{4}-T)^4\left((\frac{s}{4}-T)^5\right)^2}\right] ds. \tag{3.5}$$

Since

$$\lim_{s\to\infty}\left(\frac{1}{s} - \frac{11520s}{3(\frac{s}{4}-T)^4\left((\frac{s}{4}-T)^5\right)^2}\right)\Big/\left(\frac{1}{s}\right) = 1 > 0 \quad\text{and}\quad \int_{T_1}^{\infty}\frac{1}{s}ds = \infty,$$

we have

$$\int_{T_1}^{\infty}\left[\frac{1}{s} - \frac{11520s}{3(\frac{s}{4}-T)^4\left((\frac{s}{4}-T)^5\right)^2}\right] ds = \infty.$$

Thus, from (3.5), we get

$$\limsup_{t\to\infty}\int_{T_1}^{t}\left[\frac{1}{s} - \frac{11520s}{3(\frac{s}{4}-T)^4\left((\frac{s}{4}-T)^5\right)^2}\right] ds = \infty,$$

which implies that (2.36) holds. Therefore, equation (3.4) is oscillatory by Theorem 2.8.

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